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based on JETP Lett. 116 (2022) 1, 61-69

J.Phys.G 50 (2023) 1, 015001 [2203.09307 [hep-ph]]

Phys.Rev.D 107 (2023) 9, 094034 [2302.12171 [hep-ph]]

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Int.J.Mod.Phys.A 40 (2025) 04, 2450175 [2404.01873 [hep-ph]]

Particles 8 (2025) 1, 29

Phys.Rev.D 113 (2026) 1, 014041 [2510.23025 [hep-ph]]

Conference “Particle physics at medium and high energies”, June 2-5, Protvino

QCD with Analytic Coupling

0. History. QED.

Consider so-called polarization operator $D(k^2)$ in QED. Leading logarithmic terms of $D(k^2)$ in the n order of perturbation theory with $|k^2| >> m^2$ (m is the electron mass) have the following form:

$$(e^2F(K^2,m^2))^n/K^2,\ K^2=-k^2\geq 0,\ F(K^2,m^2)=\frac{1}{3\pi}\ln\left(\frac{K^2}{4m^2}\right).$$

Resummation of the large logarithms leads to (Landau,Abrikosov,Khalatnikov:1954):

$$D_{\rm per}(k^2)=\frac{1}{K^2}\frac{1}{1-\frac{e^2}{3\pi}\ln\left(\frac{K^2}{4m^2}\right)}.$$

Then, there is the pole *(so-called Landau pole)* at K_p^2:

$$K_p^2=4m^2e^{3\pi/e^2}$$

and QED is not applicable at $K^2\geq K_p^2$ (Landau,Pomeranchuk:1955).

With another side, there is so-called Kallen-Lehmann representation:

$$D(k^2) = \frac{1}{K^2} + \int_{4m^2}^{\infty} dz \, \frac{I(z)}{z+K^2}, \quad I(z) = ImD(i\varepsilon - K^2)$$

and $D_{\rm per}(k^2)$ is not in agreement with the Kallen-Lehmann representation.

Combination of the Kallen-Lehmann representation and perturbation theory *(or same, perturbation theory for $I(z)$)* has been considered in (Redmond:1958), (Redmond,Uretsky:1958), (Bogolyubov,Logunov,Shirkov:1959): $I(z) \to I_{\rm per}(z)$.

We follow (Bogolyubov,Logunov,Shirkov:1959).

From calculation (Landau,Abrikosov,Khalatnikov:1954) they obtained that $I_{\text{per}}(z) = 0$ for $z < 4m^2$ and for $z \geq 4m^2$:

$$I_{\text{per}}(z) = \frac{e^2}{3\pi z} \frac{1}{\left(\left(1 - \frac{e^2}{3\pi} \ln\left(\frac{z-4m^2}{4m^2}\right)\right)^2 + \frac{e^2}{9}\right)} .$$

Using $I_{\text{per}}(z)$ in the Kallen-Lehmann representation they obtained at $|k^2| >> m^2$

$$D(k^2) = \frac{1}{K^2} \frac{1}{1 - \frac{e^2}{3\pi} \ln\left(\frac{K^2}{4m^2}\right)} + \frac{(3\pi)/e^2}{K^2 - K_p^2} .$$

The additional term cancels exacly Landau pole at $K^2 = K_p^2$. Moreover, it cannot be obtained in the framework of perturbation theory, since it cannot be expanded in e^2-series.

Thus, the combination of perturation theory and Kallen-Lehmann representation (i.e. perturbation theory for spectral function) does not lead to the Landau problem in QED.

In the general case the QCD couplant is defined as a product of propagators and a vertex function. Therefore, one might pose a question concerning the analytic properties of this quantity. This matter has been examined (Ginzburg,Shirkov:1965).

It was shown that in this case the integral representation of the Kallen-Lehmann type holds for the running coupling, too. Proceeding from these motivations, the analytic approach was lately extended to Quantum Chromodynamics by D.V. Shirkov and I.L. Solovtsov.

1. Analytic coupling constant

According to the general principles of (local) quantum field theory (QFT) (Bogolyubov,Shirkov:1959); (Oehme:1994) observables in the spacelike domain can have singularities only with negative values of their argument Q^2.

On the other hand, for large values of Q^2, these observables are usually represented as power series expansion by the running coupling constant (couplant) $\alpha_s(Q^2)$, which, in turn, has a ghost singularity, the so-called Landau pole, for $Q^2 = \Lambda^2$.

To restore analyticity, this pole must be removed.

1.1 Strong coupling constant

Strong couplant $\alpha_s(Q^2)$ obeys the renormalized group equation

$$L \equiv \ln\frac{Q^2}{\Lambda^2} = \int^{\overline{a}_s(Q^2)} \frac{da}{\beta(a)}, \quad \overline{a}_s(Q^2) = \frac{\alpha_s(Q^2)}{4\pi}, \quad a_s(Q^2) = \beta_0\, \overline{a}_s(Q^2)$$

with some boundary condition and the QCD β-function:

$$\beta(a_s) \;=\; -\sum_{i=0} \beta_i \overline{a}_s^{i+2} = -\beta_0 \overline{a}_s^2 \,(1 + \sum_{i=1} b_i a_s^i), \quad b_i = \frac{\beta_i}{\beta_0^{i+1}}\,,$$

where the first fifth coefficients, i.e. β_i with $i \leq 4$, are exactly known (Baikov,Chetyrkin,Kuhn: 2017).

So, already at leading order (LO), when $a_s(Q^2) = a_s^{(1)}(Q^2)$, we have

$$a_s^{(1)}(Q^2) = \frac{1}{L}\,,$$

i.e. $a_s^{(1)}(Q^2)$ does contain a pole at $Q^2 = \Lambda^2$.

1.2 Beyond LO

Following (Cvetic, Valenzuela: 2006), we introduce here the derivatives (in the k-order of perturbation theory (PT))

$$\tilde{a}_{n+1}^{(k)}(Q^2) = \frac{(-1)^n}{n!} \frac{d^n a_s^{(k)}(Q^2)}{(dL)^n}, a_s^{(k)}(Q^2) = \frac{\beta_0 \alpha_s^{(k)}(Q^2)}{4\pi} = \beta_0 \, \overline{a}_s^{(k)}(Q^2),$$

which are very convenient in the case of analytic QCD. β_0 is the first coefficient of the QCD β-function:

$$\beta(\overline{a}_s^{(k)}) = -\left(\overline{a}_s^{(k)}\right)^2 (\beta_0 + \sum_{i=1}^{k} \beta_i \left(\overline{a}_s^{(k)}\right)^i),$$

where β_i are known up to $k=4$ (Baikov, Chetyrkin, Kuhn: 2008).

The series of derivatives $\tilde{a}_n(Q^2)$ can successfully replace the corresponding series of a_s-powers (see, e.g. (Kotikov, Zemlyakov: 2022)). Indeed, each derivative reduces the a_s power but is accompanied by an additional β-function $\sim a_s^2$. Thus, each application of a derivative yields an additional a_s, and thus it is indeed possible to use a series of derivatives instead of a series of a_s-powers.

In LO, the series of derivatives $\tilde{a}_n(Q^2)$ are exactly the same as a_s^n. Beyond LO, the relationship between $\tilde{a}_n(Q^2)$ and a_s^n was established in (Cvetic, Valenzuela: 2006), (Cvetic, Kogerler, Valenzuela: 2010) and extended to the fractional case, where $n \to$ is a non-integer ν, in (Cvetic, Kotikov: 2012)

Now we consider the $1/L$ expansion of $\tilde{a}_{\nu}^{(k)}(Q^2)$. After some calculations, we have

$$\tilde{a}_{\nu,0}^{(1)}(Q^2) = \left(a_{s,0}^{(1)}(Q^2)\right)^{\nu} = \frac{1}{L_0^{\nu}},$$

$$\tilde{a}_{\nu,i}^{(i+1)}(Q^2) = \tilde{a}_{\nu,i}^{(1)}(Q^2) + \sum_{m=1}^{i} C_m^{\nu+m}\, \tilde{\delta}_{\nu,i}^{(m+1)}(Q^2),$$

$$\tilde{\delta}_{\nu,i}^{(m+1)}(Q^2) = \hat{R}_m \frac{1}{L_i^{\nu+m}}, \quad C_m^{\nu+m} = \frac{\Gamma(\nu+m)}{m!\Gamma(\nu)},$$

where

$$\hat{R}_1 = b_1[\hat{Z}_1(\nu) + \frac{d}{d\nu}], \quad \hat{R}_2 = b_2 + b_1^2[\frac{d^2}{(d\nu)^2} + 2\hat{Z}_1(\nu+1)\frac{d}{d\nu} + \hat{Z}_2(\nu+1)].$$

The representation of the $\tilde{\delta}_{\nu,i}^{(m+1)}(Q^2)$ corrections as $\hat{R}_m$-operators is very important to use. This will make it possible to present high-order results for the analytic couplant in a similar way.

Here

$$Z_2(\nu) = S_1^2(\nu) - S_2(\nu),$$
$$Z_1(\nu) \equiv S_1(\nu) = \Psi(1+\nu) + \gamma_{\mathrm{E}}, \quad S_2(\nu) = \zeta_2 - \Psi'(1+\nu),$$

and

$$S_m(N) = \sum_{k=1}^{N} \frac{1}{k^m}, \quad \hat{Z}_1(\nu) = Z_1(\nu) - 1, \quad \hat{Z}_2(\nu) = Z_2(\nu) - 2Z_1(\nu) + 1.$$

1.3. The case $\nu = 1$

For the case $\nu = 1$,

$$a^{(1)}_{s,0}(Q^2) = \frac{1}{L_0}, \quad a^{(i+1)}_{s,i}(Q^2) = a^{(1)}_{s,i}(Q^2) + \sum_{m=2}^{i} \delta^{(m)}_{s,i}(Q^2), \quad L_i = \ln \frac{Q^2}{\Lambda_i^2},$$

where the corrections $\delta^{(m)}_{s,k}(Q^2)$ can be represented as follows

$$\delta^{(2)}_{s,k}(Q^2) = -\frac{b_1 \ln L_k}{L_k^2}, \quad \delta^{(3)}_{s,k}(Q^2) = \frac{1}{L_k^3} \left[b_1^2 (\ln^2 L_k - \ln L_k - 1) + b_2 \right].$$

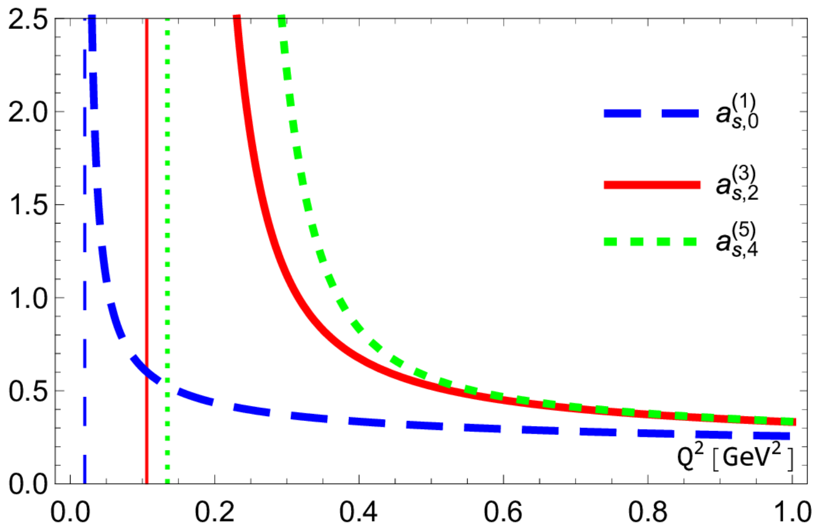

Figure 1: The results for $a_{s,i}^{(i+1)}(Q^2)$ and $(\Lambda_i^{f=3})^2$ (vertical lines) with $i = 0, 2, 4$.

In Fig. 1 one can see that the strong couplants $a_{s,i}^{(i+1)}(Q^2)$ become to be singular at $Q^2 = \Lambda_i^2$. The Λ_0 and Λ_i $(i \geq 1)$ values are rather different (Chen,Liu,Wang,Waqas,Peng: 2021):

$$\Lambda_0^{f=3} = 142 \text{ MeV}, \quad \Lambda_1^{f=3} = 367 \text{ MeV}, \quad \Lambda_2^{f=3} = 324 \text{ MeV},$$
$$\Lambda_3^{f=3} = 328 \text{ MeV}.$$

2. MA coupling

There are several ways to obtain analytical versions of the strong couplant a_s (see, e.g. (Bakulev: 2008)).

In a series of papers (Shirkov,Solovtsov: 1996,1997); (Milton,Solovtsov,Solovtsova: 1997); (Shirkov: 2001) authors have developed an effective approach to eliminate the Landau singularity without introducing extraneous IR regulators.

The idea: the dispersion relation, which connects the new analytic couplant $A_{\rm MA}(Q^2)$ with the spectral function $r_{\rm pt}(s)$, obtained in the framework of perturbative theory. In LO

$$A^{(1)}_{\rm MA}(Q^2) = \frac{1}{\pi} \int_0^{+\infty} \frac{ds}{(s+t)} \, r^{(1)}_{\rm pt}(s)\,, \quad r^{(1)}_{\rm pt}(s) = {\rm Im}\; a^{(1)}_s(-s-i\epsilon)\,,$$

So, let's repeat once again: the spectral function is taken directly from perturbation theory, but the analytic couplant $A_{\rm MA}(Q^2)$ is restored using dispersion relations.
This approach is called $Minimal\ Approach$ (MA) (Cvetic, Valenzuela: 2008) or $Analytic\ Perturbation\ Theory$ (APT) (Shirkov, Solovtsov:1996,1997); (Milton,Solovtsov,Solovtsova:1997); (Shirkov:2001)

Thus, MA QCD is a very convenient approach that combines the general (analytical) properties of quantum field quantities and the results obtained within the framework of perturbative QCD, leading to the appearance of the MA couplant $A_{\rm MA}(Q^2)$, which is close to the usual strong couplant $a_s(Q^2)$ in the limit of large values of its argument and completely different at $Q^2 \leq \Lambda^2$.

A further development of APT is the so-called fractional APT (FAPT), which extends the principles of constructing to non-integer powers of couplant, which arise for many quantities having non-zero anomalous dimensions (Bakulev,Mikhailov,Stefanis: 2005,2008,2010), with some privious study (Karanikas,Stefanis: 2001)
and reviews (Bakulev: 2008), (Stefanis: 2013).

The results in FATP have a very simple form in LO perturbation theory, but they are quite complicated in higher orders.

2.1 LO

The LO minimal analytic coupling $A^{(1)}_{\text{MA},\nu}$ have the form (Bakulev,Mikhailov,Stefanis: 2005)

$$A^{(1)}_{\text{MA},\nu,0}(Q^2) = \left(a^{(1)}_{\nu,0}(Q^2)\right)^{\nu} - \frac{\text{Li}_{1-\nu}(z_0)}{\Gamma(\nu)} \equiv \frac{1}{L_0^{\nu}} - \Delta^{(1)}_{\nu,0}\,,$$

where

$$\text{Li}_{\nu}(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^{\nu}} = \frac{z}{\Gamma(\nu)} \int_0^{\infty} \frac{dt\; t^{\nu-1}}{(e^t - z)}, \quad z_i = \frac{\Lambda_i^2}{Q^2}$$

is the Polylogarithmic function.

For $\nu = 1$ we recover the famous Shirkov-Solovtsov result (Shirkov, Solovtsov: 1996)

$$A^{(1)}_{\text{MA},0}(Q^2) \equiv A^{(1)}_{\text{MA},\nu=1,0}(Q^2) = a^{(1)}_{s,0}(Q^2) - \frac{z_0}{1-z_0} = \frac{1}{L_0} - \frac{z_0}{1-z_0}\,.$$

2.2 Beyond LO

Following to the LO analytic couplant, we consider the difference between the derivatives of usual and MA couplants:

$$\tilde{A}_{\mathrm{MA},n+1}(Q^2) = \frac{(-1)^n}{n!} \frac{d^n A_{\mathrm{MA}}(Q^2)}{(dL)^n} \,.$$

For the differences of fracted derivatives of usual and MA couplants

$$\tilde{\Delta}^{(i+1)}_{\nu,i} \equiv \tilde{a}^{(i+1)}_{\nu,i} - \tilde{A}^{(i+1)}_{\mathrm{MA},\nu,i}$$

we have the following results

$$\tilde{\Delta}^{(i+1)}_{\nu,i} = \tilde{\Delta}^{(1)}_{\nu,i} + \sum_{m=1}^{i} C^{\nu+m}_{m} \, \hat{R}_m \left(\frac{\mathrm{Li}_{-\nu-m+1}(z_i)}{\Gamma(\nu+m)} \right) ,$$

where the operators $\hat{R}_i$ $(i = 1, 2, 3, 4)$ are shown above.

After some evaluations, we obtain

$$\tilde{\Delta}^{(i+1)}_{\nu,i} = \tilde{\Delta}^{(1)}_{\nu,i} + \sum_{m=1}^{i} C^{\nu+m}_{m} \, \overline{R}_m(z_i) \left(\frac{\mathrm{Li}_{-\nu-m+1}(z_i)}{\Gamma(\nu+m)} \right) ,$$

where

$$\overline{R}_1(z) = b_1[\gamma_{\mathrm{E}} - 1 + \mathrm{M}_{-\nu,1}(z)],$$
$$\overline{R}_2(z) = b_2 + b_1^2[\mathrm{M}_{-\nu-1,2}(z) + 2(\gamma_{\mathrm{E}} - 1)\mathrm{M}_{-\nu-1,1}(z) + (\gamma_{\mathrm{E}} - 1)^2 - \zeta_2],$$

and

$$\mathrm{Li}_{\nu,k}(z) = (-1)^k \frac{d^k}{(d\nu)^k} \mathrm{Li}_\nu(z) = \sum_{m=1}^{\infty} \frac{z^m \ln^k m}{m^\nu}, \quad \mathrm{M}_{\nu,k}(z) = \frac{\mathrm{Li}_{\nu,k}(z)}{\mathrm{Li}_\nu(z)}.$$

So, we have for MA analytic couplants $\tilde{A}^{(i+1)}_{\rm MA,\nu}$ the following expressions:

$$\tilde{A}^{(i+1)}_{\rm MA,\nu,i}(Q^2) = \tilde{A}^{(1)}_{\rm MA,\nu,i}(Q^2) + \sum_{m=1}^{i} C_m^{\nu+m} \tilde{\delta}^{(m+1)}_{\rm MA,\nu,i}(Q^2)$$

where

$$\tilde{A}^{(1)}_{\rm MA,\nu,i}(Q^2) = \tilde{a}^{(1)}_{\nu,i}(Q^2) - \frac{{\rm Li}_{1-\nu}(z_i)}{\Gamma(\nu)},$$

$$\tilde{\delta}^{(m+1)}_{\rm MA,\nu,i}(Q^2) = \tilde{\delta}^{(m+1)}_{\nu,i}(Q^2) - \overline{R}_m(z_i)\, \frac{{\rm Li}_{-\nu+1-m}(z_i)}{\Gamma(\nu+m)}$$

and $\tilde{\delta}^{(k+1)}_{\nu,m}(Q^2)$ are given above.

There are three more representations for $\tilde{A}^{(1)}_{\rm MA,\nu,i}(Q^2)$ (see (Kotikov, Zemlyakov: 2005)) that give exactly the same numerical results. Each of the representations is useful in its own kinematic range.

2.3. The case $\nu = 1$

For the case $\nu = 1$,

$$A^{(i+1)}_{\mathrm{MA},i}(Q^2) \equiv \tilde{A}^{(i+1)}_{\mathrm{MA},\nu=1,i}(Q^2) = A^{(1)}_{\mathrm{MA},i}(Q^2) + \sum_{m=1}^{i} \tilde{\delta}^{(m+1)}_{\mathrm{MA},1,i}(Q^2)$$

where

$$A^{(1)}_{\mathrm{MA},i}(Q^2) = \tilde{a}^{(1)}_{\nu=1,i}(Q^2) - \mathrm{Li}_0(z_i) = a^{(1)}_{s,i}(Q^2) - \mathrm{Li}_0(z_i),$$
$$\tilde{\delta}^{(m+1)}_{\mathrm{MA},1,i}(Q^2) = \tilde{\delta}^{(m+1)}_{1,i}(Q^2) - \overline{R}_m(z_i)\,\frac{\mathrm{Li}_{-m}(z_i)}{m!}$$

and

$$\mathrm{Li}_0(z) = \frac{z}{1-z}, \quad \mathrm{Li}_{-1}(z) = \frac{z}{(1-z)^2}, \quad \mathrm{Li}_{-2}(z) = \frac{z(1+z)}{(1-z)^3}\,.$$

The results can be used for phenomenological studies beyond LO in the framework of the minimal analytic QCD.

Here we apply the inverse logarithmic expansion of the MA couplants, recently obtained in (Kotikov, Zemlyakov: 2023) for any PT order. This approach is very convenient: for LO the MA couplants have simple representations (see (Bakulev, Mikhailov, Stefanis: 2007,2007,2010)), while beyond LO the MA couplants are very close to LO ones, especially for $Q^2 \to \infty$ and $Q^2 \to 0$, where the differences between MA couplants of various PT orders become insignificant. Moreover, for $Q^2 \to \infty$ and $Q^2 \to 0$ the (fractional) derivatives of the MA couplants with $n \geq 2$ tend to zero, and therefore only the first term in perturbative expansions makes a valuable contribution.

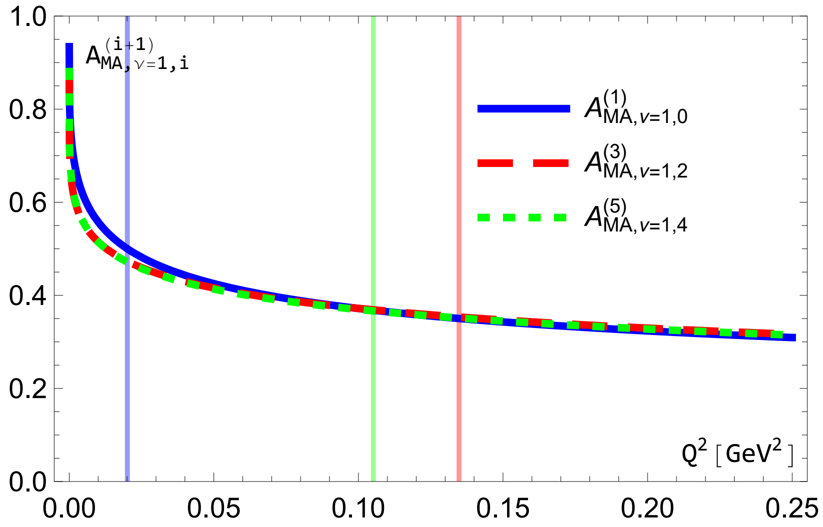

Figure 2: The results for $A^{(i+1)}_{\mathrm{MA},\nu=1,\mathrm{i}}(Q^2)$ with $i = 0, 2, 4$.

From Fig. 2 we can see differences between $A^{(i+1)}_{\mathrm{MA},\nu=1,\mathrm{i}}(Q^2)$ with $i = 0, 2, 4$, which are rather small and have nonzero values around the position $Q^2 = \Lambda_i^2$.

3. Bjorken and Gross-Llewellyn Smith sum rules

The polarized BSR is defined as the difference between the proton and neutron polarized SFs, integrated over the entire interval x

$$\Gamma_1^{p-n}(Q^2) = \int_0^1 dx\, [g_1^p(x, Q^2) - g_1^n(x, Q^2)].$$

The GLS sum rule is defined by the integral

$$C_{GLS}^{p+n}(Q^2) = \frac{1}{2} \int_0^1 \left[F_3^{\overline{\nu}p}(x, Q^2) + F_3^{\nu n}(x, Q^2)\right] dx\,,$$

which in the quark-parton model counts the number of valence quarks in the proton.

Perturbatively computed radiative corrections for the Bjorken and Gross-Llewellyn-Smith sum rules serve as an important test of the suitability of QCD as a theory of strong interactions.

Theoretically, the quantities $\Gamma_1^{p-n}(Q^2)$ and $C_{GLS}^{p+n}(Q^2)$ can be written in the Operator Product Expansion (OPE) forms

$$\Gamma_1^{p-n}(Q^2) = \frac{g_A}{6}\,(1 - D_{\rm BS}(Q^2)) + \sum_{i=1} \frac{\mu_{2i+2}^{\rm BS}}{Q^{2i}}\,,$$

$$C_{GLS}^{p+n}(Q^2) = 3\,(1 - D_{\rm GLS}(Q^2)) + \sum_{i=1} \frac{\mu_{2(i+1)}^{\rm GLS}}{Q^{2n}}\,,$$

where g_A=1.2762 $\pm$ 0.0005 is the nucleon axial charge, $(1 - D_{BS}(Q^2))$ and $(1 - D_{\rm GLS}(Q^2)$ are the leading-twist (or twist-two) contributions, and $\mu_{2i+2}^{\rm BS}/Q^{2i}$ and $\mu_{2i+2}^{\rm GLS}/Q^{2i}$ $(i \geq 1)$ are the higher-twist (HT) contributions.

Since we plan to consider in particular small Q^2 values here, these HT representations are not so convenient. So, it is preferable to use the so-called "massive" twist-four representations, (Teryaev: 2013), (Khandramai, Teryaev, Gabdrakhmanov: 2016)) which includes a part of the above HT contributions.

$$\Gamma_1^{p-n}(Q^2) = \frac{g_A}{6}\,(1 - D_{\rm BS}(Q^2)) + \frac{\hat{\mu}_4^{\rm BS} M_{\rm BS}^2}{Q^2 + M_{\rm BS}^2}\,,$$

$$\overline{C}_{GLS}^{p+n}(Q^2) = 3\,(1 - D_{GLS}(Q^2)) + \frac{\hat{\mu}_4^{GLS} M_{GLS}^2}{Q^2 + M_{GLS}^2}.$$

For $Q^2 >> M_{\rm a}^2$ $(a = BS, GLS)$, the "massive" twist-four representation can be expanded in powers of $M_{\rm a}^2/Q^2$, and the obtained results will have the form shown on the standart OPE form.

Up to the k-th PT order, the twist-two parts have the form

$$D_{\rm a}^{(1)}(Q^2) = \frac{4}{\beta_0}\, a_s^{(1)}, \quad D_{\rm a}^{(k\geq 2)}(Q^2) = \frac{4}{\beta_0}\, a_s^{(k)} \left(1 + \sum_{m=1}^{k-1} d_m^{\rm a} (a_s^{(k)})^m\right), \qquad ({\rm a} = {\rm BS}, {\rm GLS}),$$

where $d_1^{\rm a}$, $d_2^{\rm a}$ and $d_3^{\rm a}$ are known from exact calculations. (Baikov,Chetyrkin,Kuehn: 2010,2012). The $d_4^{\rm BS}$ value was estimated in (Ayala,Pineda: 2022).

Converting the coupling powers into its derivatives, we have

$$D_{\rm a}^{(1)}(Q^2) = \frac{4}{\beta_0}\, \tilde{a}_1^{(1)}, \quad D_{\rm a}^{(k\geq 2)}(Q^2) = \frac{4}{\beta_0} \left(\tilde{a}_1^{(k)} + \sum_{m=2}^{k} \tilde{d}_{m-1}^{\rm a} \tilde{a}_m^{(k)}\right), \qquad ({\rm a} = {\rm BS}, {\rm GLS}),$$

where ($b_i = \beta_i/\beta_0^{i+1}$)

$$\tilde{d}_1^{\rm a} = d_1^{\rm a}, \quad \tilde{d}_2^{\rm a} = d_2^{\rm a} - b_1 d_1^{\rm a}, \quad \tilde{d^{\rm a}}_3 = d_3^{\rm a} - \frac{5}{2} b_1 d_2^{\rm a} - (b_2 - \frac{5}{2} b_1^2)\, d_1^{\rm a},$$

$$\tilde{d}_4^{\rm a} = d_4^{\rm a} - \frac{13}{3} b_1 d_3^{\rm a} - (3b_2 - \frac{28}{3} b_1^2)\, d_2^{\rm a} - (b_3 - \frac{22}{3} b_1 b_2 + \frac{28}{3} b_1^3)\, d_1^{\rm a}\,.$$

In MA QCD, the above results become as follows

$$\Gamma_{\mathrm{MA},1}^{p-n}(Q^2) = \frac{g_A}{6}\,(1 - D_{\mathrm{MA,BS}}(Q^2)) + \frac{\hat{\mu}_{\mathrm{MA},4}^{\mathrm{BS}} M_{\mathrm{BS}}^2}{Q^2 + M_{\mathrm{BS}}^2},$$

$$\overline{C}_{\mathrm{MA,GLS}}^{p+n}(Q^2) = 3\,(1 - D_{\mathrm{MA,GLS}}(Q^2)) + \frac{\hat{\mu}_{\mathrm{MA},4}^{\mathrm{GLS}} M_{\mathrm{GLS}}^2}{Q^2 + M_{\mathrm{GLS}}^2},$$

where the perturbative parts $D_{\mathrm{BS,MA}}(Q^2)$ and $D_{\mathrm{GLS,MA}}(Q^2)$ take the same forms, however, with analytic coupling $\tilde{A}_{\mathrm{MA},\nu}^{(k)}$

$$D_{\mathrm{MA,a}}^{(1)}(Q^2) = \frac{4}{\beta_0}\,A_{\mathrm{MA}}^{(1)}, \quad D_{\mathrm{MA,a}}^{k\geq 2}(Q^2) = \frac{4}{\beta_0}\,(A_{\mathrm{MA}}^{(1)}$$

$$+ \sum_{m=2}^{k} \tilde{d}_{m-1}^{\mathrm{a}}\,\tilde{A}_{\mathrm{MA},\nu=\mathrm{m}}^{(k)})\,, \quad (\mathrm{a} = \mathrm{BS}, \mathrm{GLS}),$$

For the case of 3 active quark flavors ($f = 3$), which is accepted here, we have

$$d_1^{\rm BS} = \tilde{d}_1^{\rm BS} = 1.59,\ d_2^{\rm BS} = 3.99\ (\tilde{d}_2^{\rm BS} = 2.73),\ d_3^{\rm BS} = 15.42\ (\tilde{d}_3^{\rm BS} = 8.61),$$
$$d_4^{\rm BS} = 63.76\ (\tilde{d}_4^{\rm BS} = 21.52),$$
$$d_1^{\rm GLS} = \tilde{d}_1^{\rm GLS} = 1.59,\ \ d_2^{\rm GLS} = 3.75\ (\tilde{d}_2^{\rm GLS} = 2.51),$$
$$d_3^{\rm GLS} = 16.77\ (\tilde{d}_3^{\rm GLS} = 10.44),$$

i.e., the coefficients in the series of derivatives are slightly smaller.

It is important to emphasize that the perturbative structure of the Bjorken sum rule is very similar to that of the GLS sum rule. Both sum rules are governed by the identical (in LO) twist-2 part. Their approximate equivalence has been discussed in (Broadhurst,Kataev: 1993).

Therefore, building on the definitions of the perturbative coefficients d_i, we can (with some precision) treat BSR and GLS as the same dependence with only different general factor:

$$C_{GLS}^{p+n}(Q^2) \approx \frac{18}{g_A} \Gamma_1^{p-n}(Q^2)$$

and, thus, for analytic QCD we have similarly

$$C_{\mathrm{A,GLS}}^{p+n}(Q^2) \approx \frac{18}{g_A} \Gamma_{\mathrm{A},1}^{p-n}(Q^2) \,.$$

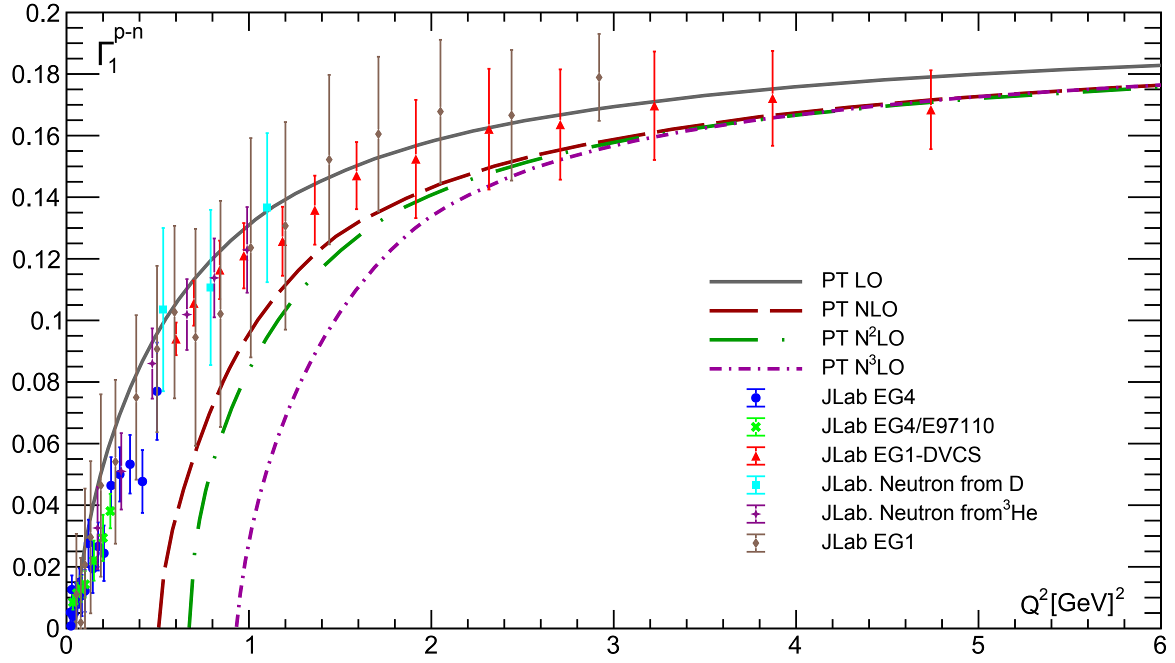

Figure 3: The results for $\Gamma_1^{p-n}(Q^2)$ in the first four orders of PT.

4. Results for Bjorken sum rule

The fitting results of experimental data obtained only with statistical uncertainties are presented in Table 1 and shown in Figs. 3 and 6. For the fits we use Q^2-independent M^2 and $\hat{\mu}_4$ and the two-twist parts for regular PT and APT, respectively.

As it can be seen in Fig. 3, with the exception of LO, the results obtained using conventional couplant are very poor. Moreover, the discrepancy in this case increases with the order of PT.

The LO results describe experimental points relatively well, since the value of $\Lambda_{\rm LO}$ is quite small compared to other Λ_i, and disagreement with the data begins at lower values of Q^2.

Thus, using the "massive" twist-four form does not improve these results, since with $Q^2 \to \Lambda_i^2$ conventional couplants become singular, which leads to large and negative results for the twist-two part (see above). So, as the PT order increases, ordinary couplants become singular for ever larger Q^2 values, while BSR tends to negative values for ever larger Q^2 values.

	M^2	$\hat{\mu}_{\mathrm{MA},4}$	$\chi^2/(\mathrm{d.o.f.})$
LO	0.472 ± 0.035	-0.212 ± 0.006	0.667
NLO	0.414 ± 0.035	-0.206 ± 0.008	0.728
$\mathrm{N^2LO}$	0.397 ± 0.034	-0.208± 0.008	0.746
$\mathrm{N^3LO}$	0.394 ± 0.034	-0.209 ± 0.008	0.754
$\mathrm{N^4LO}$	0.397 ± 0.035	-0.208 ± 0.007	0.753

Table 1: The values of the fit parameters.

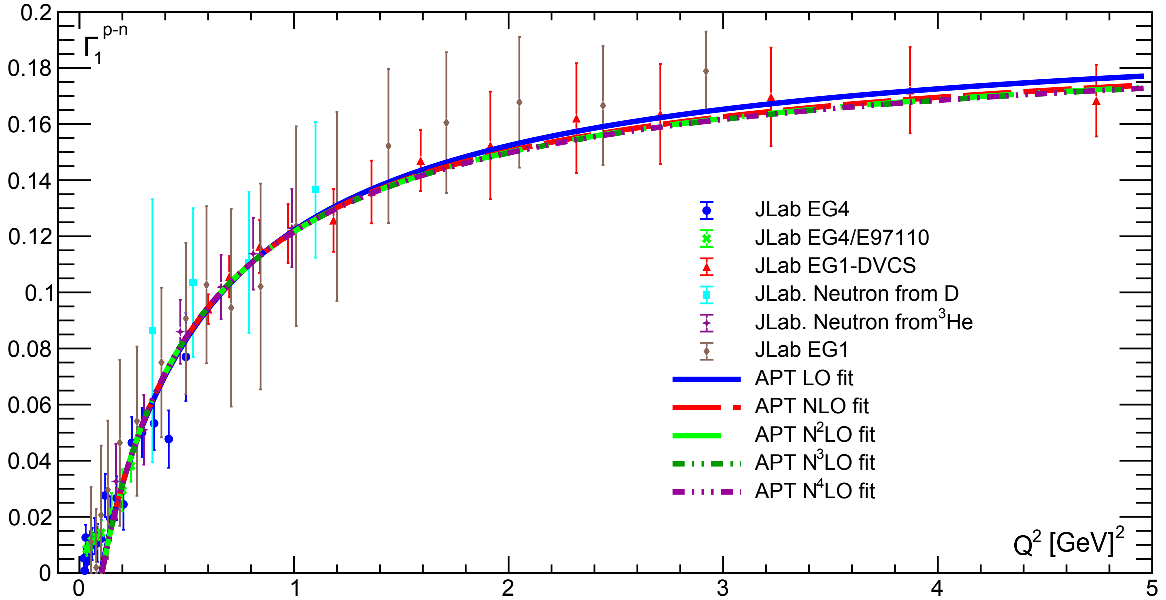

Figure 4: The results for $\Gamma_1^{p-n}(Q^2)$ in the first four orders of APT.

In contrast, our results obtained for different APT orders are practically equivalent: the corresponding curves become indistinguishable when Q^2 approaches 0 and slightly different everywhere else. As can be seen in Fig. 6, the fit quality is pretty high, which is demonstrated by the values of the corresponding $\chi^2/(\mathrm{d.o.f.})$ (see Table 1).

4. Results for Gross-Llewellyn Smith sum rule

By analogy with the previous subsection, we perform a fitting of experimental data for the Gross-Llewellyn Smith (GLS) sum rule within the framework of standard and analytical QCD with the "massive" forms of the twist-four terms.

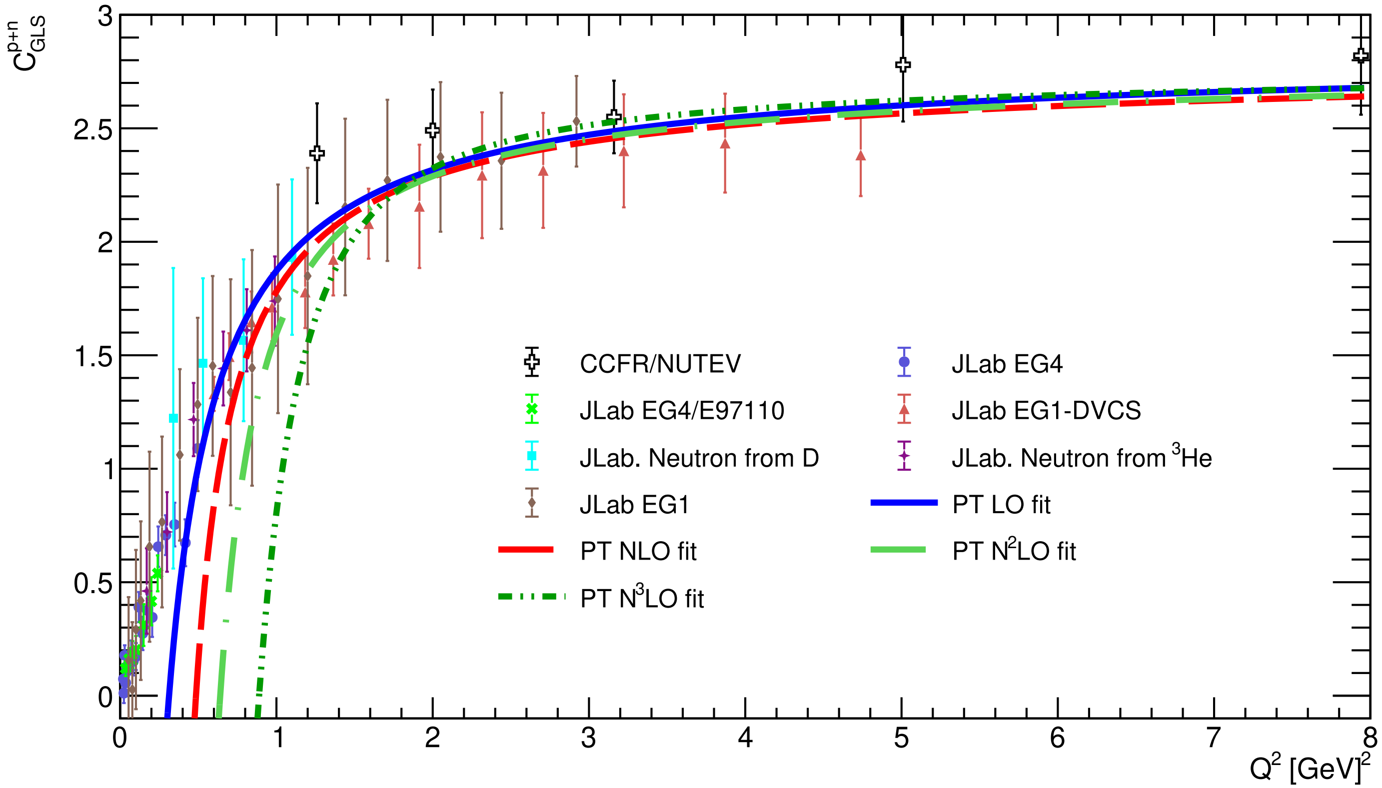

Figure 5: As in Fig. 3 but for the GLS sum rule

As in the Bjorken sum rule case, we see a lack of agreement between the standard QCD predictions and the experimental data.

Moreover, we see that the discrepancy increases with increasing PT order. This is the same as in the Bjorken sum rule case, and the reason is the same. With increasing PT order, the Landau pole of the strong coupling constant shifts toward higher Q^2 values. Thus, the results for GLS sum rule shifts toward negative values, since the PT corrections have a negative sign.

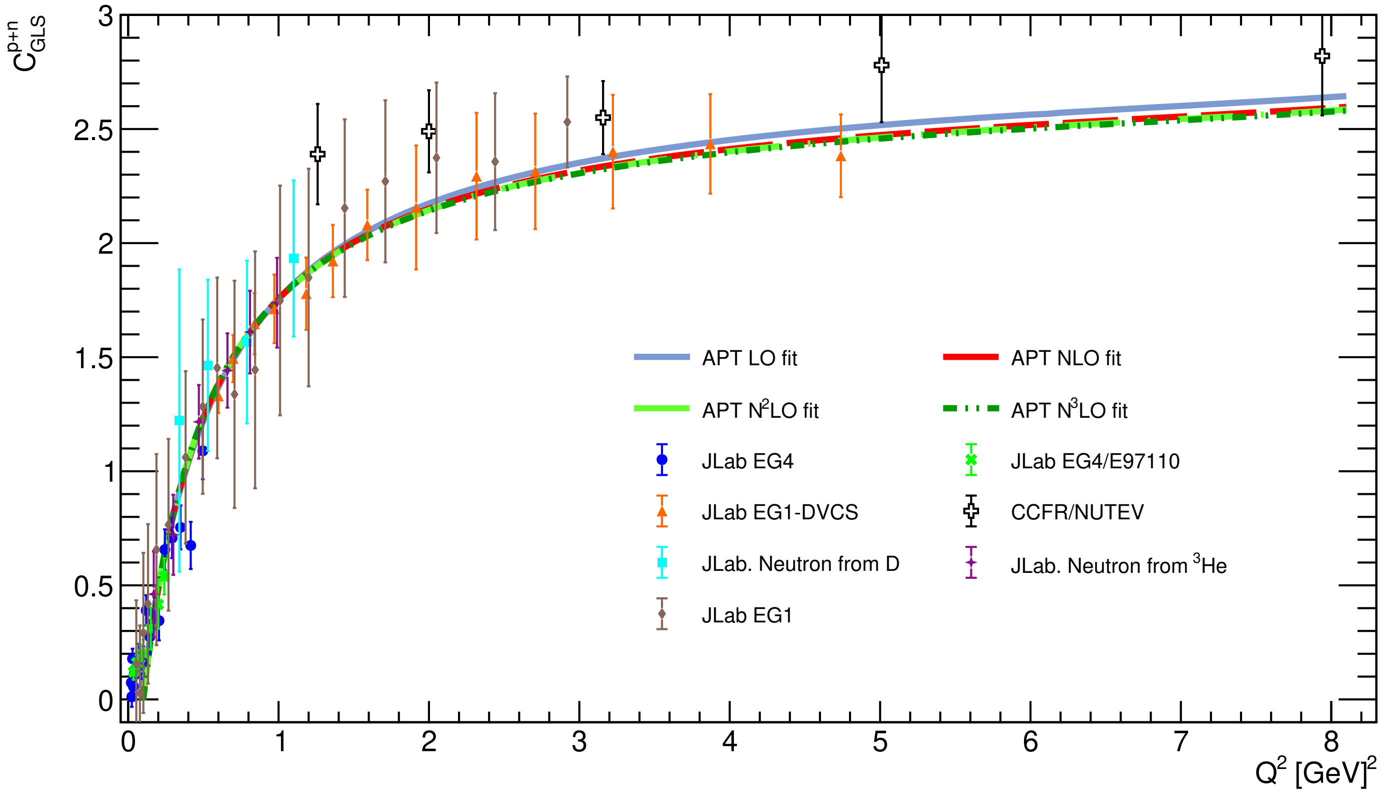

Figure 6: As in Fig. 4 but for the APT result.

The corresponding results for the APT case are presented in Fig. 6. So, we have good agreement between the QCD predictions and the experimental data.

Since the number of experimental points for the GLS sum rule is small we have added in Figs. 5 and 6 the experimental BSR data rescaled by a factor of $18/g_A$. We see that the experimental data for the GLS sum rule lie slightly above these rescaled experimental points for Bjorken sum rule. This observation is consistent with the results of (Londergan,Thomas:2010), where the possibility was discussed that the experimental data for the GLS sum rule should be smaller due to some unconsidered corrections.

The available experimental information on the GLS sum rule still has large uncertainties, and the number of experimental points is very small. For this reason, it is useful to supplement the comparison with recent lattice QCD results (Can et al. [QCDSF]:2025).

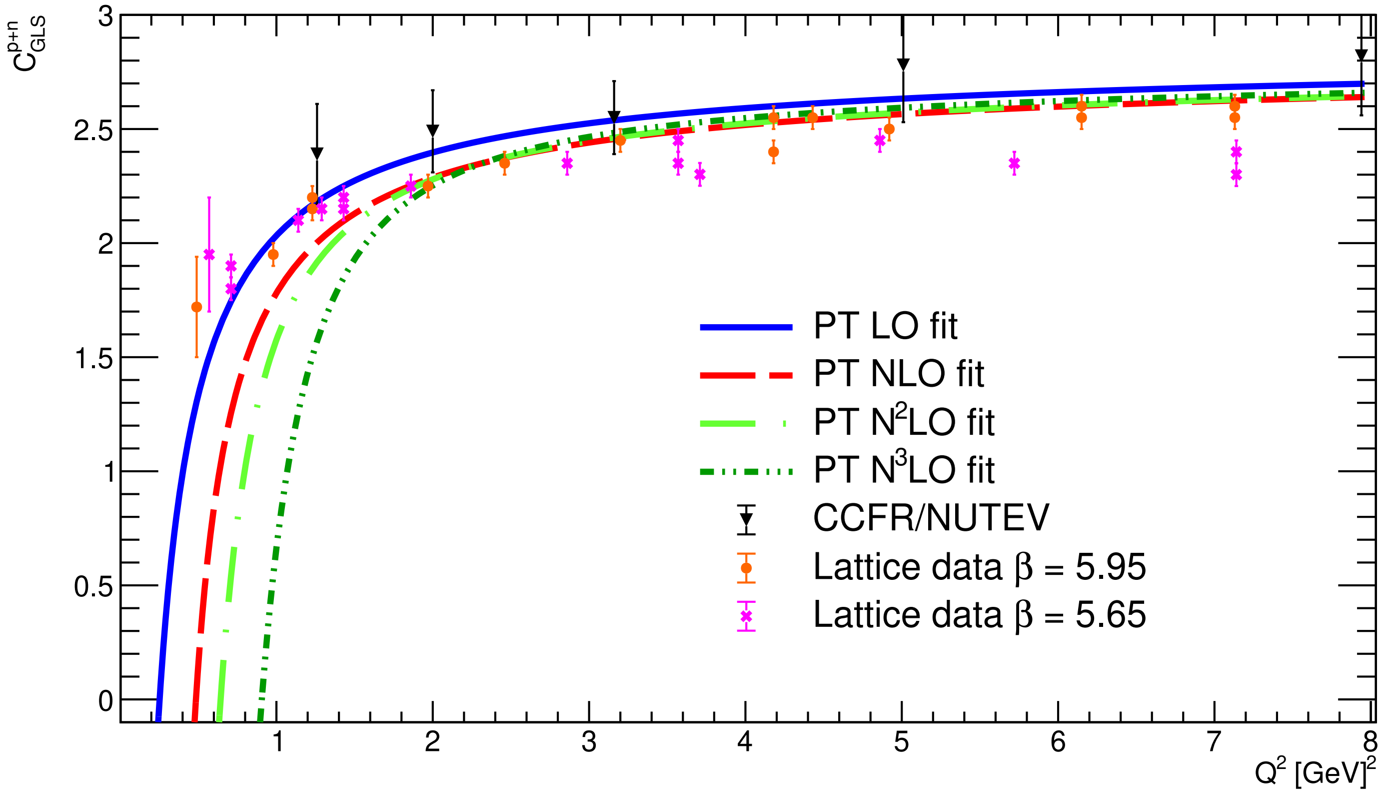

Figure 7: As in Fig. 5 but with lattice data.

In the case of standard PT, the results obtained by fitting these lattice "data" are presented in Fig. 7. It can be seen that the LO results agree quite well with the lattice "data", which cannot be said about the results for higher PT orders.

Moreover, as already discussed for the case of the experimental data for the GLS sum rule, the discrepancy increases with increasing PT order. The reasons are the same as discussed above.

	M^2 [GeV^2]	$\hat{\mu}_{\mathrm{A},4}$	$\chi^2/(\mathrm{d.o.f.})$
LO	0.879 $\pm$ 0.338	-1.37 $\pm$ 0.31	4.84
NLO	0.408 $\pm$ 0.277	-1.54 $\pm$ 0.32	3.50
N^{2}LO	0.400 $\pm$ 0.258	-1.68$\pm$ 1.06	3.24
N^{3}LO	0.370 $\pm$ 0.247	-1.78 $\pm$ 1.20	3.12

Table 2: The values of the parameters obtained from the lattice data.

In the case of analytic QCD, we have good agreement with the lattice "data". The results obtained by fitting these lattice data are presented in Table 2 and Fig. 8.

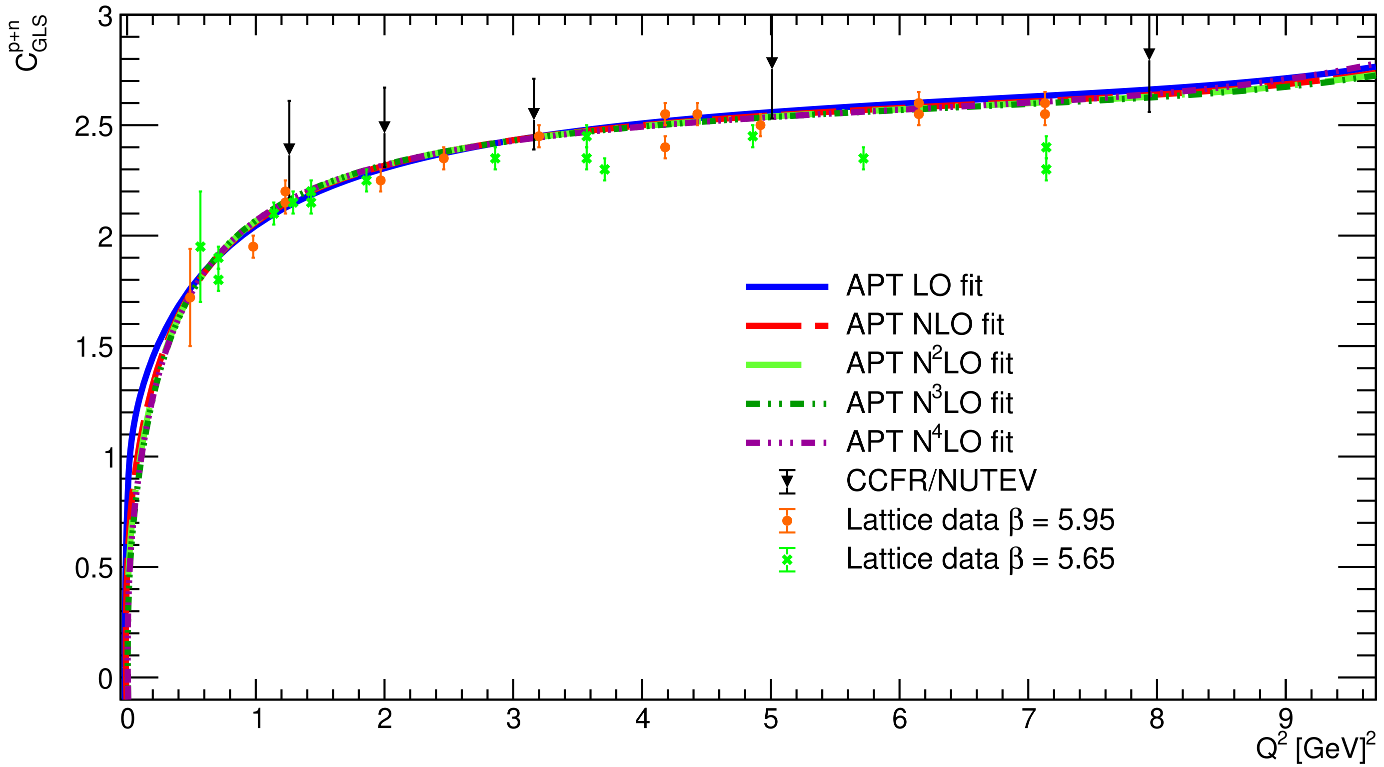

Figure 8: As in Fig. 6 but with the lattice data.

Note that we obtain rather large χ^2 values for the fits because the uncertainties in the lattice "data" are very small.

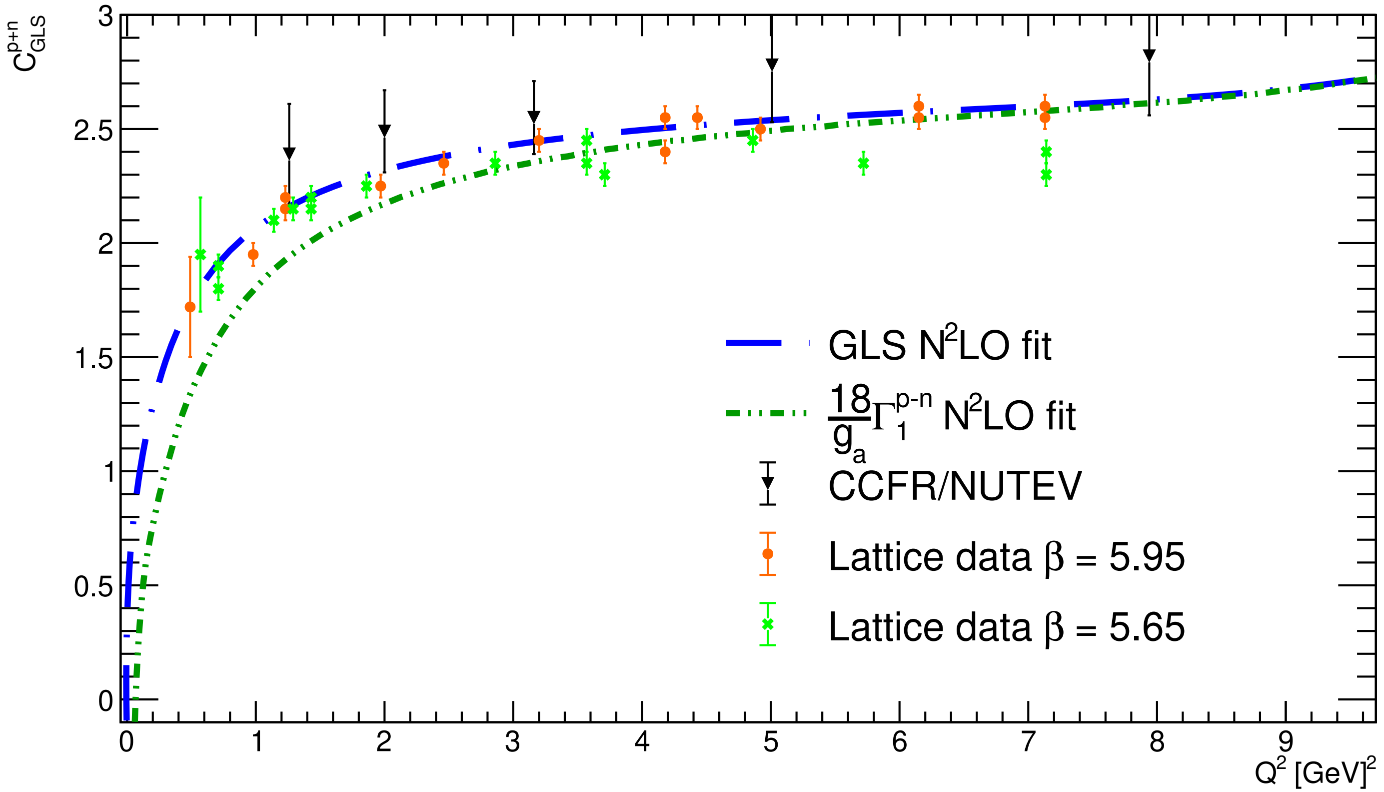

Figure 9: As in Fig. 8 with the rescaled BSR curve added.

In Fig. 9, we show the results for the GLS sum rule obtained in Table 2, along with the BSR results scaled by a factor of $18/g_A$. As can be seen from Fig. 9, the above ratio is valid only for large values of Q^2: $Q^2 \geq 5$ GeV^2. For smaller values of Q^2, $C^{p+n}_{\mathrm{A,GLS}}(Q^2)$ and $(18/g_A)\,\Gamma^{p-n}_{\mathrm{A},1}(Q^2)$ become different.

The difference arises due to the different parameter values in the twist-four terms in the cases of the GLS and Bjorken sum rules. Thus, the ratio

$$C^{p+n}_{\mathrm{A,GLS}}(Q^2) \approx \frac{18}{g_A}\,\Gamma^{p-n}_{\mathrm{A},1}(Q^2)\,,$$

based on the assumption in (Kataev:2005), seems to be fulfilled only for rather large values of Q^2, $Q^2 \geq 5$ GeV^2.

5. Conclusions

In this talk, we have focused on the introduction of the Shirkov-Solovtsov and Bakulev-Mikhailov-Stefanis approaches and their recent extension beyond the leading order of perturbation theory.

We have considered $1/L$-expansions of the ν-derivatives of the strong couplant a_s expressed as combinations of operators $\hat{R}_m$ applied to the LO couplant $a_s^{(1)}$.

Applying these operators to the ν-derivatives of the LO MA couplant $A_{\rm MA}^{(1)}$, we have got representations for the ν-derivatives of the MA couplant: $\tilde{A}_{\rm MA,\nu}^{(i)}$, i.e. , in each i-order of PT.

The high-order corrections are negligible in the $Q^2 \to 0$ and $Q^2 \to \infty$ asymptotics and are nonzero in a neighborhood of the point $Q^2 = \Lambda^2$. Thus, in fact, they are really only small corrections to the LO MA couplant $A^{(1)}_{\text{MA},\nu}(Q^2)$.

All our results have a compact form and do not contain complicated special functions, such as the Lambert W-function (Magradze: 1999), which already appears in two-loop order as an exact solution to the usual couplant and which was used to estimate the MA couplants in (Bakulev,Mikhailov,Stefanis: 2010).

I would like to point out that I have shown only one type of representations for the MA couplant of $A_{\rm MA}(Q^2)$ in the space-like domain. There are three other types of representations for $A_{\rm MA}(Q^2)$, but they are beyond the scope of the talk. I would like to note also that two of them are in integral form and can be extended to the non-minimal case, where the corresponding spectral functions become non-purely perturbative.

There are also two types of representations for $U_{\rm MA}(s)$ in the time-like domain. One of them is in integral form and can be extended to the non-minimal case.

As a example, we considered the Bjorken and GLS sum rules. In the case of BSR, obtained results similar to previous studies in (Pasechnik,Shirkov,Teryaev,Solovtsova,Khandramai: 2008,2009,2011), (Ayala,Cvetic,Kotikov,Shaikhatdenov: 2018)

The results based on usual perturbation theory do not not agree with the experimental data at $Q^2 \leq 1.5$ GeV^2. MA APT leads to good agreement with the data when we used the "massive" version for high-twist contributions.