

Covariant reggeization framework for diffraction

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Plan

- ❑ Introduction
- ❑ Basics of the framework
- ❑ Calculations
- ❑ Further study

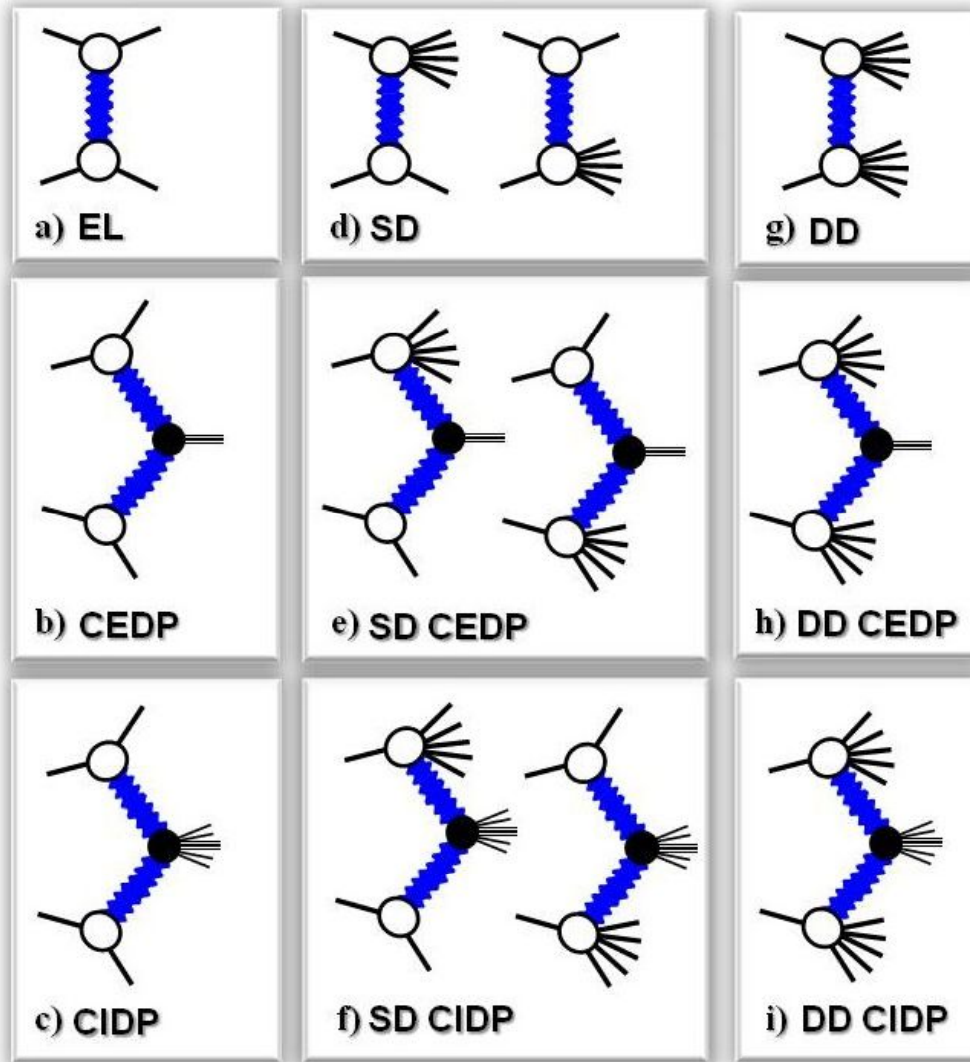
[V.A. Petrov, R.A. Ryutin, A.E. Sobol and J.-P. Guillaud, *Azimuthal Angular Distributions in EDDE as Spin-Parity Analyser and Glueball Filter for LHC*, JHEP06(2005)007]

[R.A. Ryutin , *Visualizations of exclusive central diffraction*, Eur. Phys. J. C 74, 3162 (2014)]

[V.A. Petrov and R.A. Ryutin, *Single and double diffractive dissociation and the problem of extraction of the proton–Pomeron cross-section*, Int. J. Mod. Phys. 31, No. 10, 1650049 (2016)]

[R. Ryutin, *Covariant reggeization framework for diffraction. Part I: Hadronic tensors in Minkovsky space-time of any dimension* arXiv:2507.16019 [hep-ph]]

Introduction



Soft diffractive processes
("bare" amplitudes)

What is a
"REGGEON"?
"POMERON"?

(trajectory, intercept,
residue, slope, ...)

What if the reggeon is
a quantum field?

What is “REGGEON”?

- Gribov Reggeons

(in 2+1 ...)

- BFKL (QCD) ...

- Classical ...

(10+ phenomenological models)

- Continuous & High Spin Particles

(Wigner, Fronsdal etc.)

- Hyper-Fields & Strings

+ Complex Momentum Plane

Basics

Current operator related to the
hadronic spin-J Heisenberg field operator

$$\left(\square + m_J^2\right) \Phi^{\mu_1 \dots \mu_J}(x) = \mathcal{J}^{\mu_1 \dots \mu_J}(x)$$

Rarita-Schwinger TST conditions

$$\begin{aligned}\partial_\mu \mathcal{J}^{\mu_1 \dots \mu_J} &= 0 ; \\ \mathcal{J}^{\mu_1 \dots \mu_J} &= \mathcal{J}^{(\mu_1 \dots \mu_J)} ; \\ g_{\mu_i \mu_k} \mathcal{J}^{\mu_1 \dots \mu_i \dots \mu_k \dots \mu_J} &= 0\end{aligned}$$

$$\begin{aligned}q_\lambda \mathcal{V}^{\mu_1 \dots \lambda \dots \mu_J} &= 0 ; \\ \mathcal{V}^{\mu_1 \dots \mu_J} &= \mathcal{V}^{(\mu_1 \dots \mu_J)} ; \\ g_{\mu_i \mu_k} \mathcal{V}^{\mu_1 \dots \mu_i \dots \mu_k \dots \mu_J} &= 0\end{aligned}$$

Basics

Scalar-Scalar-Spin-J vertex

$$\mathcal{V}^{\mu_1 \dots \mu_J}(p, q) = \langle p - q | \mathcal{J}^{\mu_1 \dots \mu_J} | p \rangle$$

Spin-J Propagator

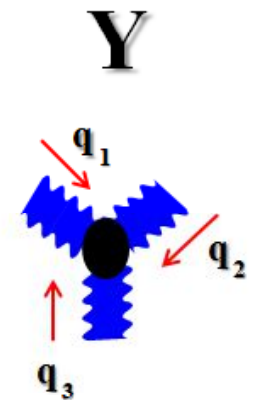
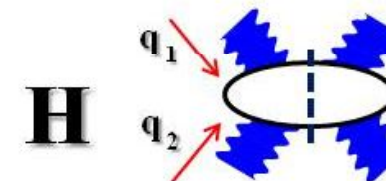
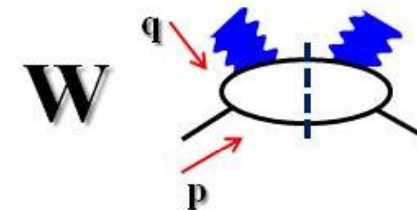
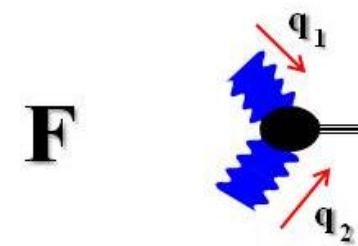
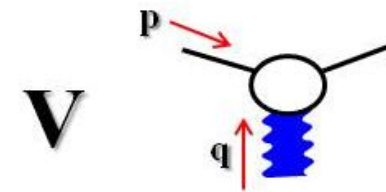
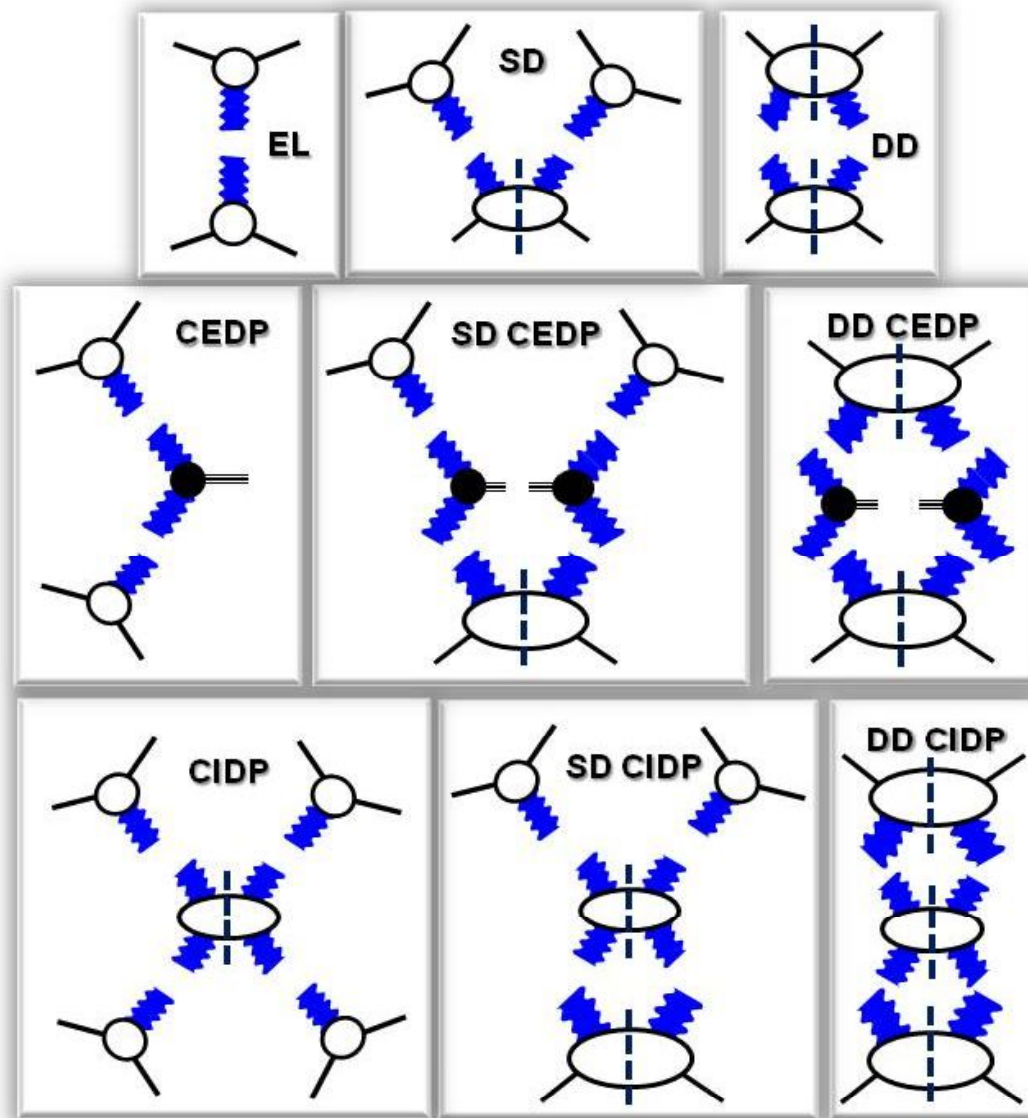
$$\begin{aligned} \mathcal{P}^{\mu_1 \dots \mu_J \nu_1 \dots \nu_J}(q) &= \\ \int d^4x e^{iq(x-y)} \langle 0 | \Phi^{\mu_1 \dots \mu_J}(x) \Phi^{\nu_1 \dots \nu_J}(y) | 0 \rangle \\ &= \Pi^{\mu_1 \dots \mu_J, \nu_1 \dots \nu_J}(q) / (m^2(J) - q^2), \end{aligned}$$

Pole at $m^2(J) - q^2 = 0$, i.e. $J = \alpha_{\mathbb{R}}(q^2)$

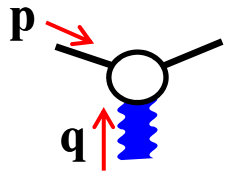
Reggeization prescription

$$\sum_J \frac{F^J}{(q^2 - m^2)} \rightarrow \frac{\alpha'_{\mathbb{R}}}{2} \eta_{\mathbb{R}}(q^2) \Gamma(-\alpha_{\mathbb{R}}(q^2)) F^{\alpha_{\mathbb{R}}(q^2)}$$

Diffractive Lego



Irreducible tensors



$$\mathcal{V}^J \equiv \mathcal{V}_{(1)}^J(p, q)$$

$$(\mu)_J \equiv (\mu_1 \mu_2 \dots \mu_J)$$

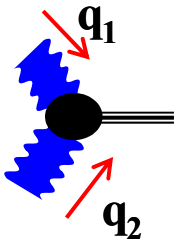
$$(1) \leftrightarrow (\mu)_{J_1}$$

$$(1') \leftrightarrow (\mu')_{J_{1'}}$$

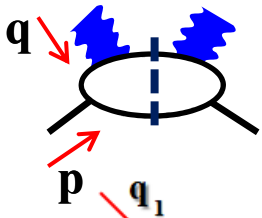
$$(2) \leftrightarrow (\nu)_{J_2}$$

$$(2') \leftrightarrow (\nu')_{J_{2'}}$$

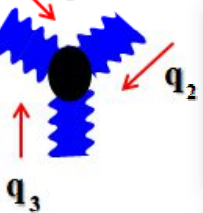
$$(3) \leftrightarrow (\rho)_{J_3}$$



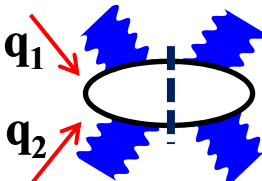
$$\mathcal{F}^{J_1, J_2} \equiv \mathcal{F}_{(1), (2)}^{J_1, J_2}(q_1, q_2)$$



$$\mathcal{W}^{J_1, J_{1'}} \equiv \mathcal{W}_{(1), (1')}^{J_1, J_{1'}}(p, q)$$



$$\mathcal{Y}^{\vec{J}} \equiv \mathcal{Y}^{\{J_1, J_2, J_3\}} \equiv \mathcal{Y}_{(1), (2), (3)}^{\vec{J}}(q_1, q_2)$$



$$\mathcal{H}^{\vec{J}} \equiv \mathcal{H}^{\{J_1, J_{1'}, J_2, J_{2'}\}} \equiv \mathcal{H}_{(1), (1'), (2), (2')}^{\vec{J}}(q_1, q_2)$$

Transverse Symmetric tensors (V,W,F)

$$P_\alpha \equiv \frac{\left(p_\alpha - \frac{pq}{q^2} q_\alpha\right)}{\sqrt{p^2 - (pq)^2/q^2}}$$

$$P_{i\alpha} \equiv \frac{\left(p_{c\alpha} - \frac{p_c q_i}{q_i^2} q_{i\alpha}\right)}{\sqrt{M_c^2 - (p_c q_i)^2/q_i^2}}$$

$$Q_i^2 = -q_i^2 \equiv -t_i$$

$$\chi_m = \frac{Q_1 Q_2}{q_1 q_2}, \quad \lambda_m = 1 - \chi_m^2$$

$$G_{(rr)} \equiv G_{(rr)\alpha_1\alpha_2} \equiv g_{\alpha_1\alpha_2} - \frac{q_{r\alpha_1} q_{r\alpha_2}}{q_r^2},$$

$$\alpha_{1,2} \in (r),$$

$$G_{(rr')} \equiv G_{(rr')\alpha\beta} \equiv g_{\alpha\beta} - \frac{q_{r\alpha} q_{r'\beta}}{q_r^2},$$

$$\alpha \in (r) \& \beta \in (r'),$$

$$\hat{G}_{(rs)} \equiv \hat{G}_{(rs)\alpha\beta} \equiv g_{\alpha\beta} - \frac{q_{s\alpha} q_{r\beta}}{q_r q_s},$$

$$r \neq s, \alpha \in (r), \beta \in (s),$$

$$G_{\alpha\beta} q_{r\alpha} = 0, \quad \hat{G}_{(rs)\alpha\beta} q_{r\alpha} = \hat{G}_{(rs)\alpha\beta} q_{s\beta} = 0$$

Transverse Symmetric tensors (V,W,F)

$$S_{n_1}^{V;J_1} \equiv \left(P_{(1)}^{J_1-2n_1} G_{(11)}^{n_1} \right)$$

$$S_{k',n_1 n_{1'}}^W \{J_1, J_{1'}\} \equiv \left(G_{(11')}^{k'} \prod_{i \in \Omega_W} P_{(i)}^{\bar{J}_i} G_{(ii)}^{n_i} \right)$$

$$\Omega_W = \{1, 1'\}, \bar{J}_i = J_i - 2n_i - k',$$

$$S_n^P J = S_{J-2n, n n}^W \{J, J\} \equiv \left(G_{(11')}^{J-2n} G_{(11)}^n G_{(1'1')}^n \right)$$

$$S_{k',n_1 n_2}^F \{J_1, J_2\} \equiv \left(\hat{G}_{(12)}^{k'} \prod_{i \in \Omega_F} P_{(i)}^{\bar{J}_i} G_{(ii)}^{n_i} \right)$$

$$\Omega_F = \{1, 2\}, \bar{J}_i = J_i - 2n_i - k',$$

Transverse Symmetric tensors (Y)

$$\mathcal{K} \equiv (q_i q_j)^2 - q_i^2 q_j^2 = \frac{1}{4} \lambda(q_1^2, q_2^2, q_3^2)$$

$$Q_i^2 = -q_i^2 \equiv -t_i, \quad i, j \in \{1, 2, 3\}, ; i \neq j,$$

$$q_3 + q_1 + q_2 = 0,$$

$$P_{i\alpha} \equiv \frac{(q_i q_s) q_{r\alpha} - (q_i q_r) q_{s\alpha}}{Q_i \sqrt{|\mathcal{K}|}},$$

$$P_{i\alpha} q_{i\alpha} = 0.$$

$$\chi_{ij} = \frac{Q_i Q_j}{|q_i q_j|}, \quad \lambda_{ij} = 1 - \kappa_i \kappa_j \chi_{ij}^2,$$

$$\hat{\lambda}_{rs}^i \equiv \prod_{\substack{a \neq b \\ a, b \in \{1, 2, 3\}}} \kappa_{ab} \cdot \eta_{is} \eta_{ir} \chi_{rs} \sqrt{|\lambda_{ir}| |\lambda_{is}|},$$

$$\tilde{\lambda}_{ir}^r \equiv \kappa_r |\lambda_{ir}|,$$

$$\bar{\chi}_{ij} = \kappa_i \kappa_{ij} \chi_{ij} \bar{\eta}_{ij}.$$

$$\kappa_{ij} = \text{sign}[q_i q_j], \quad \kappa_i = \text{sign}[-q_i^2], \quad \kappa = \text{sign}[\mathcal{K}],$$

$$\eta_{ij} \equiv \begin{cases} +1 & i > j \\ -1 & i < j \end{cases}, \quad \bar{\eta}_{ij} \equiv \begin{cases} \eta_{si} & j = r \\ \eta_{ir} & j = s \\ r < s, & r, s \neq i \end{cases},$$

$$r, s \in \{1, 2, 3\},$$

$$Q_1 P_1 + Q_3 P_3 = Q_2 P_2,$$

$$Q_s \frac{\sqrt{|\lambda_{is}|}}{\chi_{is}} = Q_r \frac{\sqrt{|\lambda_{ir}|}}{\chi_{ir}},$$

$$P_i^2 = \kappa \kappa_i \equiv \bar{\kappa}_i.$$

Transverse Symmetric tensors (Y)

$$S_{\vec{k}', \vec{n}}^Y \vec{J} \equiv \left(\prod_{i \in \Omega_Y} P_{(i)}^{\bar{J}_i} G_{(ii)}^{n_i} \prod_{\substack{\forall r \neq s \\ r, s \in \Omega_Y}} \hat{G}_{(rs)}^{k'_{rs}} \right),$$

$$\Omega_Y = \{1, 2, 3\}, \quad \bar{J}_i = J_i - 2n_i - \sum_{\substack{r \neq i \\ r \in \Omega_Y}} k'_{ri},$$

Transverse Symmetric tensors (H)

$$S_{\vec{k}', \vec{n}}^H \vec{J} \equiv \left(G_{(11')}^{k'_{11'}} G_{(22')}^{k'_{22'}} \prod_{i \in \Omega_H} P_{(i)}^{\bar{J}_i} G_{(ii)}^{n_i} \prod_{\substack{\forall r \neq r', s \\ r, s \in \Omega_H}} \hat{G}_{(rs)}^{k'_{rs}} \right),$$

$$\Omega_H = \{1, 1', 2, 2'\}, \quad \bar{J}_i = J_i - 2n_i - \sum_{\substack{r \neq i \\ r \in \Omega_H}} k'_{ri}$$

$$\begin{aligned} G_{(ii)\alpha\beta} &= G_{(i'i')\alpha\beta} = G_{(ii')\alpha\beta}, \\ G_{(ij)\alpha\beta} &= G_{(ij')\alpha\beta} = G_{(i'j)\alpha\beta} = G_{(i'j')\alpha\beta}, \\ i, j &\in \{1, 1', 2, 2'\}, \quad i \neq j. \end{aligned}$$

Irreducible tensors (spur $S^\vee$)

$$Sp_1 S_{n_1}^{V;J_1} = g_{\mu_1 \mu_2} \left[P_{(1)}^{\mu_1} P_{(1)}^{\mu_2} S_{n_1}^{V;J_1-2} + \right. \\ \left. + \sum_i \left(P_{(1)}^{\mu_1} G_{(11)}^{\mu_2 \mu_i} + P_{(1)}^{\mu_2} G_{(11)}^{\mu_1 \mu_i} \right) S_{n_1-1}^{V;J_1-3} + \right. \\ \left. + \sum_{i < j} G_{(11)}^{\mu_1 \mu_i} G_{(11)}^{\mu_2 \mu_j} S_{n_1-2}^{V;J_1-4} + G_{(11)}^{\mu_1 \mu_2} S_{n_1-1}^{V;J_1-2} \right]$$

$$\mathcal{N}_n^J = \frac{J!}{2^n n! (J-2n)!}$$

$$S_{n_1}^{V;J_1-2} + 2 \frac{(J_1-2) \mathcal{N}_{n_1-1}^{J_1-3}}{\mathcal{N}_{n_1-1}^{J_1-2}} S_{n_1-1}^{V;J_1-2} + \\ + \frac{(J_1-2)(J_1-3) \mathcal{N}_{n_1-2}^{J_1-4}}{\mathcal{N}_{n_1-1}^{J_1-2}} S_{n_1-1}^{V;J_1-2} + (D-1) S_{n_1-1}^{V;J_1-2} = \\ S_{n_1}^{V;J_1-2} + (2(J_1-2n_1) + 2(n_1-1) + D-1) S_{n_1-1}^{V;J_1-2},$$

Irreducible tensors (V)

$$\mathcal{V}^J(p, q) = \hat{v}_0(t) \sum_{n=0}^{[J/2]} v_n^J S_n^{V;J}$$

$$\begin{cases} v_{n-1}^J - 2(c^J + n - 1)v_n^J = 0, & n > 0, \\ v_0^J = 1, & c^J = -(J + (D - 5)/2) \end{cases}$$

$$v_n^J = \frac{1}{2^n (c^J)_n}$$

Irreducible tensors (F)

$$\mathcal{F}^{J_1, J_2}(q_{(1)}, q_{(2)}) = \sum_{k=0}^{\min(J_1, J_2)} \hat{f}_k^{J_1, J_2}(t_1, t_2) \bar{\mathcal{F}}_k^{J_1, J_2}$$

$$\bar{\mathcal{F}}_k^{J_1, J_2} = \sum_{k'=0}^k \sum_{n_{1,2}=0}^{[(J_{1,2}-k')/2]} f_{n_1 n_2}^{k'}(k; J_1, J_2) S_{k', n_1 n_2}^{F; J_1, J_2},$$

$$\left\{ \begin{array}{l} f_{n_1-1 n_2}^{k'} - 2(c^{J_1} + n_1 - 1)f_{n_1 n_2}^{k'} + 2\chi_m \bar{J}_2 f_{n_1-1 n_2}^{k'+1} + \\ + 2n_2 f_{n_1-1 n_2-1}^{k'+2} - \lambda_m \bar{J}_2 (\bar{J}_2 - 1) f_{n_1-1 n_2}^{k'+2} = 0, \\ \\ f_{n_1 n_2-1}^{k'} - 2(c^{J_2} + n_2 - 1)f_{n_1 n_2}^{k'} + 2\chi_m \bar{J}_1 f_{n_1 n_2-1}^{k'+1} + \\ + 2n_1 f_{n_1-1 n_2-1}^{k'+2} - \lambda_m \bar{J}_1 (\bar{J}_1 - 1) f_{n_1 n_2-1}^{k'+2} = 0, \\ \\ c^{J_i} = -(J_i + (D - 5)/2), \bar{J}_i = (J_i - 2n_i - k'), \end{array} \right.$$

Irreducible tensors (F)

$$f_{n_1 n_2}^{k'}(k; J_1, J_2) =$$

$$= 2^{k-k'-n_1-n_2-1} \chi_m^{k-k'} \times$$

$$\left\{ \frac{n_2! (J_2 - 2n_2 - k')!}{(c^{J_1})_{n_1} \chi_m^{2n_2}} \Lambda_{R \ k-k' \ n_1 n_2}^{J_1 J_2 k}(\hat{x}) + \right.$$

$$\left. \frac{n_1! (J_1 - 2n_1 - k')!}{(c^{J_2})_{n_2} \chi_m^{2n_1}} \Lambda_{R \ k-k' \ n_2 n_1}^{J_2 J_1 k}(\hat{x}) \right\},$$

$$\mathcal{R}_N^L(\hat{x}) = \sum_{i=0}^N \mathbb{C}_N^{N-L+2i} \mathbb{C}_{N-L+2i}^i (\hat{x}/4)^i$$

$$\hat{x} = -\lambda_m / \chi_m^2, \mathcal{R}_N^L(0) = \mathbb{C}_N^L$$

$$\alpha_{l \ n}^{J_1 J_2 k} = \frac{(J_1 - k + l)!}{(J_1 - k)!} \frac{1}{(J_2 - k + l - 2n)! n! (c^{J_2})_n}$$

$$\Lambda_{R \ l \ n_1 n_2}^{J_1 J_2 k}(\hat{x}) = \sum_{i=0}^{n_2} \mathbb{C}_{n_1}^{n_2-i} \sum_{j=0}^l \alpha_{j \ i}^{J_1 J_2 k} \mathcal{R}_i^j(\hat{x}) \mathcal{R}_{n_1-n_2+i}^{l-2n_2+2i-j}(\hat{x})$$

Irreducible tensors (W)

$$\mathcal{W}^{J_1, J_{1'}}(p_{(1)}, q_{(1)}) = \sum_{k=0}^{\min(J_1, J_{1'})} \hat{w}_k^{J_1, J_{1'}}(t_1) \bar{\mathcal{W}}_k^{J_1, J_{1'}}$$

$$\bar{\mathcal{W}}_k^{J_1, J_{1'}} = \sum_{k'=0}^k \sum_{n_{1,1'}=0}^{\lfloor (J_{1,1'} - k')/2 \rfloor} w_{n_1 n_{1'}}^{k'(k; J_1, J_{1'})} S_{k', n_1 n_2}^{W; J_1, J_{1'}}$$

$$\left\{ \begin{array}{l} w_{n_1-1, n_{1'}}^{k'} - 2(c^{J_1} + n_1 - 1)w_{n_1 n_{1'}}^{k'} + 2\bar{J}_1' w_{n_1-1, n_{1'}}^{k'+1} + \\ + 2n_{1'} w_{n_1-1, n_{1'}-1}^{k'+2} = 0, \\ \\ w_{n_1, n_{1'}-1}^{k'} - 2(c^{J_{1'}} + n_{1'} - 1)w_{n_1 n_{1'}}^{k'} + 2\bar{J}_1 w_{n_1, n_{1'}-1}^{k'+1} + \\ + 2n_1 w_{n_1-1, n_{1'}-1}^{k'+2} = 0, \\ \\ c^{J_1} = -(J_1 + (D-5)/2), \bar{J}_1 = (J_1 - 2n_1 - k'), \\ c^{J_{1'}} = -(J_{1'} + (D-5)/2), \bar{J}_1' = (J_{1'} - 2n_{1'} - k'), \end{array} \right.$$

Irreducible tensors (W)

$$w_{n_1 n_{1'}}^{k'}(k; J_1, J_{1'}) =$$

$$= 2^{k-k'-n_1-n_{1'}-1} \times$$

$$\left\{ \frac{n_{1'}! (J_{1'} - 2n_{1'} - k')!}{(c^{J_1})_{n_1}} \Lambda_{\mathbb{C} \, k-k' \, n_1 n_{1'}}^{J_1 J_{1'} k} + \right.$$

$$\left. \frac{n_1! (J_1 - 2n_1 - k')!}{(c^{J_{1'}})_{n_{1'}}} \Lambda_{\mathbb{C} \, k-k' \, n_{1'} n_1}^{J_{1'} J_1 k} \right\},$$

$$\mathcal{P}^J(q) = \sum_{n=0}^{[J/2]} \frac{n!}{(c^J)_n} S_n^{P \, J}$$

$$\Lambda_{C \, l \, n_1 n_2}^{J_1 J_2 k} = \lim_{\hat{x} \rightarrow 0} \Lambda_{R \, l \, n_1 n_2}^{J_1 J_2 k}(\hat{x}) =$$

$$= \sum_{i=0}^{n_2} \mathbb{C}_{n_1}^{n_2-i} \sum_{j=0}^l \alpha_{j \, i}^{J_1 J_2 k} \mathbb{C}_i^j \mathbb{C}_{n_1-n_2+i}^{l-2n_2+2i-j}$$

Generating function method (Y)

$$\begin{aligned}\bar{\mathcal{Y}}_g(\{x_r\}, \{y_r\}, \{y_{rs}\}) &= \bar{\mathcal{Y}}^{\vec{J}} \prod_{r \in \Omega_Y} \omega_{(r)}^{J_r} = \\ &= \sum_{\tilde{\Omega}_{\vec{k}', \vec{n}}^Y} y_{\vec{n}}^{\vec{k}', (\vec{k}; \vec{J})} \mathcal{N}_{\vec{k}, \vec{n}}^{Y, \vec{J}} \times \prod_{r \in \Omega_Y} x_r^{\bar{J}_r} y_r^{n_r} \prod_{\substack{\bar{r} \neq \bar{s} \\ \bar{r}, \bar{s} \in \Omega_Y}} y_{\bar{r}\bar{s}}^{k'_{\bar{r}\bar{s}}}\end{aligned}$$

$$x_r = P_r \omega_r, \quad y_r = \omega_r^2 - \frac{(\omega_r q_r)^2}{q_r^2},$$

$$y_{rs} = \omega_r \omega_s - \frac{(\omega_s q_r)(\omega_r q_s)}{q_r q_s}, \quad y_{rs} \equiv y_{sr}, \quad r \neq s$$

$$\mathcal{N}_{\vec{k}', \vec{n}}^{Y, \vec{J}} = \prod_{r \in \Omega_Y} \frac{J_r!}{2^{n_r} n_r! \bar{J}_r} \prod_{\substack{\bar{r} \neq \bar{s} \\ \bar{r}, \bar{s} \in \Omega_Y}} \frac{1}{k'_{\bar{r}\bar{s}}!}$$

Generating function method (Y)

$$\begin{aligned} g_{\alpha\beta} \partial_{\omega_{i\alpha}} \partial_{\omega_{i\beta}} = & \\ & \bar{\kappa}_i \partial_{x_i x_i}^2 + 4x_i \partial_{x_i y_i}^2 + 4y_i \partial_{y_i y_i}^2 + 2 \sum_{r \neq i} \bar{\chi}_{ir} x_r \partial_{x_i y_{ir}}^2 + \\ & + 4 \sum_{r \neq i} y_{ir} \partial_{y_i y_{ir}}^2 + \sum_{r \neq i} (y_r - \tilde{\lambda}_{ir}^r x_r^2) \partial_{y_{ir} y_{ir}}^2 + \\ & + 2(y_{rs} + \hat{\lambda}_{rs}^i x_r x_s)_{r \neq s \neq i} \partial_{y_{ir} y_{is}}^2 + 2(D-1) \partial_{y_i}. \end{aligned}$$

Generating function method (Y)

$$\begin{aligned}
 y_{\vec{n}}^{\vec{k}', (\vec{k}; \vec{J})} \mathcal{N}_{\vec{k}', \vec{n}}^Y \vec{J} & \left[\bar{\kappa}_i \bar{J}_i (\bar{J}_i - 1) x_i^{\bar{J}_i - 2} \dots + 4 \bar{J}_i n_i y_i^{n_i - 1} \dots + \right. \\
 & + 4 n_i (n_i - 1) y_i^{n_i - 1} \dots + 2 \bar{J}_i \sum_{r \neq i} \bar{\chi}_{ir} k'_{ir} x_i^{\bar{J}_i - 1} x_r^{\bar{J}_r + 1} y_{ir}^{k'_{ir} - 1} \dots + \\
 & + 4 n_i \sum_{r \neq i} k'_{ir} y_i^{n_i - 1} \dots + \\
 & + \sum_{r \neq i} k'_{ir} (k'_{ir} - 1) \left(y_r^{n_r + 1} - \tilde{\lambda}_{ir}^r x_r^{\bar{J}_r + 2} \right) y_{ir}^{k'_{ir} - 2} \dots + \\
 & + 2 k'_{ir} k'_{is} \left(y_{rs}^{k'_{rs} + 1} + \hat{\lambda}_{rs}^i x_r^{\bar{J}_r + 1} x_s^{\bar{J}_s + 1} \right)_{r \neq s \neq i} y_{ir}^{k'_{ir} - 1} y_{is}^{k'_{is} - 1} \dots + \\
 & \left. + 2(D - 1) n_i y_i^{n_i - 1} \dots \right] = 0,
 \end{aligned}$$

Irreducible tensors (Y)

$$\mathcal{Y}^{\vec{J}}(q_1, q_2) = \sum_{\bar{\Omega}_{\vec{k}}^Y} \hat{y}_{\vec{k}}^{\vec{J}}(t_1, t_2) \bar{\mathcal{Y}}_{\vec{k}}^{\vec{J}},$$

$$\bar{\mathcal{Y}}_{\vec{k}}^{\vec{J}} = \sum_{\bar{\Omega}_{\vec{k}', \vec{n}}^Y} y_{\vec{n}}^{\vec{k}'}(\vec{k}; \vec{J}) S_{\vec{k}', \vec{n}}^{Y; \vec{J}},$$

$$\bar{\Omega}_{\vec{k}', \vec{n}}^Y$$

$$\left\{ \begin{array}{l} \bar{J}_r \geq 0, r \in \Omega_Y \\ \sum_{\substack{s \in \Omega_Y \\ s \neq r}} k'_{rs} \leq \sum_{\substack{s \in \Omega_Y \\ s \neq r}} k_{rs} \\ n_r, k'_{rs} \in \mathbb{Z}_+ \end{array} \right.$$

$$\bar{\Omega}_{\vec{k}}^Y$$

$$\left\{ \begin{array}{l} 0 \leq k_{12} + k_{13} \leq J_1 \\ 0 \leq k_{12} + k_{23} \leq J_2 \\ 0 \leq k_{13} + k_{23} \leq J_3 \\ k_{rs} \in \mathbb{Z}_+, \end{array} \right.$$

Irreducible tensors (Y)

$$\begin{aligned} \hat{\mathcal{A}}_i^Y &= \hat{O}_{n_i}^{-1} \times \\ &\times \left(\bar{\kappa}_i + \left[2k'_{rs} \hat{O}_{k'_{ir}}^{+1} \hat{O}_{k'_{is}}^{+1} \hat{O}_{k'_{rs}}^{-1} + 2\hat{\lambda}_{rs}^i \bar{J}_r \bar{J}_s \hat{O}_{k'_{ir}}^{+1} \hat{O}_{k'_{is}}^{+1} \right] + \right. \\ &\left. + \sum_{\substack{j \neq i \\ j \in \Omega_Y}} \left[2\bar{\chi}_{ij} \bar{J}_j \hat{O}_{k'_{ij}}^{+1} + 2n_j \hat{O}_{k'_{ij}}^{+2} \hat{O}_{n_j}^{-1} - \tilde{\lambda}_{ij}^j \bar{J}_j (\bar{J}_j - 1) \hat{O}_{k'_{ij}}^{+2} \right] \right) \\ r, s, i, j &\in \Omega_Y; i \neq r \neq s. \end{aligned}$$

$$y_{\vec{n}}^{\vec{k}'} = \frac{1}{2(c^{J_i} + n_i - 1)} \hat{\mathcal{A}}_i^Y y_{\vec{n}}^{\vec{k}'}$$

Generating function method (H)

$$\bar{\mathcal{H}}_g(\{x_r\}, \{y_r\}, \{y_{rs}\}) = \bar{\mathcal{H}}^{\vec{J}} \prod_{r \in \Omega_H} \omega_{(r)}^{J_r} =$$

$$\sum_{\tilde{\Omega}_{\vec{k}', \vec{n}}^H} h_{\vec{n}}^{\vec{k}', (\vec{k}; \vec{J})} \mathcal{N}_{\vec{k}, \vec{n}}^{H, \vec{J}} \times \prod_{r \in \Omega_H} x_r^{\bar{J}_r} y_r^{n_r} \prod_{\substack{\hat{r} \neq \hat{s} \\ \hat{r}, \hat{s} \in \Omega_H}} y_{\hat{r} \hat{s}}^{k'_{\hat{r} \hat{s}}}$$

$$x_r = P_r \omega_r, \quad y_r = \omega_r^2 - \frac{(\omega_r q_r)^2}{q_r^2},$$

$$y_{rs} = \omega_r \omega_s - \frac{(\omega_s q_r)(\omega_r q_s)}{q_r q_s}, \quad y_{rs} \equiv y_{sr}, \quad r \neq s,$$

$$y_{ii'} = \omega_i \omega_{i'} - \frac{(\omega_i q_i)(\omega_{i'} q_i)}{q_i^2}, \quad i = 1, 2.$$

$$\mathcal{N}_{\vec{k}', \vec{n}}^{H, \vec{J}} = \prod_{r \in \Omega_H} \frac{J_r!}{2^{n_r} n_r! \bar{J}_r} \prod_{\substack{\bar{r} \neq \bar{s} \\ \bar{r}, \bar{s} \in \Omega_H}} \frac{1}{k'_{\bar{r} \bar{s}}!},$$

Generating function method (H)

$$\begin{aligned} g_{\alpha\beta} \partial_{\omega_{(i)\alpha}} \partial_{\omega_{(i)\beta}} = & \\ & \partial_{x_i x_i}^2 + 4x_i \partial_{x_i y_i}^2 + 4y_i \partial_{y_i y_i}^2 + 2x_{i'} \partial_{x_i y_{ii'}}^2 + \\ & + 2\chi_m \sum_{r \neq i, i'} x_r \partial_{x_i y_{ir}}^2 + 4 \sum_{r \neq i} y_{ir} \partial_{y_i y_{ir}}^2 + y_{i'} \partial_{y_{ii'} y_{ii'}}^2 + \\ & + \sum_{r \neq i, i'} (y_r - \lambda_m x_r^2) \partial_{y_{ir} y_{ir}}^2 + 2(y_{jj'} - \lambda_m x_j x_{j'}) \partial_{y_{ij} y_{ij'}}^2 + \\ & + \sum_{r \neq i, i'} 2y_{i'r} \partial_{y_{ii'} y_{ir}}^2 + 2(D-1) \partial_{y_i}, \\ & \{j, j'\} \neq \{i, i'\}, j \equiv \bar{i}, j' \equiv \bar{i}'. \end{aligned}$$

Generating function method (H)

$$\begin{aligned}
 h_{\vec{n}}^{\vec{k}', (\vec{k}; \vec{J})} \mathcal{N}_{\vec{k}, \vec{n}}^{H, \vec{J}} & \left[\bar{J}_i (\bar{J}_i - 1) x_i^{\bar{J}_i - 2} \dots + 4 \bar{J}_i n_i y_i^{n_i - 1} \dots + \right. \\
 & + 4 n_i (n_i - 1) y_i^{n_i - 1} \dots + 2 \bar{J}_i k'_{ii'} x_i^{\bar{J}_i - 1} x_{i'}^{\bar{J}_{i'} + 1} y_{ii'}^{k'_{ii'} - 1} \dots + \\
 & + 2 \chi_m \bar{J}_i \sum_{r \neq i, i'} k'_{ir} x_i^{\bar{J}_i - 1} x_r^{\bar{J}_r + 1} y_{ir}^{k'_{ir} - 1} \dots + \\
 & + 4 n_i \sum_{r \neq i} k'_{ir} y_i^{n_i - 1} \dots + k'_{ii'} (k'_{ii'} - 1) y_{i'}^{n_{i'} + 1} y_{ii'}^{k'_{ii'} - 2} \dots + \\
 & + \sum_{r \neq i, i'} k'_{ir} (k'_{ir} - 1) \left(y_r^{n_r + 1} - \lambda_m x_r^{\bar{J}_r + 2} \right) y_{ir}^{k'_{ir} - 2} \dots + \\
 & + 2 k'_{ij} k'_{ij'} \left(y_{jj'}^{k'_{jj'} + 1} - \lambda_m x_j^{\bar{J}_j + 1} x_{j'}^{\bar{J}_{j'} + 1} \right) y_{ij}^{k'_{ij} - 1} y_{ij'}^{k'_{ij'} - 1} \dots + \\
 & + \sum_{r \neq i, i'} 2 k'_{ii'} k'_{ir} y_{ii'}^{k'_{ii'} - 1} y_{ir}^{k'_{ir} - 1} y_{i'r}^{k'_{i'r} + 1} \dots + \\
 & \left. + 2 (D - 1) n_i y_i^{n_i - 1} \dots \right] = 0,
 \end{aligned}$$

Irreducible tensors (H)

$$\mathcal{H}^{\vec{J}}(q_1, q_2) = \sum_{\bar{\Omega}_{\vec{k}}^H} \hat{h}_{\vec{k}}^{\vec{J}}(t_1, t_2) \bar{\mathcal{H}}_{\vec{k}}^{\vec{J}},$$

$$\bar{\mathcal{H}}_{\vec{k}}^{\vec{J}} = \sum_{\tilde{\Omega}_{\vec{k}', \vec{n}}^H} h_{\vec{n}}^{\vec{k}'(\vec{k}; \vec{J})} S_{\vec{k}', \vec{n}}^{H; \vec{J}},$$

$$\tilde{\Omega}_{\vec{k}', \vec{n}}^H \left\{ \begin{array}{l} \bar{J}_r \geq 0, r \in \Omega_H \\ \sum_{\substack{s \in \Omega_H \\ s \neq r}} k'_{rs} \leq \sum_{\substack{s \in \Omega_H \\ s \neq r}} k_{rs} \\ n_r, k'_{rs} \in \mathbb{Z}_+ \end{array} \right.$$

$$\bar{\Omega}_{\vec{k}}^H \left\{ \begin{array}{l} 0 \leq k_{11'} + k_{12} + k_{12'} \leq J_1 \\ 0 \leq k_{11'} + k_{1'2} + k_{1'2'} \leq J_{1'} \\ 0 \leq k_{22'} + k_{12} + k_{1'2} \leq J_2 \\ 0 \leq k_{22'} + k_{12'} + k_{1'2'} \leq J_{2'} \\ k_{rs} \in \mathbb{Z}_+, \end{array} \right.$$

Irreducible tensors (H)

$$\begin{aligned}
 \hat{\mathcal{A}}_i^H = & \hat{O}_{n_i}^{-1} \left(1 + 2\bar{J}_i \hat{O}_{k'_{ii'}}^{+1} + 2n_{i'} \hat{O}_{k'_{ii'}}^{+2} \hat{O}_{n_{i'}}^{-1} + \right. \\
 & + 2k'_{jj'} \hat{O}_{k'_{ij}}^{+1} \hat{O}_{k'_{ij'}}^{+1} \hat{O}_{k'_{jj'}}^{-1} - 2\lambda_m \bar{J}_j \bar{J}_{j'} \hat{O}_{k'_{ij}}^{+1} \hat{O}_{k'_{ij'}}^{+1} + \\
 & + \sum_{\substack{r \neq i, i' \\ r \in \Omega_H}} \left[2\chi_m \bar{J}_r \hat{O}_{k'_{ir}}^{+1} + 2k'_{i'r} \hat{O}_{k'_{ii'}}^{+1} \hat{O}_{k'_{ir}}^{+1} \hat{O}_{k'_{i'r}}^{-1} \right. \\
 & \left. \left. + 2n_r \hat{O}_{k'_{ir}}^{+2} \hat{O}_{n_r}^{-1} - \lambda_m \bar{J}_r (\bar{J}_r - 1) \hat{O}_{k'_{ir}}^{+2} \right] \right), \\
 & r, i, i', j, j' \in \Omega_H; \\
 & \{j, j'\} \neq \{i, i'\}, j \equiv \bar{i}, j' \equiv \bar{i}'.
 \end{aligned}$$

$$h_{\vec{n}}^{\vec{k}'} = \frac{1}{2(c^{J_i} + n_i - 1)} \hat{\mathcal{A}}_i^H h_{\vec{n}}^{\vec{k}'}$$

General multivariate recurrence

$$\begin{aligned} b_n^{\vec{\kappa}'} &= \left(\gamma + \sum_{i=1}^L \omega_i \hat{O}^{\vec{\Delta}_i} \right) b_{n-1}^{\vec{\kappa}'} \\ b_0^{\vec{\kappa}'} &= B^{\vec{\kappa}'} \end{aligned}$$

$$\begin{aligned} \vec{\Delta}_i &= \{\Delta_{i,1}, \Delta_{i,2}, \dots, \Delta_{i,N}\} \\ (\mathcal{M}_\Delta)_{ij} &\equiv \left(\vec{\Delta}_i \right)_j \end{aligned}$$

$$\begin{aligned} \vec{\kappa} &= \{k_1, k_2, \dots, k_N\} \\ \vec{\kappa}' + \vec{\Delta} &= \{k'_1 + \Delta_1, k'_2 + \Delta_2, \dots, k'_N + \Delta_N\} \end{aligned}$$

$$b_n^{\vec{\kappa}'} = \sum_{m=0}^n \frac{n!}{(n-m)!} \gamma^{n-m} \sum_{\substack{\sum_{j=1}^L m_j = m \\ m_i \geq 0}} \prod_{i=1}^L \frac{\omega_i^{m_i}}{m_i!} B^{\vec{\kappa}' + \sum_{r=1}^L m_r \vec{\Delta}_r} \equiv \hat{\mathcal{S}}^{\vec{\omega}, \mathcal{M}_\Delta} \left[B^{\vec{\kappa}'} \right]$$

$$\hat{\mathcal{S}}_{a_1, \dots, a_{N'}}^{\vec{\omega}, \mathcal{M}_\Delta} \left[B^{\vec{\kappa}'} \right] = \hat{\mathcal{S}}^{\vec{\omega}, \mathcal{M}_\Delta} \left[B^{\vec{\kappa}'} \right] \Big|_{m_{a_1} = \dots = m_{a_{N'}} = 0}$$

Solution (Y)

Change of variables and initial condition

$$y_{\vec{n}}^{\vec{k}'} \equiv y_{\vec{n}}^{\vec{k}', (\vec{k}; \vec{J})} = \frac{2^{\sum_{\substack{r, s \in \Omega_Y \\ r \neq s}} (k_{rs} - k'_{rs}) - \sum_{r \in \Omega_Y} n_r}}{\prod_{r \in \Omega_Y} (c^{J_r})_{n_r}} y_{\vec{n}}^{\vec{k}'},$$

$$y_{\vec{n}}^{\vec{k}} = 1, \quad y_{\vec{0}}^{\vec{k}'} = \delta_{\vec{k}', \vec{k}} \equiv \prod_{\substack{r, s \in \Omega_Y \\ r \neq s}} \delta_{k'_{rs} k_{rs}}.$$

$$y_{\vec{n}}^{\vec{k}'} = y_{i; \vec{n}}^{\vec{k}'} \mathbf{A}_{i; \vec{n}}^{\vec{k}'},$$

$$\mathbf{A}_{i; \vec{n}}^{\vec{k}'} = \prod_{\substack{r \neq i \\ r \in \Omega_Y}} \bar{J}_r! n_r! \left(c^{J_r} \right)_{n_r} \cdot k'_{\hat{r} \hat{s}}!,$$

$$i, \hat{r}, \hat{s} \in \Omega_Y, \quad i \neq \hat{r} \neq \hat{s}.$$

$$\vec{\kappa}_i'^Y = \{k'_{ir}, k'_{is}, k'_{rs}, n_r, n_s\},$$

$$\vec{\kappa}_i^Y = \{k_{ir}, k_{is}, k_{rs}, n_r, n_s\},$$

$$i, r, s \in \Omega_Y, \quad i \neq r \neq s.$$

Solution (Y)

Parameters for the general recurrence

$$\vec{\omega}_i^Y = \{1, \hat{\lambda}_{rs}^i/2, \bar{\chi}_{ir}, -\tilde{\lambda}_{ir}^j/4, \bar{\chi}_{is}, -\tilde{\lambda}_{is}^j/4, 1, 1\},$$

$$\mathcal{M}_{\Delta}^Y = \begin{pmatrix} 1 & 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & -1 & 0 \\ 0 & 2 & 0 & 0 & -1 \end{pmatrix}$$

Solution (Y)

$$\begin{aligned}
 \mathbf{y}_{j; \vec{n}}^{\vec{k}'} \Big|_{n_i \neq 0}^{n_{r,s}=0} &= \frac{\mathbf{A}_{i; \vec{n}}^{\vec{k}'} \Big|_{n_i \neq 0}^{n_{r,s}=0}}{\mathbf{A}_{j; \vec{n}}^{\vec{k}'} \Big|_{n_i \neq 0}^{n_{r,s}=0} \mathbf{A}_{i; \vec{0}}^{\vec{k}'}} \times \\
 &\times \hat{\mathcal{S}}_{4,5}^{\vec{\omega}_i^Y, \mathcal{M}_\Delta^Y} [\delta_{\vec{k}' \vec{k}}], \\
 j = r, s; i, r, s \in \Omega_Y, i \neq r \neq s,
 \end{aligned}$$

Solution (Y)

$$\mathbf{y}_{j; \vec{n}}^{\vec{k}'} \Big|_{n_i \neq 0}^{n_{r,s}=0} = \frac{\mathbf{A}_{i; \vec{n}}^{\vec{k}'} \Big|_{n_i \neq 0}^{n_{r,s}=0}}{\mathbf{A}_{j; \vec{n}}^{\vec{k}'} \Big|_{n_i \neq 0}^{n_{r,s}=0} \mathbf{A}_{i; \vec{0}}^{\vec{k}'}} \times$$

$$\times \hat{\mathcal{S}}_{4,5}^{\vec{\omega}_i^Y, \mathcal{M}_\Delta^Y} [\delta_{\vec{k}' \vec{k}}],$$

$$j = r, s; i, r, s \in \Omega_Y, i \neq r \neq s,$$

$$\mathbf{y}_{i; \vec{n}}^{\vec{k}'} \Big|_{i,j \neq 0}^{n_a=0, a \neq i,j} = \hat{\mathcal{S}}_{a_1}^{\vec{\omega}_i^Y, \mathcal{M}_\Delta^Y} \left[\mathbf{y}_{i; \vec{n}}^{\vec{k}'} \Big|_{n_j \neq 0, n_i=0}^{n_a=0, a \neq i,j} \right],$$

$$\{i, j\} = \{1, 2\}, \{2, 1\}, \{1, 3\}, \{3, 1\}, \{2, 3\}, \{3, 2\},$$

$$a_1 = \begin{cases} 4, & j = s, a = r \\ 5, & j = r, a = s. \end{cases}$$

Solution (Y)

$$\mathbf{y}_{j; \vec{n}}^{\vec{k}'} \Big|_{n_i \neq 0}^{n_{r,s}=0} = \frac{\mathbf{A}_{i; \vec{n}}^{\vec{k}'} \Big|_{n_i \neq 0}^{n_{r,s}=0}}{\mathbf{A}_{j; \vec{n}}^{\vec{k}'} \Big|_{n_i \neq 0}^{n_{r,s}=0} \mathbf{A}_{i; \vec{0}}^{\vec{k}'}} \times$$

$$\times \hat{\mathcal{S}}_{4,5}^{\vec{\omega}_i^Y, \mathcal{M}_\Delta^Y} [\delta_{\vec{k}', \vec{k}}],$$

$$j = r, s; i, r, s \in \Omega_Y, i \neq r \neq s,$$

$$\mathbf{y}_{i; \vec{n}}^{\vec{k}'} \Big|_{i,j \neq 0}^{n_a=0, a \neq i,j} = \hat{\mathcal{S}}_{a_1}^{\vec{\omega}_i^Y, \mathcal{M}_\Delta^Y} \left[\mathbf{y}_{i; \vec{n}}^{\vec{k}'} \Big|_{n_j \neq 0, n_i=0}^{n_a=0, a \neq i,j} \right],$$

$$\{i, j\} = \{1, 2\}, \{2, 1\}, \{1, 3\}, \{3, 1\}, \{2, 3\}, \{3, 2\},$$

$$a_1 = \begin{cases} 4, & j = s, a = r \\ 5, & j = r, a = s. \end{cases}$$

$$\mathbf{y}_{\vec{n}}^{\vec{k}'} \Big|_{n_{i,j} \neq 0}^{n_a=0, a \neq i,j} =$$

$$= \frac{1}{2} \sum_{\hat{j} \in \{i,j\}} \mathbf{y}_{\hat{j}; \vec{n}}^{\vec{k}'} \Big|_{n_{i,j} \neq 0}^{n_a=0, a \neq i,j} \mathbf{A}_{\hat{j}; \vec{n}}^{\vec{k}'} \Big|_{n_{i,j} \neq 0}^{n_a=0, a \neq i,j}$$

Solution (Y)

$$\mathbf{y}_{j; \vec{n}}^{\vec{k}'} \Big|_{n_i \neq 0}^{n_{r,s}=0} = \frac{\mathbf{A}_{i; \vec{n}}^{\vec{k}'} \Big|_{n_i \neq 0}^{n_{r,s}=0}}{\mathbf{A}_{j; \vec{n}}^{\vec{k}'} \Big|_{n_i \neq 0}^{n_{r,s}=0} \mathbf{A}_{i; \vec{0}}^{\vec{k}'}} \times$$

$$\times \hat{\mathcal{S}}_{4,5}^{\vec{\omega}_i^Y, \mathcal{M}_\Delta^Y} [\delta_{\vec{k}', \vec{k}}],$$

$$j = r, s; i, r, s \in \Omega_Y, i \neq r \neq s,$$

$$\mathbf{y}_{i; \vec{n}}^{\vec{k}'} \Big|_{i,j \neq 0}^{n_a=0, a \neq i,j} = \hat{\mathcal{S}}_{a_1}^{\vec{\omega}_i^Y, \mathcal{M}_\Delta^Y} \left[\mathbf{y}_{i; \vec{n}}^{\vec{k}'} \Big|_{n_j \neq 0, n_i=0}^{n_a=0, a \neq i,j} \right],$$

$$\{i, j\} = \{1, 2\}, \{2, 1\}, \{1, 3\}, \{3, 1\}, \{2, 3\}, \{3, 2\},$$

$$a_1 = \begin{cases} 4, & j = s, a = r \\ 5, & j = r, a = s. \end{cases}$$

$$\mathbf{y}_{\vec{n}}^{\vec{k}'} \Big|_{n_{i,j} \neq 0}^{n_a=0, a \neq i,j} =$$

$$= \frac{1}{2} \sum_{\hat{j} \in \{i,j\}} \mathbf{y}_{\hat{j}; \vec{n}}^{\vec{k}'} \Big|_{n_{i,j} \neq 0}^{n_a=0, a \neq i,j} \mathbf{A}_{\hat{j}; \vec{n}}^{\vec{k}'} \Big|_{n_{i,j} \neq 0}^{n_a=0, a \neq i,j}$$

$$\mathbf{y}_{i; \vec{n}}^{\vec{k}'} \Big|_{n_{r,s} \neq 0}^{n_i=0} = \frac{\mathbf{y}_{\vec{n}}^{\vec{k}'} \Big|_{n_{r,s} \neq 0}^{n_i=0}}{\mathbf{A}_{i; \vec{n}}^{\vec{k}'} \Big|_{n_{r,s} \neq 0}^{n_i=0}},$$

$$i, r, s \in \Omega_Y, i \neq r \neq s,$$

Solution (Y)

$$\mathbf{y}_{j; \vec{n}}^{\vec{k}'} \Big|_{n_i \neq 0}^{n_{r,s}=0} = \frac{\mathbf{A}_{i; \vec{n}}^{\vec{k}'} \Big|_{n_i \neq 0}^{n_{r,s}=0}}{\mathbf{A}_{j; \vec{n}}^{\vec{k}'} \Big|_{n_i \neq 0}^{n_{r,s}=0} \mathbf{A}_{i; \vec{0}}^{\vec{k}'}} \times$$

$$\times \hat{\mathcal{S}}_{4,5}^{\vec{\omega}_i^Y, \mathcal{M}_\Delta^Y} [\delta_{\vec{k}' \vec{k}}],$$

$$j = r, s; i, r, s \in \Omega_Y, i \neq r \neq s,$$

$$\mathbf{y}_{i; \vec{n}}^{\vec{k}'} \Big|_{i,j \neq 0}^{n_a=0, a \neq i,j} = \hat{\mathcal{S}}_{a_1}^{\vec{\omega}_i^Y, \mathcal{M}_\Delta^Y} \left[\mathbf{y}_{i; \vec{n}}^{\vec{k}'} \Big|_{n_j \neq 0, n_i=0}^{n_a=0, a \neq i,j} \right],$$

$$\{i, j\} = \{1, 2\}, \{2, 1\}, \{1, 3\}, \{3, 1\}, \{2, 3\}, \{3, 2\},$$

$$a_1 = \begin{cases} 4, & j = s, a = r \\ 5, & j = r, a = s. \end{cases}$$

$$\mathbf{y}_{\vec{n}}^{\vec{k}'} \Big|_{n_{i,j} \neq 0}^{n_a=0, a \neq i,j} =$$

$$= \frac{1}{2} \sum_{\hat{j} \in \{i,j\}} \mathbf{y}_{\hat{j}; \vec{n}}^{\vec{k}'} \Big|_{n_{i,j} \neq 0}^{n_a=0, a \neq i,j} \mathbf{A}_{\hat{j}; \vec{n}}^{\vec{k}'} \Big|_{n_{i,j} \neq 0}^{n_a=0, a \neq i,j}$$

$$\mathbf{y}_{i; \vec{n}}^{\vec{k}'} \Big|_{n_{r,s} \neq 0}^{n_i=0} = \frac{\mathbf{y}_{\vec{n}}^{\vec{k}'} \Big|_{n_{r,s} \neq 0}^{n_i=0}}{\mathbf{A}_{i; \vec{n}}^{\vec{k}'} \Big|_{n_{r,s} \neq 0}^{n_i=0}},$$

$$i, r, s \in \Omega_Y, i \neq r \neq s,$$

$$\mathbf{y}_{i; \vec{n}}^{\vec{k}'} = \hat{\mathcal{S}}^{\vec{\omega}_i^Y, \mathcal{M}_\Delta^Y} \left[\mathbf{y}_{i; \vec{n}}^{\vec{k}'} \Big|_{n_{r,s} \neq 0}^{n_i=0} \right]$$

Solution (Y)

$$\mathbf{y}_{j; \vec{n}}^{\vec{k}'} \Big|_{n_i \neq 0}^{n_{r,s}=0} = \frac{\mathbf{A}_{i; \vec{n}}^{\vec{k}'} \Big|_{n_i \neq 0}^{n_{r,s}=0}}{\mathbf{A}_{j; \vec{n}}^{\vec{k}'} \Big|_{n_i \neq 0}^{n_{r,s}=0} \mathbf{A}_{i; \vec{0}}^{\vec{k}'}} \times$$

$$\times \hat{\mathcal{S}}_{4,5}^{\vec{\omega}_i^Y, \mathcal{M}_\Delta^Y} [\delta_{\vec{k}' \vec{k}}],$$

$$j = r, s; i, r, s \in \Omega_Y, i \neq r \neq s,$$

$$\mathbf{y}_{i; \vec{n}}^{\vec{k}'} \Big|_{i,j \neq 0}^{n_a=0, a \neq i,j} = \hat{\mathcal{S}}_{a_1}^{\vec{\omega}_i^Y, \mathcal{M}_\Delta^Y} \left[\mathbf{y}_{i; \vec{n}}^{\vec{k}'} \Big|_{n_j \neq 0, n_i=0}^{n_a=0, a \neq i,j} \right],$$

$$\{i, j\} = \{1, 2\}, \{2, 1\}, \{1, 3\}, \{3, 1\}, \{2, 3\}, \{3, 2\},$$

$$a_1 = \begin{cases} 4, & j = s, a = r \\ 5, & j = r, a = s. \end{cases}$$

$$\mathbf{y}_{\vec{n}}^{\vec{k}'} \Big|_{n_{i,j} \neq 0}^{n_a=0, a \neq i,j} =$$

$$= \frac{1}{2} \sum_{\hat{j} \in \{i,j\}} \mathbf{y}_{\hat{j}; \vec{n}}^{\vec{k}'} \Big|_{n_{i,j} \neq 0}^{n_a=0, a \neq i,j} \mathbf{A}_{\hat{j}; \vec{n}}^{\vec{k}'} \Big|_{n_{i,j} \neq 0}^{n_a=0, a \neq i,j}$$

$$\mathbf{y}_{i; \vec{n}}^{\vec{k}'} \Big|_{n_{r,s} \neq 0}^{n_i=0} = \frac{\mathbf{y}_{\vec{n}}^{\vec{k}'} \Big|_{n_{r,s} \neq 0}^{n_i=0}}{\mathbf{A}_{i; \vec{n}}^{\vec{k}'} \Big|_{n_{r,s} \neq 0}^{n_i=0}},$$

$$i, r, s \in \Omega_Y, i \neq r \neq s,$$

$$\mathbf{y}_{i; \vec{n}}^{\vec{k}'} = \hat{\mathcal{S}}^{\vec{\omega}_i^Y, \mathcal{M}_\Delta^Y} \left[\mathbf{y}_{i; \vec{n}}^{\vec{k}'} \Big|_{n_{r,s} \neq 0}^{n_i=0} \right]$$

$$\mathbf{y}_{\vec{n}}^{\vec{k}'} = \frac{1}{3} \sum_{i \in \Omega_Y} \mathbf{y}_{i; \vec{n}}^{\vec{k}'} \mathbf{A}_{i; \vec{n}}^{\vec{k}'}$$

Solution (H)

Change of variables and initial condition

$$h_{\vec{n}}^{\vec{k}'} \equiv h_{\vec{n}}^{\vec{k}', (\vec{k}; \vec{J})} = \frac{2^{\sum_{\substack{r, s \in \Omega_H \\ r \neq s}} (k_{rs} - k'_{rs}) - \sum_{r \in \Omega_H} n_r}}{\prod_{r \in \Omega_H} (c^{J_r})_{n_r}} h_{\vec{n}}^{\vec{k}'},$$

$$h_{\vec{n}}^{\vec{k}} = 1, \quad h_{\vec{0}}^{\vec{k}'} = \delta_{\vec{k}' \vec{k}} \equiv \prod_{\substack{r, s \in \Omega_H \\ r \neq s}} \delta_{k'_{rs} k_{rs}}.$$

$$h_{\vec{n}}^{\vec{k}'} = h_{i; \vec{n}}^{\vec{k}'} A_{i; \vec{n}}^{\vec{k}'},$$

$$A_{i; \vec{n}}^{\vec{k}'} = \prod_{\substack{r \neq i \\ r \in \Omega_H}} \bar{J}_r! n_r! \left(c^{J_r} \right)_{n_r} \prod_{\substack{r, s \in \Omega_H \\ r \neq s \neq i}} k'_{rs}!, \quad i \in \Omega_H.$$

$$\begin{aligned} \vec{\kappa}_i'^H &= \{k'_{ii'}, k'_{ij}, k'_{ij'}, k'_{i'j}, k'_{i'j'}, k'_{jj'}, n_{i'}, n_j, n_{j'}\}, \\ \vec{\kappa}_i^H &= \{k_{ii'}, k_{ij}, k_{ij'}, k_{i'j}, k_{i'j'}, k_{jj'}, n_{i'}, n_j, n_{j'}\}. \end{aligned}$$

$$\begin{aligned} r, i, i', j, j' &\in \Omega_H; \\ \{j, j'\} &\neq \{i, i'\}, \quad j \equiv \bar{i}, \quad j' \equiv \bar{i}'. \end{aligned}$$

Solution (H)

Parameters for the general recurrence

$$\vec{\omega}^H = \{1, 1, -\lambda_m/2, \chi_m, 1, -\lambda_m/4, \chi_m, 1, -\lambda_m/4, 1, 1, 1\},$$

$$\mathcal{M}_{\Delta}^H = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

Solution (H)

$$\mathbf{h}_{r; \vec{n}}^{\vec{k}'} \Big|_{n_i \neq 0}^{n_{i'}, j, j' = 0} = \frac{\mathbf{A}_{i; \vec{n}}^{\vec{k}'} \Big|_{n_i \neq 0}^{n_{i'}, j, j' = 0}}{\mathbf{A}_{r; \vec{n}}^{\vec{k}'} \Big|_{n_i \neq 0}^{n_{i'}, j, j' = 0} \mathbf{A}_{i; \vec{0}}^{\vec{k}'}} \times \hat{\mathcal{S}}_{10,11,12}^{\vec{\omega}^H, \mathcal{M}_{\Delta}^H} [\delta_{\vec{k}' \vec{k}}],$$

$r = i', j, j'; j \equiv \bar{i}, j' \equiv \bar{i}'$

Solution (H)

$$\mathbf{h}_{\vec{r}; \vec{n}}^{\vec{k}'} \Big|_{n_i \neq 0}^{n_{i'}, j, j' = 0} = \frac{\mathbf{A}_{\vec{i}; \vec{n}}^{\vec{k}'} \Big|_{n_i \neq 0}^{n_{i'}, j, j' = 0}}{\mathbf{A}_{\vec{r}; \vec{n}}^{\vec{k}'} \Big|_{n_i \neq 0}^{n_{i'}, j, j' = 0} \mathbf{A}_{\vec{i}; \vec{0}}^{\vec{k}'}} \times \hat{\mathcal{S}}_{10,11,12}^{\vec{\omega}^H, \mathcal{M}_{\Delta}^H} [\delta_{\vec{k}' \vec{k}}],$$

$r = i', j, j'; j \equiv \bar{i}, j' \equiv \bar{i}'$

$$\mathbf{h}_{\vec{r}_i; \vec{n}}^{\vec{k}'} \Big|_{n_{r_1, r_2} \neq 0}^{n_a = 0, a \neq r_{1,2}} = \hat{\mathcal{S}}_{a_1, a_2}^{\vec{\omega}^H, \mathcal{M}_{\Delta}^H} \left[\mathbf{h}_{\vec{r}_i; \vec{n}}^{\vec{k}'} \Big|_{n_{r_j} \neq 0, n_{r_i} = 0}^{n_a = 0, a \neq r_{1,2}} \right] \quad \{a_1, a_2\} = \begin{cases} \{11, 12\}, & r_j = r'_i, \\ \{10, 12\}, & r_j = \bar{r}_i, \\ \{10, 11\}, & r_j = \bar{r}'_i. \end{cases}$$

$\{i, j\} = \{1, 2\}, \{2, 1\},$

Solution (H)

$$\mathbf{h}_{\vec{r}; \vec{n}}^{\vec{k}'} \Big|_{n_i \neq 0}^{n_{i'}, j, j' = 0} = \frac{\mathbf{A}_{\vec{i}; \vec{n}}^{\vec{k}'} \Big|_{n_i \neq 0}^{n_{i'}, j, j' = 0}}{\mathbf{A}_{\vec{r}; \vec{n}}^{\vec{k}'} \Big|_{n_i \neq 0}^{n_{i'}, j, j' = 0} \mathbf{A}_{\vec{i}; \vec{0}}^{\vec{k}'}} \times \hat{\mathcal{S}}_{10,11,12}^{\vec{\omega}^H, \mathcal{M}_{\Delta}^H} [\delta_{\vec{k}' \vec{k}}],$$

$r = i', j, j'; j \equiv \bar{i}, j' \equiv \bar{i}'$

$$\mathbf{h}_{\vec{r}_i; \vec{n}}^{\vec{k}'} \Big|_{n_{r_1, r_2} \neq 0}^{n_a = 0, a \neq r_{1,2}} = \hat{\mathcal{S}}_{a_1, a_2}^{\vec{\omega}^H, \mathcal{M}_{\Delta}^H} \left[\mathbf{h}_{\vec{r}_i; \vec{n}}^{\vec{k}'} \Big|_{n_{r_j} \neq 0, n_{r_i} = 0}^{n_a = 0, a \neq r_{1,2}} \right] \quad \{a_1, a_2\} = \begin{cases} \{11, 12\}, & r_j = r'_i, \\ \{10, 12\}, & r_j = \bar{r}_i, \\ \{10, 11\}, & r_j = \bar{r}'_i. \end{cases}$$

$\{i, j\} = \{1, 2\}, \{2, 1\},$

$$\mathbf{h}_{\vec{n}}^{\vec{k}'} \Big|_{n_{r_1, r_2} \neq 0}^{n_a = 0, a \neq r_{1,2}} = \frac{1}{2} \sum_{i \in \{1, 2\}} \mathbf{h}_{\vec{r}_i; \vec{n}}^{\vec{k}'} \Big|_{n_{r_1, r_2} \neq 0}^{n_a = 0, a \neq r_{1,2}} \mathbf{A}_{\vec{r}_i; \vec{n}}^{\vec{k}'} \Big|_{n_{r_1, r_2} \neq 0}^{n_a = 0, a \neq r_{1,2}}$$

Solution (H)

$$\mathbf{h}_{\vec{r}; \vec{n}}^{\vec{k}'} \Big|_{n_i \neq 0}^{n_{i'}, j, j' = 0} = \frac{\mathbf{A}_{\vec{i}; \vec{n}}^{\vec{k}'} \Big|_{n_i \neq 0}^{n_{i'}, j, j' = 0}}{\mathbf{A}_{\vec{r}; \vec{n}}^{\vec{k}'} \Big|_{n_i \neq 0}^{n_{i'}, j, j' = 0} \mathbf{A}_{\vec{i}; \vec{0}}^{\vec{k}'}} \times \hat{\mathcal{S}}_{10,11,12}^{\vec{\omega}^H, \mathcal{M}_{\Delta}^H} [\delta_{\vec{k}' \vec{k}}],$$

$r = i', j, j'; j \equiv \bar{i}, j' \equiv \bar{i}'$

$$\mathbf{h}_{\vec{r}_i; \vec{n}}^{\vec{k}'} \Big|_{n_{r_1, r_2} \neq 0}^{n_a = 0, a \neq r_{1,2}} = \hat{\mathcal{S}}_{a_1, a_2}^{\vec{\omega}^H, \mathcal{M}_{\Delta}^H} \left[\mathbf{h}_{\vec{r}_i; \vec{n}}^{\vec{k}'} \Big|_{n_{r_j} \neq 0, n_{r_i} = 0}^{n_a = 0, a \neq r_{1,2}} \right] \quad \{a_1, a_2\} = \begin{cases} \{11, 12\}, & r_j = r'_i, \\ \{10, 12\}, & r_j = \bar{r}_i, \\ \{10, 11\}, & r_j = \bar{r}'_i. \end{cases}$$

$\{i, j\} = \{1, 2\}, \{2, 1\},$

$$\mathbf{h}_{\vec{n}}^{\vec{k}'} \Big|_{n_{r_1, r_2} \neq 0}^{n_a = 0, a \neq r_{1,2}} = \frac{1}{2} \sum_{i \in \{1, 2\}} \mathbf{h}_{\vec{r}_i; \vec{n}}^{\vec{k}'} \Big|_{n_{r_1, r_2} \neq 0}^{n_a = 0, a \neq r_{1,2}} \mathbf{A}_{\vec{r}_i; \vec{n}}^{\vec{k}'} \Big|_{n_{r_1, r_2} \neq 0}^{n_a = 0, a \neq r_{1,2}}$$

Solution (H)



$$\mathbf{h}_{r_i; \vec{n}}^{\vec{k}'} \Big|_{\substack{n_a=0, a \neq r_{1,2,3} \\ n_{r_1, r_2, r_3} \neq 0}} = \hat{\mathcal{S}}_{a_1}^{\vec{\omega}^H, \mathcal{M}_\Delta^H} \left[\mathbf{h}_{r_i; \vec{n}}^{\vec{k}'} \Big|_{\substack{n_a=0, a \neq r_{i,j,k} \\ n_{r_j, r_k} \neq 0, n_{r_i}=0}} \right], a_1 = \begin{cases} 12, r_j = r'_i, r_k = \bar{r}_i \\ 11, r_j = r'_i, r_k = \bar{r}'_i, \\ 10, r_j = \bar{r}_i, r_k = \bar{r}'_i. \end{cases}$$

Solution (H)

$$\mathbf{h}_{\vec{k}'; \vec{n}} \Big|_{\substack{n_a=0, a \neq r_{1,2,3} \\ n_{r_1, r_2, r_3} \neq 0}} = \hat{\mathcal{S}}_{a_1}^{\vec{\omega}^H, \mathcal{M}_\Delta^H} \left[\mathbf{h}_{\vec{k}'; \vec{n}} \Big|_{\substack{n_a=0, a \neq r_{i,j,k} \\ n_{r_j, r_k} \neq 0, n_{r_i}=0}} \right], a_1 = \begin{cases} 12, r_j = r'_i, r_k = \bar{r}_i \\ 11, r_j = r'_i, r_k = \bar{r}'_i, \\ 10, r_j = \bar{r}_i, r_k = \bar{r}'_i. \end{cases}$$

$$\mathbf{h}_{\vec{k}'; \vec{n}} \Big|_{\substack{n_a=0, a \neq r_{1,2,3} \\ n_{r_1, r_2, r_3} \neq 0}} = \frac{1}{3} \sum_{i \in \{1,2,3\}} \mathbf{h}_{\vec{k}'; \vec{n}} \Big|_{\substack{n_a=0, a \neq r_{1,2,3} \\ n_{r_1, r_2, r_3} \neq 0}} \times \mathbf{A}_{\vec{k}'; \vec{n}} \Big|_{\substack{n_a=0, a \neq r_{1,2,3} \\ n_{r_1, r_2, r_3} \neq 0}}$$

Solution (H)

$$\mathbf{h}_{\vec{k}'; \vec{n}} \Big|_{\substack{n_a=0, a \neq r_{1,2,3} \\ n_{r_1, r_2, r_3} \neq 0}} = \hat{\mathcal{S}}_{a_1}^{\vec{\omega}^H, \mathcal{M}_{\Delta}^H} \left[\mathbf{h}_{\vec{k}'; \vec{n}} \Big|_{\substack{n_a=0, a \neq r_{i,j,k} \\ n_{r_j, r_k} \neq 0, n_{r_i}=0}} \right], a_1 = \begin{cases} 12, r_j = r'_i, r_k = \bar{r}_i \\ 11, r_j = r'_i, r_k = \bar{r}'_i, \\ 10, r_j = \bar{r}_i, r_k = \bar{r}'_i. \end{cases}$$

$$\mathbf{h}_{\vec{k}'; \vec{n}} \Big|_{\substack{n_a=0, a \neq r_{1,2,3} \\ n_{r_1, r_2, r_3} \neq 0}} = \frac{1}{3} \sum_{i \in \{1,2,3\}} \mathbf{h}_{\vec{k}'; \vec{n}} \Big|_{\substack{n_a=0, a \neq r_{1,2,3} \\ n_{r_1, r_2, r_3} \neq 0}} \times \mathbf{A}_{\vec{k}'; \vec{n}} \Big|_{\substack{n_a=0, a \neq r_{1,2,3} \\ n_{r_1, r_2, r_3} \neq 0}}$$

$$\mathbf{h}_{\vec{k}'; \vec{n}} = \hat{\mathcal{S}}^{\vec{\omega}^H, \mathcal{M}_{\Delta}^H} \left[\mathbf{h}_{\vec{k}'; \vec{n}} \Big|_{\substack{n_r=0 \\ n_a \neq 0, a \neq r}} \right]$$

Solution (H)

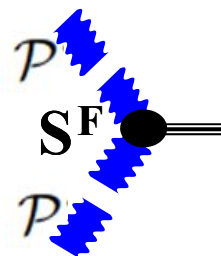
$$\mathbf{h}_{r_i; \vec{n}}^{\vec{k}'} \Big|_{\substack{n_a=0, a \neq r_{1,2,3} \\ n_{r_1, r_2, r_3} \neq 0}} = \hat{\mathcal{S}}_{a_1}^{\vec{\omega}^H, \mathcal{M}_\Delta^H} \left[\mathbf{h}_{r_i; \vec{n}}^{\vec{k}'} \Big|_{\substack{n_a=0, a \neq r_{i,j,k} \\ n_{r_j, r_k} \neq 0, n_{r_i}=0}} \right], a_1 = \begin{cases} 12, r_j = r'_i, r_k = \bar{r}_i \\ 11, r_j = r'_i, r_k = \bar{r}'_i, \\ 10, r_j = \bar{r}_i, r_k = \bar{r}'_i. \end{cases}$$

$$\mathbf{h}_{\vec{n}}^{\vec{k}'} \Big|_{\substack{n_a=0, a \neq r_{1,2,3} \\ n_{r_1, r_2, r_3} \neq 0}} = \frac{1}{3} \sum_{i \in \{1,2,3\}} \mathbf{h}_{r_i; \vec{n}}^{\vec{k}'} \Big|_{\substack{n_a=0, a \neq r_{1,2,3} \\ n_{r_1, r_2, r_3} \neq 0}} \times \mathbf{A}_{r_i; \vec{n}}^{\vec{k}'} \Big|_{\substack{n_a=0, a \neq r_{1,2,3} \\ n_{r_1, r_2, r_3} \neq 0}}$$

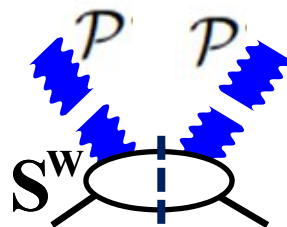
$$\mathbf{h}_{r; \vec{n}}^{\vec{k}'} = \hat{\mathcal{S}}^{\vec{\omega}^H, \mathcal{M}_\Delta^H} \left[\mathbf{h}_{r; \vec{n}}^{\vec{k}'} \Big|_{\substack{n_r=0 \\ n_a \neq 0, a \neq r}} \right]$$

$$\mathbf{h}_{\vec{n}}^{\vec{k}'} = \frac{1}{4} \sum_{r \in \Omega_H} \mathbf{h}_{r; \vec{n}}^{\vec{k}'} \mathbf{A}_{r; \vec{n}}^{\vec{k}'}$$

Other method to find solutions

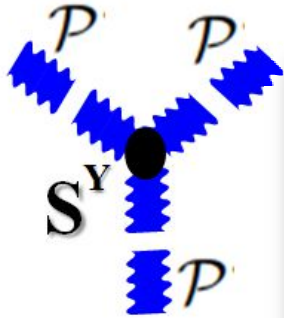


$$\bar{\mathcal{F}}_{k(\mu)_{J_1}(\nu)_{J_2}}^{J_1, J_2} = S_{k,00}^F \{J_1, J_2\}_{(\alpha)_{J_1}(\beta)_{J_2}} \mathcal{P}_{(\alpha)_{J_1}(\mu)_{J_1}}^{J_1} \mathcal{P}_{(\beta)_{J_2}(\nu)_{J_2}}^{J_2}$$

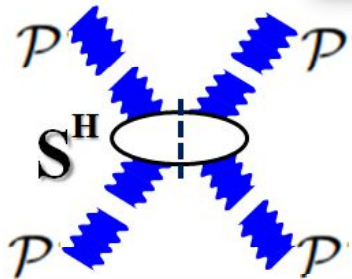


$$\bar{\mathcal{W}}_{k(\mu)_{J_1}(\nu)_{J_2}}^{J_1, J_{1'}} = S_{k,00}^W \{J_1, J_{1'}\}_{(\alpha)_{J_1}(\beta)_{J_{1'}}} \mathcal{P}_{(\alpha)_{J_1}(\mu)_{J_1}}^{J_1} \mathcal{P}_{(\beta)_{J_{1'}}(\mu')_{J_{1'}}}^{J_{1'}}$$

Other method to find solutions



$$\bar{y}_{\vec{k}}^{\vec{J}}(\mu)_{J_1}(\nu)_{J_2}(\rho)_{J_3} = S_{\vec{k}, \vec{0}}^{Y; \vec{J}}(\alpha)_{J_1}(\alpha')_{J_2}(\alpha'')_{J_3} \times \\ \times \mathcal{P}_{(\alpha)_{J_1}(\mu)_{J_1}}^{J_1} \mathcal{P}_{(\alpha')_{J_2}(\nu)_{J_2}}^{J_2} \mathcal{P}_{(\alpha'')_{J_3}(\rho)_{J_3}}^{J_3},$$



$$\bar{\mathcal{H}}_{\vec{k}}^{\vec{J}}(\mu)_{J_1}(\mu')_{J_1'}(\nu)_{J_2}(\nu')_{J_2'} = S_{\vec{k}, \vec{0}}^{H; \vec{J}}(\alpha)_{J_1}(\alpha')_{J_1'}(\beta)_{J_2}(\beta')_{J_2'} \times \\ \times \mathcal{P}_{(\alpha)_{J_1}(\mu)_{J_1}}^{J_1} \mathcal{P}_{(\alpha')_{J_1'}(\mu')_{J_1'}}^{J_1'} \mathcal{P}_{(\beta)_{J_2}(\nu)_{J_2}}^{J_2} \mathcal{P}_{(\beta')_{J_2'}(\nu')_{J_2'}}^{J_2'},$$

Non-conserved vertexes expansions

Symmetric Traceless

$$V^J \equiv \sum_{r=0}^J \bar{v}_r \left[\mathcal{V}^{J-r} q^r \right]$$

$$\left[\mathcal{V}^{J-r} q^r \right] = \sum_{u=0}^{[r/2]} \tilde{v}_u^J \left(\mathcal{V}_{(1)}^{J-r} q_{(1)}^r g_{(11)}^u \right)$$

Recurrence + solution

Orthogonality

$$\left[\mathcal{V}^{J-r} q^r \right] \otimes \left[\mathcal{V}^{J-s} q^s \right] = 0, \quad r \neq s$$

$$\tilde{v}_{u-1}^J - 2\tilde{v}_u^J (\tilde{c}^J + u - 1) = 0,$$

$$\tilde{c}^J = -(J + (D - 4)/2).$$

$$\tilde{v}_u^J = \frac{q^{2u}}{2^u (\tilde{c}^J)_u}.$$

Non-conserved vertexes expansions

$$W^{J_1, J_{1'}} \equiv \sum_{r=0}^{J_1} \sum_{r'=0}^{J_{1'}} \bar{\omega}_{rr'} \left[\mathcal{W}_{(1), (1')}^{J_1-r, J_{1'}-r'} q_{(1)}^r q_{(1')}^{r'} \right]$$

Symmetric Traceless

$$\left[\mathcal{W}_{(1), (1')}^{J_1-r, J_{1'}-r'} q_{(1)}^r q_{(1')}^{r'} \right] = \sum_{u=0}^{[r/2]} \sum_{u'=0}^{[r'/2]} \tilde{\omega}_{u, u'}^{J_1, J_{1'}} \times \\ \left(q_{(1)}^{r-2u} g_{(11)}^u \mathcal{W}_{(1), (1')}^{J_1-r, J_{1'}-r'} g_{(1'1')}^{u'} q_{(1')}^{r'-2u'} \right)$$

Orthogonality

$$\left[\mathcal{W}_{(1), (1')}^{J_1-r, J_{1'}-r'} q_{(1)}^r q_{(1')}^{r'} \right] \otimes \left[\mathcal{W}_{(1), (1')}^{J_1-s, J_{1'}-s'} q_{(1)}^s q_{(1')}^{s'} \right] = 0 \\ r \neq s, r' \neq s';$$

Solution

$$\tilde{\omega}_{u, u'}^{J_1, J_{1'}} = \tilde{v}_u^{J_1} \tilde{v}_{u'}^{J_{1'}}$$

Non-conserved vertexes expansions

$$F^{J_1, J_2} \equiv \sum_{r=0}^{J_1} \sum_{s=0}^{J_2} \bar{f}_{rs} \left[\mathcal{F}_{(1),(2)}^{J_1-r, J_2-s} q_{1(1)}^r q_{2(2)}^s \right]$$

**Symmetric
Traceless**

$$\left[\mathcal{F}_{(1),(2)}^{J_1-r, J_2-s} q_{1(1)}^r q_{2(2)}^s \right] = \sum_{u_1=0}^{[r/2]} \sum_{u_2=0}^{[s/2]} \tilde{f}_{u_1, u_2}^{J_1, J_2} \times \\ \left(q_{1(1)}^{r-2u_1} g_{(11)}^{u_1} \mathcal{F}_{(1),(2)}^{J_1-r, J_2-s} g_{(22)}^{u_2} q_{2(2)}^{s-2u_2} \right)$$

Orthogonality

$$\left[\mathcal{F}_{(1),(2)}^{J_1-r_1, J_2-r_2} q_{1(1)}^{r_1} q_{2(2)}^{r_2} \right] \otimes \left[\mathcal{F}_{(1),(2)}^{J_1-s_1, J_2-s_2} q_{1(1)}^{s_1} q_{2(2)}^{s_2} \right] = 0 \\ r_1 \neq s_1, r_2 \neq s_2$$

Solution

$$\tilde{f}_{u_1, u_2}^{J_1, J_2} = \tilde{v}_{u_1}^{J_1} \tilde{v}_{u_2}^{J_2}$$

Non-conserved vertexes expansions

$$Y^{\vec{J}} \equiv \sum_{\vec{r}=\vec{0}}^{\vec{J}} \bar{y}_{\vec{r}} \left[\mathcal{Y}_{(1),(2),(3)}^{\vec{J}-\vec{r}} q_{1(1)}^{r_1} q_{2(2)}^{r_2} q_{3(3)}^{r_3} \right]$$

Orthogonality

Symmetric Traceless

$$\left[\mathcal{Y}_{(1),(2),(3)}^{\vec{J}-\vec{r}} q_{1(1)}^{r_1} q_{2(2)}^{r_2} q_{3(3)}^{r_3} \right] = \sum_{\vec{u}=\vec{0}}^{[\vec{r}/2]} \tilde{y}_{\vec{u}}^{\vec{J}} \left(\mathcal{Y}_{(1),(2),(3)}^{\vec{J}-\vec{r}} g_{(11)}^{u_1} q_{1(1)}^{r_1-2u_1} \times g_{(22)}^{u_2} q_{2(2)}^{r_2-2u_2} g_{(33)}^{u_3} q_{3(3)}^{r_3-2u_3} \right)$$

$$\left[\mathcal{Y}_{(1),(2),(3)}^{\vec{J}-\vec{r}} q_{1(1)}^{r_1} q_{2(2)}^{r_2} q_{3(3)}^{r_3} \right] \otimes \left[\mathcal{Y}_{(1),(2),(3)}^{\vec{J}-\vec{s}} q_{1(1)}^{s_1} q_{2(2)}^{s_2} q_{3(3)}^{s_3} \right] = 0, \\ \vec{r} \neq \vec{s},$$

Solution

$$\tilde{y}_{\vec{u}}^{\vec{J}} = \prod_{r \in \Omega_Y} \tilde{v}_{u_r}^{J_r}$$

Non-conserved vertexes expansions

$$H^{\vec{J}} \equiv \sum_{\vec{r}=\vec{0}}^{\vec{J}} \bar{h}_{\vec{r}} \left[\mathcal{H}_{(1),(1'),(2),(2')}^{\vec{J}-\vec{r}} q_{1(1)}^{r_1} q_{1(1')}^{r_{1'}} q_{2(2)}^{r_2} q_{2(2')}^{r_{2'}} \right]$$

$$\begin{aligned} & \left[\mathcal{H}_{(1),(1'),(2),(2')}^{\vec{J}-\vec{r}} q_{1(1)}^{r_1} q_{1(1')}^{r_{1'}} q_{2(2)}^{r_2} q_{2(2')}^{r_{2'}} \right] = \\ & \sum_{\vec{u}=\vec{0}}^{[\vec{r}/2]} \tilde{h}_{\vec{u}}^{\vec{J}} \left(\mathcal{H}_{(1),(1'),(2),(2')}^{\vec{J}-\vec{r}} g_{(11)}^{u_1} q_{1(1)}^{r_1-2u_1} \times \right. \\ & \left. g_{(1'1')}^{u_{1'}} q_{1(1')}^{i_{1'}-2u_{1'}} g_{(22)}^{u_2} q_{2(2)}^{i_2-2u_2} g_{(2'2')}^{u_{2'}} q_{2(2')}^{i_{2'}-2u_{2'}} \right) \end{aligned}$$

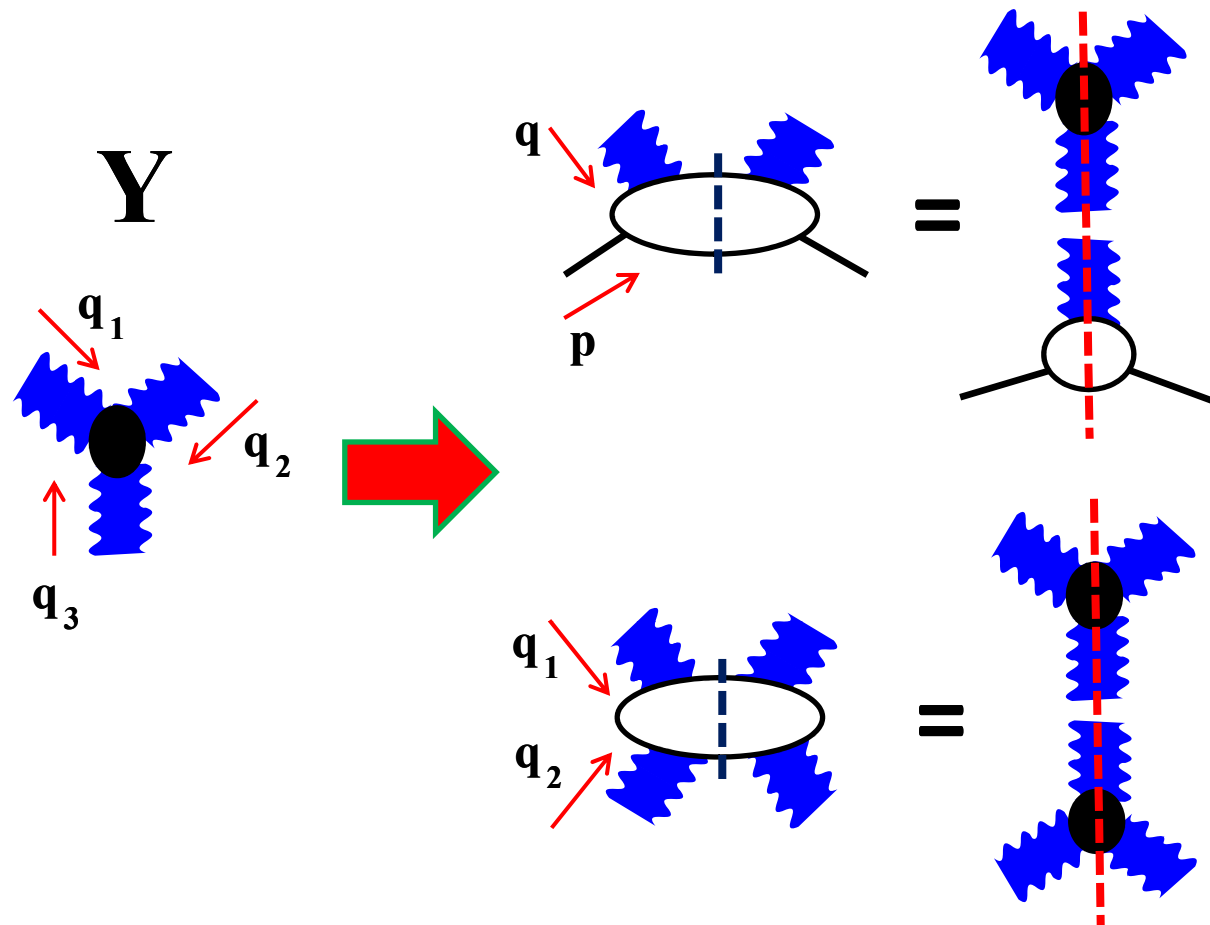
Symmetric Traceless
Solution

$$\tilde{h}_{\vec{u}}^{\vec{J}} = \prod_{r \in \Omega_H} \tilde{v}_{u_r}^{J_r}$$

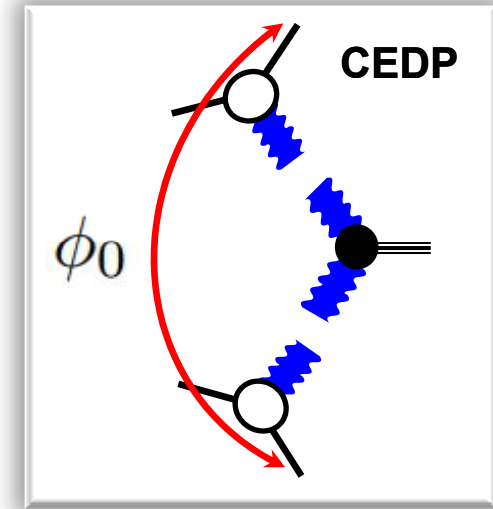
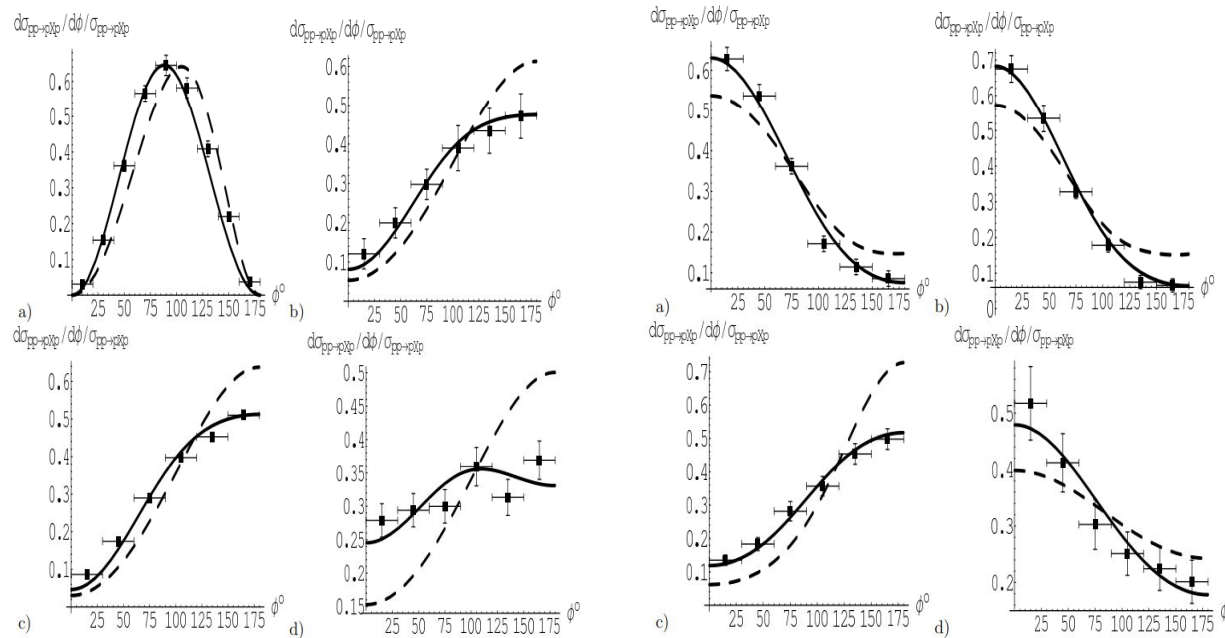
Orthogonality

$$\begin{aligned} & \left[\mathcal{H}_{(1),(1'),(2),(2')}^{\vec{J}-\vec{r}} q_{1(1)}^{r_1} q_{1(1')}^{r_{1'}} q_{2(2)}^{r_2} q_{2(2')}^{r_{2'}} \right] \otimes \\ & \left[\mathcal{H}_{(1),(1'),(2),(2')}^{\vec{J}-\vec{s}} q_{1(1)}^{s_1} q_{1(1')}^{s_{1'}} q_{2(2)}^{s_2} q_{2(2')}^{s_{2'}} \right] = 0, \vec{r} \neq \vec{s}. \end{aligned}$$

3-Reggeon vertex contractions



Applications: spin-parity analyzer

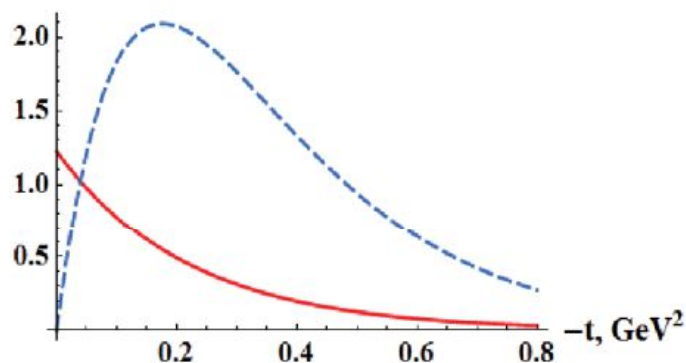


[A.B. Kaidalov, V.A. Khoze, A.D. Martin et al, *Central exclusive diffractive production as a spin-parity analyser: from hadrons to Higgs. Eur. Phys. J. C* 31, 387–396 (2003)]

[V.A. Petrov, R.A. Ryutin, A.E. Sobol and J.-P. Guillaud, *Azimuthal angular distributions in EDDE as spin-parity analyser and glueball filter for LHC, JHEP*06(2005)007]

Problems with conserved currents

$\frac{1}{\sigma_B} \frac{d\sigma}{dt}, \text{GeV}^{-2}$



[R.A. Ryutin , *Visualizations of exclusive central diffraction*,
Eur. Phys. J. C 74, 3162 (2014)]

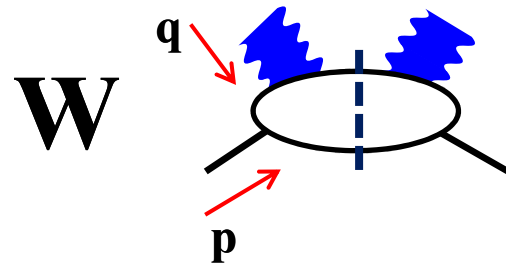
Fig. 2. The unitarization of the cross-section $|t|e^{-2B|t|}$ ($B \simeq 2.85 \text{ GeV}^{-2}$, $\sqrt{s} = 7 \text{ TeV}$) corresponding to the amplitude (89) in the Appendix C. The dashed curve represents the “bare” term and the solid one represents the unitarized result. σ_B is the integrated “bare” cross-section. The zero at $t = 0$ disappears in the unitarized cross-section.

“Non-Conserved” =
Unitarization?
Extra Dimension?
Both?

[F.E. Close, G.A. Schuller, *Evidence that the pomeron transforms as a nonconserved vector current*,
Phys. Lett. B 464, 279 (1999)]

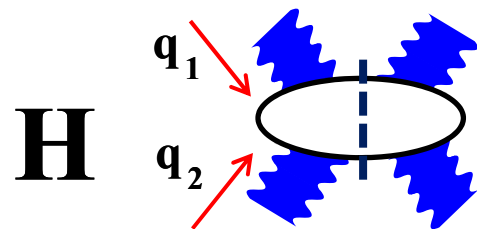
[F.E. Close, G.A. Schuller, *Central production of mesons: Exotic states versus pomeron structure*,
Phys. Lett. B 458, 127 (1999)]

Applications: extract cross-sections



**Hadron-Reggeon
cross-section**

**Hadron-Hadron
cross-section**



**Reggeon-Reggeon
cross-section**

1 mb – 100 mb?

~10 – 100 mb



H - "Reggeon"?

**What is the meaning
of these values?**

Further study

- ❑ Spin-parity analysis in CEDP
- ❑ Half-integer spins, helicity amplitudes
- ❑ Curved spaces, “non-conserved” currents
- ❑ Hyper-fields, strings
- ❑ Non-diffractive processes
- ❑ Unitarization
- ❑ Wolfram Mathematica package
- ❑ Universal framework for diffraction

To be continued ...

**THANK
YOU !**