Studying the transitions of pseudoscalar mesons into $\nu\bar{\nu}\gamma$ using anomaly sum rules approach

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Visible mode

The axial anomaly results in the non-conservation of the axial current [Adler Phys. Rev. 177 (1969) 2426, Phys. Rev. 182 (1969) 1517, Bell and Jackiw Nuovo Cim. A51 (1969) 47]. Isovector ($a = 3$) and octet ($a = 8$) components of the octet of axial currents $J^{(a)}_{\mu 5} = (1/\sqrt{2}) \sum_i \bar{\psi}_i \gamma_\mu \gamma_5 \lambda^a \psi_i$ acquire only the EM anomaly, while the singlet axial current $J^{(0)}_{\mu 5} = (1/\sqrt{3}) \sum_i \bar{\psi}_i \gamma_\mu \gamma_5 \psi_i$ has both EM and strong (non-Abelian) anomalies:

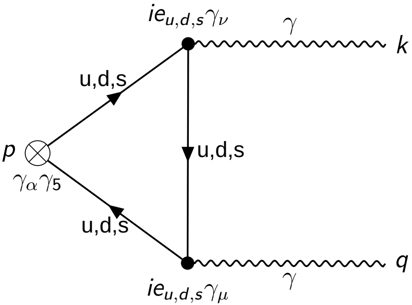
$$\partial^\mu J^{(a)}_{\mu 5} = \frac{2i}{\sqrt{2}} \sum_i m_i \bar{\psi}_i \gamma_5 \lambda^a \psi_i + \frac{e^2}{8\pi^2} C^{(a)} N_c F \widetilde{F}, \; a = 3, 8, \tag{1}$$

$$\partial^\mu J^{(0)}_{\mu 5} = \frac{2i}{\sqrt{3}} \sum_i m_i \bar{\psi}_i \gamma_5 \psi_i + \frac{e^2}{8\pi^2} C^{(0)} N_c F \widetilde{F} + \frac{n_f \alpha_s}{4\pi\sqrt{3}} G \widetilde{G}. \tag{2}$$

Here F and G are EM and gluon field strength tensors respectively, $\widetilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$ and $\widetilde{G}^{\mu\nu,t} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} G^t_{\rho\sigma}$ are their duals, $N_c = 3$ is a number of colors, $n_f = 3$ is the number of flavors, α_s is a strong coupling constant, $C^{(a)}$ are the charge factors (e_i are quark charges in units of the electron charge e):

$$C^{(3)} = \frac{1}{\sqrt{2}}(e_u^2 - e_d^2) = \frac{1}{3\sqrt{2}}, \; C^{(8)} = \frac{1}{\sqrt{6}}(e_u^2 + e_d^2 - 2e_s^2) = \frac{1}{3\sqrt{6}},$$
$$C^{(0)} = \frac{1}{\sqrt{3}}(e_u^2 + e_d^2 + e_s^2) = \frac{2}{3\sqrt{3}},$$

stem from the definition of axial currents. The sum is over $i = u, d, s$ quarks; λ^a are the



The vector-vector-axial (VVA) amplitude can be rewritten [Rosenberg Phys.Rev.1963.Vol.129.P.2786]

$$T_{\alpha\mu\nu}(k,q) = e^2 \int d^4x d^4y e^{(ikx+iqy)} \langle 0| T\{J_{\alpha 5}(0) J_\mu(x) J_\nu(y)\}|0\rangle =$$

$$= F_1\ \varepsilon_{\alpha\mu\nu\rho}k^\rho + F_2\ \varepsilon_{\alpha\mu\nu\rho}q^\rho + F_3\ k_\nu \varepsilon_{\alpha\mu\rho\sigma}k^\rho q^\sigma \quad (3)$$

$$+ F_4\ q_\nu \varepsilon_{\alpha\mu\rho\sigma}k^\rho q^\sigma + F_5\ k_\mu \varepsilon_{\alpha\nu\rho\sigma}k^\rho q^\sigma + F_6\ q_\mu \varepsilon_{\alpha\nu\rho\sigma}k^\rho q^\sigma, \quad (4)$$

where the coefficients $F_j = F_j(p^2, q^2)$, $j = 1, \ldots, 6$ are the corresponding Lorentz invariant amplitudes constrained by current conservation and Bose symmetry. The electromagnetic currents are defined as $J_\mu = \sum_i e_i \bar{\psi}_i \gamma_\mu \psi_i$, $i = u, d, s$.

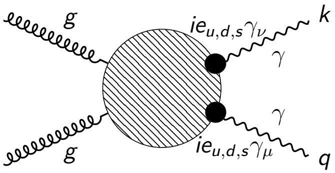
The anomaly sum rules (ASR) for the isovector ($a = 3$) and octet ($a = 8$) currents when one of the photons is real ($k^2 = 0$) and another is real or virtual ($Q^2 = -q^2 \geq 0$) read (in what follows we put $m_u = m_d = m_s = 0$) [Horejsi,Teryaev, Z.Phys. C65 1995.,Klopot,Oganesian,Teryaev Phys.Lett. B695 (2011)]

$$\frac{1}{\pi}\int_0^\infty A_3^{(3,8)}(s, Q^2)ds = e^2\frac{C^{(3,8)}N_c}{2\pi^2}, \tag{5}$$

where the spectral density function is defined as $A_3^{(3,8)} \equiv \frac{1}{2}Im(F_3 - F_6)$ The r.h.s. of (5) is exactly the Abelian (electromagnetic) anomaly constant stemmed from the matrix element $\langle 0|F\tilde{F}|\gamma\gamma^*\rangle$. Note that (5) itself is a pure theoretical result obtained directly from dispersion representation of axial anomaly [Dolgov, Zakharov, Nuclear Physics B27 1971.,Horejsi,Teryaev, Z.Phys. C65 1995.]

ASR for the singlet current has an additional part stemmed from the matrix element

$$\langle 0|\frac{\sqrt{3}\alpha_s}{4\pi}G\tilde{G}|\gamma\gamma\rangle = e^2 C^{(0)} N_c N(p^2, k^2, q^2)\epsilon^{\mu\nu\rho\sigma} k_\mu q_\nu \epsilon_\rho^{(k)} \epsilon_\sigma^{(q)}. \quad (6)$$



The dashed circle denotes all possible pert. and non-pert. strong interactions in which finally one gets two photons coupled to the quark line with corresponding EM quark vertices $e_{u,d,s}$. One need to sum over all u, d, s quark charges and so the matrix element (6) has the same charge factor coefficient $C^{(0)}$ as the matrix element $\langle 0|F\tilde{F}|\gamma\gamma^*\rangle$. The factor N_c is written explicitly for convenience. The corresponding non-Abelian contribution in the dispersive form requires a subtraction, so the singlet current ASR in the considered kinematics $N(p^2, k^2 = 0, q^2 = -Q^2) \equiv N(p^2, Q^2)$ [Khlebtsov, Klopot, Oganesian, Teryaev Phys. Rev. D Vol. 99., no.1. P. 016008 (2019), Phys. Rev. D Vol. 104., no. 1. P. 016011 (202

$$\frac{1}{\pi}\int_0^\infty A_3^{(0)}(s, Q^2)ds = \frac{e^2 C^{(0)} N_c}{2\pi^2} + e^2 C^{(0)} N_c \left(N(0, Q^2) - \frac{1}{\pi}\int_0^\infty ImR(s, Q^2)ds \right), \quad (7)$$

where

$$R(p^2, Q^2) = \frac{1}{p^2}(N(p^2, Q^2) - N(0, Q^2)).$$

We saturate the l.h.s. of ASRs (5) and (7) with a full set of resonances and single out the lowest-lying contributing states in each channel in terms of the corresponding decay constants $f_P^{(a)}$ and the form factors $F_{P\gamma}$ of the transitions $\gamma\gamma^* \to P$

$$\langle 0|J_{\alpha 5}^{(a)}(0)|P(p)\rangle = ip_\alpha f_P^{(a)}(p^2), \tag{8}$$

$$\int d^4 x e^{ikx} \langle P(p)|T\{J_\mu(x)J_\nu(0)\}|0\rangle = e^2 \epsilon_{\mu\nu\rho\sigma} k^\rho q^\sigma F_{P\gamma}(p^2, Q^2)\ . \tag{9}$$

the ASRs (5) and (7) read (we omit factor e^2 for brevity),

$$\Sigma f_P^{(3,8)}(m_P^2) F_{P\gamma}(m_P^2, Q^2) + \frac{1}{\pi}\int_{s_{3,8}}^\infty A_3^{(3,8)}(s, Q^2) ds = \frac{C^{(3,8)} N_c}{2\pi^2}, \tag{10}$$

$$\Sigma f_P^{(0)}(m_P^2) F_{P\gamma}(m_P^2, Q^2) + \frac{1}{\pi}\int_{s_0}^\infty A_3^{(0)} ds =$$
$$\frac{C^{(0)} N_c}{2\pi^2} + C^{(0)} N_c \left(N(0, Q^2) - \frac{1}{\pi}\int_0^\infty ImR(s, Q^2) ds \right). \tag{11}$$

The lower integration limits s_3, s_8, s_0 in (10), (11), emerging as free parameters of the ASR approach, strictly speaking, can be the functions of Q^2. Their values can be obtained from comparison with experiment or from comparison with other theoretical approaches.
It is natural to assume that the duality intervals of the isovector and octet currents cannot essentially differ: $s_8 \simeq s_3$ within 20% uncertainty of the $SU(3)$ symmetry braking.
Put $s_8 \simeq s_3 = 0.6$ GeV^2.
The duality interval of the singlet current s_0 is different from s_3 and s_8, we put $s_0 \simeq 1$ GeV^2.
The one-loop approximation for the spectral densities of the isovector and octet currents $A_3^{(3,8)}(s, Q^2)$

$$A_3^{(3,8)}(s, Q^2) = \frac{C^{(3,8)} N_c}{2\pi} \frac{Q^2}{(s + Q^2)^2}, \tag{12}$$

so that the integration in the ASRs (10) leads to the following expressions for the hadron contributions,

$$\Sigma f_P^{(3,8)} F_{P\gamma}(Q^2) = \frac{C^{(3,8)} N_c}{2\pi^2} \frac{s_{3,8}}{s_{3,8} + Q^2}. \tag{13}$$

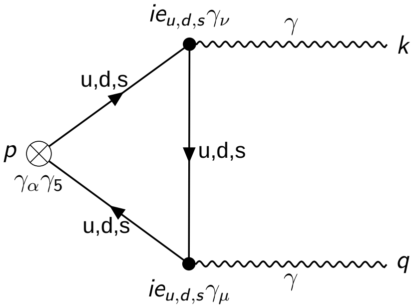


Figure: $A^{(0)}_{QED}$.

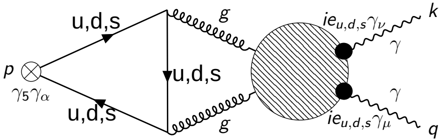


Figure: $A^{(0)}_{QCD}$.

The singlet current differs from the isovector and octet by a new type of diagrams involving virtual gluons. In order to single out EM contribution, we split the spectral density into two parts,

$$A_3^{(0)} = A^{(0)}_{QED} + A^{(0)}_{QCD}.$$

The $A^{(0)}_{QED}$ represents the contribution of pure QED diagrams. The second part $A^{(0)}_{QCD}$ is the contribution of diagrams with virtual gluons coupled to two photons through all possible perturbative and non-perturbative strong interactions

For the $A_{QED}^{(0)}$ contribution lowest one-loop part is given by a similar expression to Eq. (12) with an appropriate charge factor $C^{(0)}$. Making use of it, we can rewrite the ASR (11) as

$$\Sigma f_P^{(0)} F_{P\gamma}(Q^2) = \frac{N_c C^{(0)}}{2\pi^2}\frac{s_0}{s_0+Q^2} - C^{(0)} N_c\left(\frac{1}{\pi}\int_{s_0}^{\infty} A_{QCD}\, ds + N(0,Q^2) - \frac{1}{\pi}\int_0^{\infty} ImR(s,Q^2)ds\right). \quad (14)$$

The first and the last three terms in r.h.s. of Eq. (14) represent the electromagnetic and the strong anomaly contributions to the ASR respectively. It is convenient to introduce a function that represents the ratio of contributions of strong and electromagnetic anomalies:

$$B(Q^2,s_0) = \frac{2\pi^2}{N_c C^{(0)}}\frac{s_0+Q^2}{s_0}\left[C^{(0)} N_c (N(0,Q^2) - \frac{1}{\pi}\int_0^{\infty} ImR(s,Q^2)ds - \frac{1}{\pi}\int_{s_0}^{\infty} A_{QCD}(s,Q^2)ds)\right].$$

As the integral of A_{QCD} is suppressed as α_s^2 at $s_0 \geq 1.0$ GeV^2, the function $B(Q^2,s_0)$ is predominantly determined by the first two terms. It reflects the properties of the non-perturbative matrix element $\langle 0|G\tilde{G}|\gamma\gamma^{(*)}\rangle$.

Solving system of equations

$$f_{\pi^0}^{(3)} F_{\pi^0\gamma}(Q^2) + f_{\eta}^{(3)} F_{\eta\gamma}(Q^2) + f_{\eta'}^{(3)} F_{\eta'\gamma}(Q^2) = \frac{N_c C^{(3)}}{2\pi} \frac{s_3}{s_3 + Q^2}$$

$$f_{\pi^0}^{(8)} F_{\pi^0\gamma}(Q^2) + f_{\eta}^{(8)} F_{\eta\gamma}(Q^2) + f_{\eta'}^{(8)} F_{\eta'\gamma}(Q^2) = \frac{N_c C^{(8)}}{2\pi} \frac{s_8}{s_8 + Q^2}$$

$$f_{\pi^0}^{(0)} F_{\pi^0\gamma}(Q^2) + f_{\eta}^{(0)} F_{\eta\gamma}(Q^2) + f_{\eta'}^{(0)} F_{\eta'\gamma}(Q^2) = \frac{N_c C^{(0)}}{2\pi^2} \frac{s_0(1 + B(Q^2))}{s_0 + Q^2}$$

The generic form of π^0, η, η' TFFs

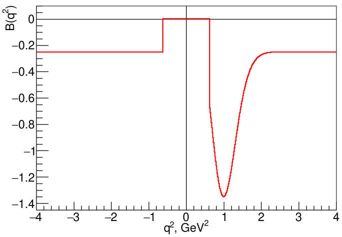
$$F_P(Q^2) = \left[\alpha_P \frac{s_3}{s_3 + Q^2} + \beta_P \frac{s_8}{s_8 + Q^2} + \gamma_P \frac{s_0}{s_0 + Q^2}(1 + B(Q^2))\right]. \tag{15}$$

where $P = \pi^0, \eta, \eta'$. The coef. $\alpha_P, \beta_P, \gamma_P$ are determined by decay constants $f_P^{(a)}$. Analytical continuation to time-like results in $Q^2 \to -q^2$

$$|F_P(q^2)| = \left|\alpha_P \frac{s_3}{s_3 - q^2} + \beta_P \frac{s_8}{s_8 - q^2} + \gamma_P \frac{s_0}{s_0 - q^2}\left[1 + B(q^2)\right]\right|. \tag{16}$$

The function $B(Q^2)$ reflects properties of non-pert. matrix element $\langle 0|G\tilde{G}|\gamma\gamma^*\rangle$, which cannot be calculated quantum theory due to its non-perturbative origin. The eq. (15) and (16) relate it to the physical observables quantites – $\pi^0, \eta, \eta' \to \gamma\gamma^{(*)}$ TFFs.

It was established



$$B(q^2) = \begin{cases} 0, & 0 < |q^2| < 0.6\ GeV^2, \\ be^{-\frac{(q^2-\mu)^2}{2c^2}} + B_{as}, & |q^2| > 0.6\ GeV^2, \end{cases}$$

where $B_{as} = -0.262$ is the asymptotic value, and b, μ, c are the parameters.

$$s_0(q^2) = \begin{cases} \approx 0.6\ GeV^2, & \text{if } 0 < |q^2| < 0.6\ GeV^2, \\ \approx 1\ GeV^2, & \text{if } |q^2| > 0.6\ GeV^2, \end{cases}$$

$$s_8(q^2) = \begin{cases} \approx 0.48\ GeV^2, & \text{if } 0 < |q^2| < 0.3\ GeV^2, \\ \approx 0.6\ GeV^2, & \text{if } |q^2| > 0.3\ GeV^2, \end{cases}$$

The $s_3(q^2)$ is varying from 0.6 GeV^2 at $q^2 \to 0$ to 0.67 GeV^2 at $q^2 \to \infty$. For the aims of this work it can be described simply by a constant $s_3 = 0.6$ GeV^2.

Daltiz decays & slopes

Slopes of TFFs

$$a_P \equiv \lim_{q^2 \to 0} \frac{\partial}{\partial q^2} \frac{|F_P(q^2)|}{|F_P(0)|} = \frac{1}{|F_P(0)|} \left(\frac{\alpha_P}{s_3} + \frac{\beta_P}{s_8} + \frac{\gamma_P(1 + B(0))}{s_0} \right), \qquad (17)$$

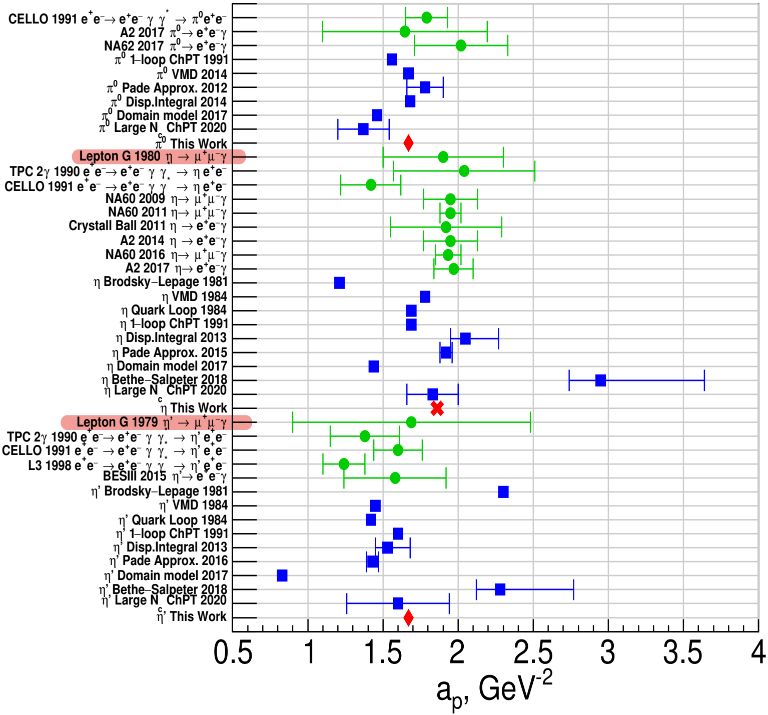
where $|F_P(0)| = \alpha_P + \beta_P + \gamma_P(1 + B(0))$, $B(0)$ the value of the gluon anomaly contribution at $q^2 = 0$.
Connected to mean squared radius

$$a_p = \frac{< r^2 >}{6}, \qquad (18)$$

and, accordingly, one can obtain an estimate of the "physical" radius of pseudoscalar mesons

$$r_p = \frac{1}{5.068} \sqrt{6 a_p}, \qquad (19)$$

where numerical factor for conversion GeV^-1 into fm.
Pioneering work by IHEP,Protvino:
SERPUKHOV-134, STUDY OF RARE ELECTROMAGNETIC DECAYS OF MESONS 1977-79.


CELLO 1991 e+e- → e+e- γ γ* → π0e+e-
A2 2017 π0 → e+e-γ
NA62 2017 π0 → e+e-γ
π0 1-loop ChPT 1991
π0 VMD 2014
π0 Pade Approx. 2012
π0 Disp.Integral 2014
π0 Domain model 2017
π0 Large Nc ChPT 2020
π0 This Work
Lepton G 1980 η → μ+μ-γ
TPC 2γ 1990 e+e- → e+e- γ γ* → η e+e-
CELLO 1991 e+e- → e+e- γ γ* → η e+e-
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NA60 2011 η → μ+μ-γ
Crystall Ball 2011 η → e+e-γ
A2 2014 η → e+e-γ
NA60 2016 η → μ+μ-γ
A2 2017 η → e+e-γ
η Brodsky-Lepage 1981
η VMD 1984
η Quark Loop 1984
η 1-loop ChPT 1991
η Disp.Integral 2013
η Pade Approx. 2015
η Domain model 2017
η Bethe-Salpeter 2018
η Large Nc ChPT 2020
η This Work
Lepton G 1979 η' → μ+μ-γ
TPC 2γ 1990 e+e- → e+e- γ γ* → η' e+e-
CELLO 1991 e+e- → e+e- γ γ* → η' e+e-
L3 1998 e+e- → e+e- γ γ* → η' e+e-
BESIII 2015 η' → e+e-γ
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η' VMD 1984
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η' 1-loop ChPT 1991
η' Disp.Integral 2013
η' Pade Approx. 2016
η' Domain model 2017
η' Bethe-Salpeter 2018
η' Large Nc ChPT 2020
η' This Work
0.5
1
1.5
2
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3
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4
ap, GeV-2

Invisible mode

In general the neutral weak current can be written as a sum of 12 terms:

$$j^0_\mu = \sum_i (g^i_L \bar{\psi}_i O^L_\mu \psi_i + g^i_R \bar{\psi}_i O^R_\mu \psi_i), \tag{20}$$

where

$$i = \nu_e,\ \nu_\mu,\ \nu_\tau,\ e,\ \mu,\ \tau,\ u,\ c,\ t,\ d,\ s,\ b, \tag{21}$$

$$O^L_\mu = \gamma_\mu(1+\gamma_5),\ O^R_\mu = \gamma_\mu(1-\gamma_5), \tag{22}$$

g^i_L and g^i_R are the numerical coefficients corresponding to weak charges

$$g^i_L = \frac{1}{2}, \quad g^i_R = 0 \ for\ \nu_e,\ \nu_\mu,\ \nu_\tau, \tag{23}$$

$$g^i_L = -\frac{1}{2} + \xi,\ g^i_R = +\xi \ for\ \ e,\ \mu,\ \tau, \tag{24}$$

$$g^i_L = \frac{1}{2} - \frac{2}{3}\xi,\ g^i_R = -\frac{2}{3}\xi \ for\ u,\ c,\ t, \tag{25}$$

$$g^i_L = -\frac{1}{2} + \frac{1}{3}\xi,\ g^i_R = \frac{1}{3}\xi \ for\ d,\ s,\ b, \tag{26}$$

$\xi = sin^2\theta_W$, where θ_W is the Weinberg angle($\xi \approx 0.23$).

The neutral weak current has both vector(V) and axial(A) parts. Let us rewrite the equations for weak currents separating V and A parts.

$$j^0_\mu = \sum_i [(g^i_L + g^i_R)\bar{\psi}_i \gamma_\mu \psi_i + (g^i_L - g^i_R)\bar{\psi}_i \gamma_\mu \gamma_5 \psi_i]. \qquad (27)$$

In the case under consideration we already have vector EM current from photon and one axial current. The AVA diagram can be neglected, so only vector part of neutral currents remains in (27).

Thus the triangle graph amplitude for the processes involving Z^0-boson and photon will be similar to the 2 photons case. Technically this diagram differs only by the charge factor at one vertex. The corresponding 3-point correlation function contains the axial current $J_{\alpha 5}$ with momentum $p = k + q$, EM vector current $J_\nu = \sum_i e_i \bar{\psi}_i \gamma_\nu \psi_i,\ i = u, d, s$ with momenta k(photon) and vector part of the neutral weak current (27) with momenta q(Z^0-boson).

The transition of isovector and octet axial currents will have the matrix element $\langle 0|F\tilde{F}|Z^0\gamma\rangle$ stemmed from the Abelian (electromagnetic) anomaly, while the singlet axial current will have an additional matrix element $\langle 0|G\tilde{G}|Z^0\gamma\rangle$ stemmed from the Non-Abelian (strong) anomaly:

$$\partial^\mu J^{(0)}_{\mu 5} = \frac{2i}{\sqrt{3}}\sum_i m_i\bar{\psi}_i\gamma_5\psi_i + \frac{v}{\sqrt{3}\pi^2}N_c F\widetilde{F} + \frac{n_f\alpha_s}{4\pi\sqrt{3}}G\widetilde{G}, \tag{28}$$

$$\partial^\mu J^{(3)}_{\mu 5} = \frac{2i}{\sqrt{2}}\sum_i m_i\bar{\psi}_i\gamma_5\lambda^3\psi_i + \frac{v}{\sqrt{2}\pi^2}N_c F\widetilde{F}, \tag{29}$$

$$\partial^\mu J^{(8)}_{\mu 5} = \frac{2i}{\sqrt{2}}\sum_i m_i\bar{\psi}_i\gamma_5\lambda^8\psi_i + \frac{v}{\sqrt{6}\pi^2}N_c F\widetilde{F}, \tag{30}$$

where $v = \sum_k e_k x_k$ and $k = u, d, s$-quarks. The e_k factors denote EM quark charges (in the electron charge e units) and x_k are the Z^0-boson coupling "charges" $x_u = \bar{g}(\frac{1}{2} - \frac{4}{3}\xi)$ and $x_{d,s} = \bar{g}(-\frac{1}{2} + \frac{2}{3}\xi)$ for $u-$ and $d, s-$quarks, respectively.

The Z^0-boson coupling "charges" for $u-$ and $d, s-$quarks are different and so the Feynman diagrams for the matrix element $\langle 0|F\tilde{F}|Z^0\gamma\rangle$ now should be changed to

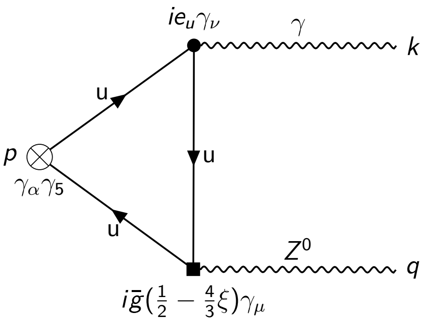


Figure: $A^{Z^0}_{QED}$ for u-quark with Z^0 boson vertex.

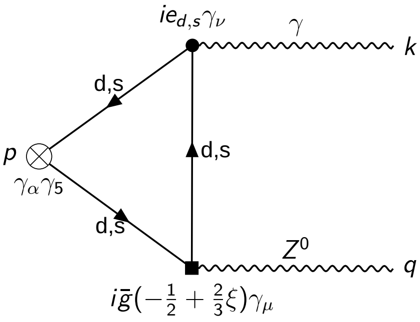


Figure: $A^{Z^0}_{QED}$ for d,s-quarks with Z^0 boson vertex.

For the charge factors we got for $C^{(0)}_{Z^0}$

$$C^{(0)}_{Z^0} = \frac{1}{\sqrt{3}}\left(e_u\bar{g}(\frac{1}{2}-\frac{4}{3}\xi)+e_d\bar{g}(-\frac{1}{2}+\frac{2}{3}\xi)+e_s\bar{g}(-\frac{1}{2}+\frac{2}{3}\xi)\right)=$$

$$=\frac{e\bar{g}}{\sqrt{3}}\left(\frac{2}{3}(\frac{1}{2}-\frac{4}{3}\xi)+(-\frac{1}{3})(-\frac{1}{2}+\frac{2}{3}\xi)+(-\frac{1}{3})(-\frac{1}{2}+\frac{2}{3}\xi)\right)=\frac{2e\bar{g}}{3\sqrt{3}}(1-2\xi).$$

It can be rewritten by using charge factor for the case of 2 photons $C^{(0)}\frac{1}{\sqrt{3}}(e_u^2+e_d^2+e_s^2)=\frac{2e^2}{3\sqrt{3}}$ as

$$C^{(0)}_{Z^0} = C^{(0)}\frac{\bar{g}}{e}(1-2\xi).$$

Similarly one can obtain such relations for isovector and octet currents

$$C^{(3)}_{Z^0} = C^{(3)}\frac{\bar{g}}{e}(\frac{1}{2}-2\xi),$$

$$C^{(8)}_{Z^0} = C^{(8)}\frac{\bar{g}}{e}(\frac{1}{2}-2\xi).$$

The term $F\tilde{F}$ differs from the one in equations for 2 photons by a charge factor constant. The interaction constant at Z^0-boson vertex doesn't affect the kinematics, one can use the same vector-vector-axial (VVA) amplitude decomposition as in $\gamma\gamma$ case. It means that the equations of the ASR for isovector ($a = 3$) and octet ($a = 8$) currents with one real photon ($k^2 = 0$) and Z^0-boson ($Q^2 = -q^2 \geq 0$) will be similar to 2 photons with the only difference in the charge factor coefficients (in what follows we put $m_u = m_d = m_s = 0$)

$$\frac{1}{\pi}\int_0^\infty A_{Z^0}^{(3,8)}(s, Q^2)ds = \frac{C_{Z^0}^{(3,8)} N_c}{2\pi^2}, \tag{31}$$

where the spectral density function is defined as $A_{Z^0}^{(3,8)} \equiv \frac{1}{2} Im(F_3 - F_6)$. The one-loop approximation

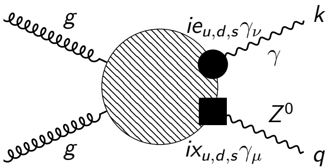
$$A_{Z^0}^{(3,8)}(s, Q^2) = \frac{C_{Z^0}^{(3,8)} N_c}{2\pi} \frac{Q^2}{(s + Q^2)^2}, \tag{32}$$

and the ASRs for the hadron contributions are

$$\Sigma f_P^{(3,8)} F_{PZ^0\gamma}(Q^2) = \frac{C_{Z^0}^{(3,8)} N_c}{2\pi^2} \frac{s_{3,8}}{s_{3,8} + Q^2}. \tag{33}$$

The non-pert. matrix element $\langle 0|G\tilde{G}|\gamma Z^0\rangle$

$$\langle 0|\frac{\sqrt{3}\alpha_s}{4\pi}G\tilde{G}|\gamma Z^0\rangle = C^{(0)}_{Z^0}N_c N_{Z^0}(p^2,k^2,q^2)\epsilon^{\mu\nu\rho\sigma}k_\mu q_\nu \epsilon^{(k)}_\rho \epsilon^{(q)}_\sigma. \quad (34)$$



The shaded circle denotes all possible pert. and non-pert. transitions of two gluons to Z^0-boson and photon. The black circle corresponds to EM vertex. The black square represents vertex corresponding to outgoing Z^0-boson, where Z^0-boson coupling "charges" are $x_u = \bar{g}(\frac{1}{2} - \frac{4}{3}\xi)$ and $x_{d,s} = \bar{g}(-\frac{1}{2} + \frac{2}{3}\xi)$ for $u-$ and $d,s-$quarks. One need to sum over all u,d,s-quark charges products, so at the leading order in $\frac{1}{m_W^2}$ the matrix element (34) has the same charge factor coefficient $C^{(0)}_{Z^0}$ as the $\langle 0|F\tilde{F}|Z^0\gamma\rangle$. The factor N_c is indicated for convenience.

The non-Abelian contribution in the dispersive form requires a subtraction, so the singlet current ASR in the considered kinematics $N(p^2,k^2=0,q^2=-Q^2)\equiv N(p^2,Q^2)$

$$\frac{1}{\pi}\int_0^\infty A^{(0)}_{Z^0}(s,Q^2)ds = \frac{C^{(0)}_{Z^0}N_c}{2\pi^2} + C^{(0)}_{Z^0}N_c\left(N_{Z^0}(0,Q^2) - \frac{1}{\pi}\int_0^\infty ImR_{Z^0}(s,Q^2)ds\right),$$

where

$$R_{Z^0}(p^2,Q^2) = \frac{1}{p^2}(N_{Z^0}(p^2,Q^2) - N_{Z^0}(0,Q^2)).$$

Introduce the function that represents the strong and EM anomalies contributions ratio:

$$B_{Z^0}(Q^2, s_0) = \frac{2\pi^2}{N_c C^{(0)}_{Z^0}} \frac{s_0 + Q^2}{s_0} \left[C^{(0)}_{Z^0} N_c \left(N_{Z^0}(0, Q^2) - \frac{1}{\pi} \int_0^\infty Im R_{Z^0}(s, Q^2) ds - \frac{1}{\pi} \int_{s_0}^\infty A^{Z^0}_{QCD}(s, Q^2) ds \right) \right].$$

As the integral of $A^{Z^0}_{QCD}$ is suppressed as α_s^2 at $s_0 \geq 1.0$ GeV^2, the function $B_{Z^0}(Q^2, s_0)$ is predominantly determined by the first two terms. It reflects the properties of the non-perturbative matrix element $\langle 0|G\tilde{G}|\gamma Z^0\rangle$.

Finally the ASR for the singlet current

$$\Sigma f^{(0)}_P F_{PZ^0\gamma}(Q^2) = \frac{N_c C^{(0)}_{Z^0}}{2\pi^2} \frac{s_0}{s_0 + Q^2} \left[1 + B_{Z^0}(Q^2, s_0)\right]. \quad (35)$$

The function $B_{Z^0}(Q^2, s_0)$ is unknown and cannot be calculated analytically due to non-pert. origin of the corresponding matrix element. It is the strong and EM contributions ratio. Therefore, as the charge factor coefficients of the corresponding matrix elements appear to be the same, the function $B_{Z^0}(Q^2, s_0)$ does not depend on these coefficients.

Strictly speaking, functions $B_{Z^0}(Q^2, s_0)$ and $B(Q^2, s_0)$ are different. But the matrix element $\langle 0|G\tilde{G}|\gamma Z^0\rangle$ up to $\frac{1}{m_W^2}$ corrections differs from the matrix element $\langle 0|G\tilde{G}|\gamma\gamma\rangle$ only by a charge factor ($N = N^0_Z$). For the matrix elements corresponding to EM anomaly contributions it is the same difference. Thus to the first order of weak corrections the functions $B_{Z^0}(Q^2, s_0)$ and $B(Q^2, s_0)$ are equal.

Solving system of equations

$$\begin{pmatrix} f^{(3)}_{\pi^0} & f^{(3)}_{\eta} & f^{(3)}_{\eta'} \\ f^{(8)}_{\pi^0} & f^{(8)}_{\eta} & f^{(8)}_{\eta'} \\ f^{(0)}_{\pi^0} & f^{(0)}_{\eta} & f^{(0)}_{\eta'} \end{pmatrix} \begin{pmatrix} F_{\pi^0}(Q^2) \\ F_{\eta}(Q^2) \\ F_{\eta'}(Q^2) \end{pmatrix} = \begin{pmatrix} \frac{N_c C^{(3)}_{Z^0}}{2\pi^2} \frac{s_3}{s_3+Q^2} \\ \frac{N_c C^{(8)}_{Z^0}}{2\pi^2} \frac{s_8}{s_8+Q^2} \\ \frac{N_c C^{(0)}_{Z^0}}{2\pi^2} \frac{s_0(1+B(Q^2,s_0))}{s_0+Q^2} \end{pmatrix},$$

one obtains expressions for the transition form factors:

$$F_{P\gamma Z^0}(Q^2) = \bar{\alpha}_P \frac{s_3}{s_3+Q^2} + \bar{\beta}_P \frac{s_8}{s_8+Q^2} + \bar{\gamma}_P \frac{s_0}{s_0+Q^2}[1 + B_{Z^0}(Q^2, s_0)],$$

where $P = \pi^0, \eta, \eta'$. The coefficients $\bar{\alpha}_P$, $\bar{\beta}_P$, $\bar{\gamma}_P$ are expressed in terms of $f^{(i)}_P$. The analytical continuation to time-like in the case of γZ^0 will be similar to the $\gamma\gamma^*$ case ($Q^2 \to -q^2$) because they differ from each other only by the charge factor coefficients, which are constants. Thus the equations TFFs in time-like domain read:

$$|F_{P\gamma Z^0}(Q^2)| = \left| \bar{\alpha}_P \frac{s_3}{s_3-q^2} + \bar{\beta}_P \frac{s_8}{s_8-q^2} + \bar{\gamma}_P \frac{s_0}{s_0-q^2}[1 + B_{Z^0}(q^2, s_0)] \right|. \quad (36)$$

Using (36) and one can predict decay widths of $\pi^0, \eta, \eta' \to \nu\bar{\nu}\gamma$

$$\Gamma_{P\to\nu\bar{\nu}\gamma} = \frac{1}{24} \frac{M^4_{Z^0}}{(M^2_{Z^0}-q^2)^2} \frac{1}{\pi^2} \frac{\alpha_{qed} G^2_F}{m^3_P} \int_0^{m^2_P} q^2(m^2_P - q^2)^3 |F_{P\gamma Z^0}(q^2)|^2 dq^2, \quad (37)$$

The (37) is for neutrino of one flavor, in order to calculate for all 3 neutrino flavors, one should multiply (37) by a factor 3.

In [L. Arnellos, W. J. Marciano and Z. Parsa, Nucl. Phys. B 196 (1982), 365-377] the estimates for $\Gamma_{\pi^0,\eta,\eta'\to\nu\bar{\nu}\gamma}$ are listed. These estimates were done under assumption that $q^2 \leqslant m^2_{\pi^0,\eta} \approx 0$. So one can calculate the same estimates with $|F_{P\gamma Z^0}(0)|^2$

$$\Gamma_{P\to\nu\bar{\nu}\gamma} = \frac{\alpha_{qed} G_F^2 m_P^7}{480\pi^2} |F_{P\gamma Z^0}(0)|^2, \quad (38)$$

where $P = \pi^0, \eta, \eta'$. The results are listed below:

$$\Gamma_{\pi^0\to\nu\bar{\nu}\gamma} = 1.6 \cdot 10^{-26} GeV, \quad (39)$$

$$\Gamma_{\eta\to\nu\bar{\nu}\gamma} = 2.6 \cdot 10^{-21} GeV. \quad (40)$$

Let us stress that the value of $\Gamma_{\pi^0\to\nu\bar{\nu}\gamma}$ (39) is calculated for one term equation for π^0 TFF, i.e. without taking into account small mixing between π^0 and $\eta - \eta'$. The value of $\Gamma_{\eta\to\nu\bar{\nu}\gamma}$ (40) is calculated taking into account strong mixing between η and η', but using old mixing scheme. Also note that the results for π^0 (39) and η (40) are calculated for the single neutrino flavor.

In order to compare the results for π^0 (39) and η (40) calculated by Arnellos et.al. in 1982 with the current calculation in ASR approach using (38), we consider two cases: without taking into account small mixing between π^0 and $\eta - \eta'$, and with taking into account this small mixing (the results correspond to the single neutrino flavor):

- MIXING OFF:

	FKS98, GeV	EF05, GeV	KOT12, GeV	EGMS16, GeV
$\Gamma_{\pi^0 \to \nu\bar{\nu}\gamma}$	$2.05 \cdot 10^{-26}$	$2.05 \cdot 10^{-26}$	$2.05 \cdot 10^{-26}$	$2.05 \cdot 10^{-26}$
$\Gamma_{\eta \to \nu\bar{\nu}\gamma}$	$2.21 \cdot 10^{-20}$	$2.30 \cdot 10^{-20}$	$2.08 \cdot 10^{-20}$	$2.12 \cdot 10^{-20}$

- MIXING ON:

	FKS98, GeV	EF05, GeV	KOT12, GeV	EGMS16, GeV
$\Gamma_{\pi^0 \to \nu\bar{\nu}\gamma}$	$5.60 \cdot 10^{-26}$	$5.42 \cdot 10^{-26}$	$5.61 \cdot 10^{-26}$	$5.55 \cdot 10^{-26}$
$\Gamma_{\eta \to \nu\bar{\nu}\gamma}$	$2.20 \cdot 10^{-20}$	$2.30 \cdot 10^{-20}$	$2.07 \cdot 10^{-20}$	$2.12 \cdot 10^{-20}$

Compare to results by Arnellos et.al. in 1982

$$\Gamma_{\pi^0 \to \nu\bar{\nu}\gamma} = 1.6 \cdot 10^{-26} GeV,$$

$$\Gamma_{\eta \to \nu\bar{\nu}\gamma} = 2.6 \cdot 10^{-21} GeV.$$

Mix. sch.	π^0			η			η'		
	$\bar{\alpha}_{\pi^0}$	$\bar{\beta}_{\pi^0}$	$\bar{\gamma}_{\pi^0}$	$\bar{\alpha}_{\eta}$	$\bar{\beta}_{\eta}$	$\bar{\gamma}_{\eta}$	$\bar{\alpha}_{\eta'}$	$\bar{\beta}_{\eta'}$	$\bar{\gamma}_{\eta'}$
MIX. OFF EGMS16	0.011	0	0	0	0.005	0.079	0	-0.0006	0.204
MIX. ON EGMS16	0.011	-1.5e-05	0.007	-6.4e-05	0.005	0.08	-0.0003	-0.0006	0.204

Table: The coefficients $\bar{\alpha_P}, \bar{\beta_P}, \bar{\gamma_P}$ in GeV^{-1} for γZ^0 processes with and without taking into account mixing between π^0 and $\eta - \eta'$.

Mix. sch.	π^0			η			η'		
	α_{π^0}	β_{π^0}	γ_{π^0}	α_{η}	β_{η}	γ_{η}	$\alpha_{\eta'}$	$\beta_{\eta'}$	$\gamma_{\eta'}$
MIX. OFF EGMS16	0.274	0	0	0	0.128	0.147	0	-0.016	0.377
MIX. ON EGMS16	0.274	-0.0004	0.013	-0.0016	0.128	0.147	-0.008	-0.016	0.377

Table: The coefficients $\alpha_P, \beta_P, \gamma_P$ in GeV^{-1} for $\gamma\gamma*$ processes with and without taking into account mixing between π^0 and $\eta - \eta'$.

When small mixing of π^0 with $\eta - \eta'$ is neglected, the decay widths calculated by ASR approach and by Arnellos et.al.coincide. However when mixing between π^0 and $\eta - \eta'$ is taken into account, we got 3 times increase of decay width $\Gamma_{\pi^0 \to \nu\bar{\nu}\gamma}$ by ASR. This effect is an essential feature of small non-zero mixing between π^0 and $\eta - \eta'$. The coefficient $\bar{\gamma}_{\pi^0}$ appears to be comparable with $\bar{\alpha}_{\pi^0}$, and so it leads to 3 times increasing of $|F_{P\gamma Z^0}(0)|^2$ and corresponding decay width. Note that it wouldn't be such increasing for $\gamma\gamma^*$ processes, where coefficient α_{π^0} dominates. But in the case γZ^0 the Salam-Weinberg $(\frac{1}{2} - 2\xi)$ coefficient leads to lowering of $\bar{\alpha}_{\pi^0}$ in such a way that it becomes comparable to $\bar{\gamma}_{\pi^0}$. Thus for γZ^0 case the small mixing between π^0 and $\eta - \eta'$ becomes crucial!

Now let us consider the results for η meson.

Mix. sch.	π^0			η			η'		
	$\bar{\alpha}_{\pi^0}$	$\bar{\beta}_{\pi^0}$	$\bar{\gamma}_{\pi^0}$	$\bar{\alpha}_\eta$	$\bar{\beta}_\eta$	$\bar{\gamma}_\eta$	$\bar{\alpha}_{\eta'}$	$\bar{\beta}_{\eta'}$	$\bar{\gamma}_{\eta'}$
MIX. OFF EGMS16	0.011	0	0	0	0.005	0.079	0	-0.0006	0.204
MIX. ON EGMS16	0.011	-1.5e-05	0.007	-6.4e-05	0.005	0.08	-0.0003	-0.0006	0.204

Table: The coefficients $\bar{\alpha_P}, \bar{\beta_P}, \bar{\gamma_P}$ in GeV^{-1} for γZ^0 processes with and without taking into account mixing between π^0 and $\eta - \eta'$.

Taking into account the effects of small mixing of π^0 with $\eta - \eta'$ does not have such a crucial role in contrast to π^0 case, and the leading coefficients of η meson TFF – $\bar{\beta}_\eta$ and $\bar{\gamma}_\eta$ do not change significantly. The coefficient $\bar{\alpha}_\eta$ appears to be much smaller than $\bar{\beta}_\eta$ and $\bar{\gamma}_\eta$, and so it has a small influence on η meson TFF and correspondingly to decay width.

At the same time, the obtained results for η meson decay widths differ from the ones estimated by Arnellos et.al. by an order of magnitude for all considered mixing schemes. This effect is a consequence of the choice of the mixing scheme. In the paper of Arnellos et.al., which was written in 1982, the old mixing scheme was used and so the values of decay constants differ from the modern ones. The use of recent mixing schemes leads to the growth of the $\Gamma_{\eta\to\nu\bar{\nu}\gamma}$ decay width by an order of magnitude.

Strictly speaking, for the case of η the interval of $dq^2 \leqslant m_\eta^2 \approx 0.3$ GeV^2 is much larger than in the π^0 case $dq^2 \leqslant m_{\pi^0}^2 \approx 0.02$ GeV^2 and so the estimation of $\Gamma_{\eta\to\nu\bar{\nu}\gamma}$ by (38) may not be accurate.
Thus we need to numerically integrate (37) for the η meson. Using instruments of ROOT numerical integration of (37) for the EGMS16 mixing scheme with interpolation formulas for $s_{3,8,0}(q^2)$ and $B(q^2)$ gives

$$\Gamma_{\eta\to\nu\bar{\nu}\gamma} = 3.45 \cdot 10^{-20}\ GeV, \tag{41}$$

for each neutrino flavor. One can see that taking into account non-trivial form of η TFF leads to increasing of the $\Gamma_{\eta\to\nu\bar{\nu}\gamma}$ by the factor 1.5 in comparison of the value calculated using (38) ($\Gamma_{\eta\to\nu\bar{\nu}\gamma} = 2.12 \cdot 10^{-20} GeV$).
For π^0 one can also perform numerical integration

$$\Gamma_{\pi^0\to\nu\bar{\nu}\gamma} = 5.68 \cdot 10^{-26}\ GeV, \tag{42}$$

for each of neutrino flavor (using (38) $\Gamma_{\pi^0\to\nu\bar{\nu}\gamma} = 5.55 \cdot 10^{-26} GeV$).

Finally, multiplying by a factor 3 results of numerical integration for π^0 and η one obtains

$$\Gamma_{\pi^0 \to \sum_i (\nu_i \bar{\nu}_i)\gamma} = 1.704 \cdot 10^{-25} GeV,$$

$$\Gamma_{\eta \to \sum_i (\nu_i \bar{\nu}_i)\gamma} = 1.04 \cdot 10^{-19}\ GeV.$$

Compare to results by Arnellos et.al. in 1982

$$\Gamma_{\pi^0 \to \nu\bar{\nu}\gamma} = 1.6 \cdot 10^{-26} GeV,$$

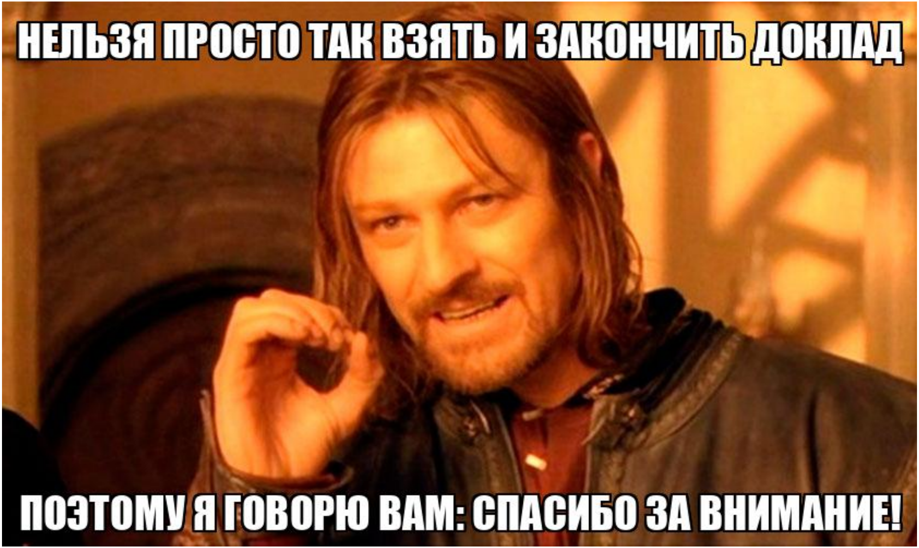
$$\Gamma_{\eta \to \nu\bar{\nu}\gamma} = 2.6 \cdot 10^{-21} GeV.$$

The modern experimental upper limit for $\Gamma_{\pi^0 \to \sum_i (\nu_i \bar{\nu}_i)\gamma}$ is $< 1.48 \cdot 10^{-17}$ GeV by NA62 collaboration (2019).

The branching for π^0 and η meson are the following:

$$\frac{\Gamma_{\pi^0 \to \sum_i (\nu_i \bar{\nu}_i)\gamma}}{\Gamma_{\pi^0 \to all}} = 2.18 \cdot 10^{-17}, \qquad (43)$$

$$\frac{\Gamma_{\eta \to \sum_i (\nu_i \bar{\nu}_i)\gamma}}{\Gamma_{\eta \to all}} = 7.94 \cdot 10^{-14}. \qquad (44)$$


НЕЛЬЗЯ ПРОСТО ТАК ВЗЯТЬ И ЗАКОНЧИТЬ ДОКЛАД
ПОЭТОМУ Я ГОВОРЮ ВАМ: СПАСИБО ЗА ВНИМАНИЕ!