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An interpolating mass formula and Regge trajectories for light and heavy quarkonia

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Abstract. Simple interpolating formula for the square of the quarkonium mass and an analytic expression for the Regge trajectories $\alpha(t)$ in a whole region of both light and heavy quarkonia are derived on the basis of the consideration of two asymptotics for the QCD inspired interquark potential. The leading trajectory functions obtained level off at -1 for $-t \to \infty$. This asymptotic value of $\alpha(t)$, $\alpha(t) \simeq -1$, implies that the cross section of the form $(1-x)^{1-2\alpha(t)}$, which is predicted by the triple Regge model, behaves like $(1-x)^3$. Is this to be attributed to the behaviour of the vector meson exchange or is it some hard scattering contribution swamping the Regge contributions? The intercepts and slopes of the leading Regge trajectories $\alpha_\rho(t)$, $\alpha_\Phi(t)$, $\alpha_\Psi(t)$ and $\alpha_\Upsilon(t)$ are calculated.

1 Introduction

It is well-known experimental fact that hadrons populate linear Regge trajectories; that is, the square of the mass of a state with orbital angular momentum l is proportional to l: $M^2(l) = \beta l + \text{const}$, with the same slope, $\beta \simeq 1.2\ \text{GeV}^2$, for all trajectories. There exists a conviction, that the Regge trajectories $\alpha(t)$ of light-flavour mesons are linear in a whole region, that is, not only in the bound-state region ($t > 0$) but in the scattering region ($t < 0$) too. However in the experiment [1] far more complicated behaviour of the ρ-meson trajectory, $\alpha_\rho(t)$, was discovered.

Presented in [1] experimental data on inclusive π^0 and η production in 100 GeV/c $\pi^\pm p$ collisions cover the kinematic region $0 \le -t \le 4\ (\text{GeV/c})^2$ and $x \ge 0.7$ and have compared in detail with the predictions of triple Regge theory [2]. So far as these reactions are theoretically clean with ρ (π^0 production) or A_2 (η production) exchange there were extracted the Regge trajectories, $\alpha(t)$, in the t range of 0 to $-4\ (\text{GeV/c})^2$. A sample of high $-t$, $-t \le 4\ (\text{GeV/c})^2$, has been fitted by the $\rho\rho P$ term, given by [2]

$$\frac{d^2\sigma}{dx\,dt} = G_\rho(t)(1-x)^{1-2\alpha_\rho(t)}, \tag{1}$$

where the pomeron intercept, $\alpha_P(0) = 1$, $G_\rho(t)$ is the residue function. There was shown that the ρ trajectory flattens off at about -0.6. The uncertainty in this asymptote can be estimated by fitting in the region $0.81 \le x \le 0.98$ which changes it up by 0.1 to $\alpha_\rho = -0.5$. This value α_ρ, $\alpha_\rho = -0.5$, implies that the cross section behaves like $(1-x)^2$.

The constituent interchange model (CIM) [3] predicts a leveling off of α_ρ at -1 or a $(1-x)^3$ cross section behaviour. However the exact value of $\alpha(t)$ in [1] is sensitive to the definition of x, where have used x as the lab energy divided by the maximum possible energy at the given t value. Changing the definition so that the denominator is just the beam energy would decrease the fitted α by about 0.2 at $-t = 4\ (\text{GeV/c})^2$. This means, that the ρ-trajectory flattens off at -0.8 or lower, that is, the trajectory level off so that the Regge exchanges *are the hard-scattering terms* [1].

The purpose of this paper is to derive an analytic expression for the quarkonium Regge trajectories, $\alpha(t)$, in the whole region, $-\infty < t < \infty$. Usually, the Regge trajectories of different hadrons are derived (in the framework of the potential models) for the bound state region, that is, at $t = E^2 > 0$. But for many purposes, for example, in the recombination [4] and fragmentation [5, 6] models and other Regge models it is necessary to know the Regge trajectories in the scattering region, that is, at $t < 0$, and, in particular, the intercepts $\alpha(0)$ and slopes α' of Regge trajectories. For example, the parameters of the leading ρ trajectory is well reproduced in the framework of quark potential models [7]

$$\alpha_\rho(0) = 0.49, \quad \alpha'_\rho = 0.88\ (\text{GeV/c})^{-2}. \tag{2}$$

As it was noted in [7] the intercepts (and slopes) of the Regge trajectories of light-quark hadrons are the fundamental constants of hadron dynamics perhaps more important to reproduce within a potential model than the mass of such or such a state.

There is another interesting application of the Regge trajectories in the scattering region. Usually, the calculations of c, b and t quark production cross sections are performed on the basis of the parton model in the framework of perturbative QCD [8]. However for this aim the recombination [4] and fragmentation [9–11] models can be used. In these models the recombination and fragmentation functions and many other physical quantities are parametrised by means of the intercepts $\alpha(0)$ of Regge trajectories. However up to now no information on the intercepts of Regge trajectories of heavy-flavour mesons has been obtained.

The question in this paper is whether approximate analytic formula for the Regge trajectories of both light and heavy quark-antiquark systems with reasonable degree of confidence can be derived in the framework of the phenomenological potential model. In order to obtain the Regge trajectories in a whole region it is necessary to known an analytic expression for the square of the total energy of quarkonium state, E^2, as a function of radial, n', and orbital, l quantum numbers. Then, if we invert $E^2(l)$ and express the angular momentum $l(E^2)$ as a function of the E^2 we obtain the Regge trajectory. In this work we use only the fact, which has good established theoretically, that the interkquark potential has two asymptotics: 1) $V(r) \propto -1/r$ at $r \to 0$ (coulomb-like behaviour motivated by the one-gluon exchange at small distances) and 2) $V(r) \propto r$ at $r \to \infty$ (linear confining behaviour which follows from lattice-gauge-theory computations). We obtain two analytic expressions for the square of the quarkonium mass corresponding these two asymptotics of the interquark potential. Then using the two-point Padé approximant we deduce an interpolating formula for the square of the quarkonium mass and (by product) an analytic expression for the Regge trajectories too. We show that the leading $S = 1$ Regge trajectories of all quarkonia flatten off at -1. Finally, to demonstrate the plausibility of the obtained mass formula, we show that our simple interpolating formula incorporating the spin-spin and spin-orbit interactions is able to reproduce the general features of the empirical quarkonium spectra.

2 The interquark potential. Heavy quarkonia

In a short time after successful classification of light-quark (u, d, s) states in the framework of $SU(3)$, or $SU(6)$, and linear Regge trajectories with quasi-universal slope, $\alpha' \simeq 0.9\ (\mathrm{GeV/c})^{-2}$, the heavy-quark (c, b) states has been discovered. The remarkable observation that the $c\bar{c}$ and $b\bar{b}$ states could be obtained as the energy levels of a nonrelativistic Hamiltonian with a universal flavour-independent confining potential has led to a number of refined and accurate calculations [12–15]. The well known question arises if the physics for light quarks and the physics for heavy ones constitute two different realms of hadron spectroscopy or if they have some common links.

Because of the intrinsically nonperturbative nature of the bound-state problem in non-Abelian gauge theories, it is, up to now, not possible to derive the forces acting between the quarks from first principles. Therefore, the corresponding interquark potential has to be determined phenomenologically. An understanding of the static potential between heavy quarks as well as spin-dependent corrections of order $1/m^2$ is of fundamental importance because it provides an intimate connections between the underlying theory of QCD and direct phenomenological consequences showing up in hadron spectroscopy. The information which goes beyond perturbation theory is, however, rather scarce.

It is shown [7, 14] that i) there is some unity (and not two different realms) in the world of meson spectroscopy, and ii) the universal flavour-independent confining potential is fixed is an extremely simple manner in terms of very small number of parameters, all of which have a direct physical interpretation. For small distances between the quarks, one expects from one-gluon exchange a Coulomb-like contribution to the potential, that is, [15]

$$V(r) \simeq -\frac{4}{3}\frac{\alpha_s}{r}, \quad r \to 0. \tag{3}$$

For large distances, in order to be able to describe confinement, the potential has to rise to infinity. From lattice-gauge-theory computations [16] follow that this rise is an approximately linear, that is,

$$V(r) \simeq \kappa r, \quad r \to \infty, \tag{4}$$

where $\kappa \simeq 0.15\ \mathrm{GeV}^2$ being the string tension.

However in an intermediate region the potential is more poorly understood. In this region, many well known potentials give reasonable results for hadron mass [7, 12, 13, 17], but these results do not depend very strong on the form of potential. The most reasonable possibility to construct an interquark potential which satisfies both of the above constraints is to simply add these two contributions. This leads to the so-called funnel-shaped (or Cornell) potential [12]:

$$V(r) = -\frac{4}{3}\frac{\alpha_s}{r} + \kappa r + V_0. \tag{5}$$

A closer inspections reveals that all phenomenologically acceptable "QCD-inspired" potentials are only variations around the funnel potential [17]. Its parameters are directly related to basic physical quantities: the universal Regge slope $\alpha' \simeq 0.9\ (\mathrm{GeV/c})^{-2}$ of light flavours and one-gluon-exchange coupling strength α_s for heavy ones. As for constant V_0, usually, it is added to the confining contribution. This constant has to be regarded as an additional arbitrary parameter in the potential, its arbitrariness being induced by the arbitrariness in the choice of the renormalization point.

An approximate framework for the description of bound states within a relativistic quantum field theory is the Bethe-Salpeter formalism. However it is hard to obtain information from this approach. An alternative is the treatment of bound states with the help of Schrödinger type eigenvalue equations [14, 15]. The employment of a nonrelativistic Schrödinger equation (with some effective potential) for the description of mass spectra not only heavy quarkonia but also light mesons remains up to now surprisingly successful [12–15, 17]. However, at least bound states consisting of light constituents should be dealt within a relativistic framework.

In order to obtain a mass formula for quarkonia note firstly that such a formula more convenient to derive not for the quarkonium mass, E, but for the square of the quarkonium mass, E^2. It is known that heavy $Q\bar{Q}$ systems one can consider nonrelativistically. For low states of heavy quarkonia (for example, $t\bar{t}$ or $b\bar{b}$) the main contribution to the boundary energy comes from one-gluon exchange (the potential (3)) and in the first approximation one can neglect the confining term. In this case Schrödinger equation has exact analytical (hydrogen-like) solution [18] and for energy eigenvalues we have approximately:

$$E'_n = -\frac{\tilde{\alpha}_s^2 m}{4(n'+l+1)^2} + V_0, \tag{6}$$

where $\tilde{\alpha}_s = \frac{4}{3}\alpha_s$, m is the quark mass and the constant V_0 have added to the potential (3); $n' = 0, 1, 2, \ldots$ and $l = 0, 1, 2, \ldots$ are the radial and orbital quantum numbers, respectively. The total energy of heavy quarkonium one can write in the form: $E = 2m + E'$, where E' is boundary energy. For the square of the energy in nonrelativistic approximation we have $E^2 \simeq 4m^2 + 4mE'$. Therefore, with the help of (6) we obtain for E_n^2:

$$E_n^2 \simeq -\frac{\tilde{\alpha}_s^2 m^2}{(n'+l+1)^2} + 4m(m+V_0). \tag{7}$$

The formula (7) is good to describe the lower states of heavy quarkonia, especially the bottomonium.

3 Light mesons

Now let us consider an extreme relativistic limit for higher excited states ($l, n' \gg 1$). In this case bound states consisting of light constituents should be dealt with in a relativistic framework and main difficulty comes from unknowing Lorentz structure of the potential. For bound states consisting of quarks and antiquarks there is very strong evidence that the static potential receives predominantly a vector and scalar contributions [17]. A standard Ansatz is to assume that the $q\bar{q}$ interaction has effective vector, $V(r)$, and scalar, $S(r)$, contributions only, i.e. that the static potential $W(r)$ may be decomposed according to the spin of exchange boson [9]:

$$L = \tilde{S}(q^2)\bar{U}U\bar{V}V + \tilde{V}(q^2)\bar{U}\gamma_\mu U\bar{V}\gamma^\mu V.$$

Two results are of fundamental importance as far as light-flavoured meson spectroscopy is concerned. The combination of relativistic kinematics and linear potential produces α) linear Regge trajectories [7, 14] which acquire some curvature at low energies as the quark masses are increased, and β) these trajectories have only even daughters.

To describe bound states consisting of light constituents let us consider a static Klein-Gordon equation of motion in which the potential has a Lorentz vector $V(r)$ and Lorentz scalar $S(r)$ parts [19, 20]:

$$[\nabla^2 + \tfrac{1}{4}(E - V(r))^2 - (m + S(r))^2]\psi(\mathbf{r}) = 0, \tag{8}$$

where the functions $V(r)$ and $S(r)$ we have chosen in the form:

$$V(r) = -\frac{\tilde{\alpha}_s}{r}(1-c) + \kappa r(1-d), \quad S(r) = -\frac{\tilde{\alpha}_s}{r}c + \kappa r d. \tag{9}$$

Here in (9) c and d are the parameters.

Let us consider (8) for higher excited states. For large angular momenta, $l \gg 1$, one may expect that the bound states will feel only the confining part of the potential. We thus assume it is justified to ignore the Coulomb term. In this limit to calculate the eigenvalues E_n^2 the WKB approximation can be used. For radial part of (8) we have:

$$\left[\frac{d^2}{dr^2} + \frac{1}{4}\left(E + \frac{\alpha_1}{r} - \kappa_1 r\right)^2 - \left(m - \frac{\alpha_2}{r} + \kappa_2 r\right)^2 - \frac{(l+\frac{1}{2})^2}{r^2}\right]R(r) = 0, \tag{10}$$

where $\alpha_1 = \tilde{\alpha}_s(1-c)$, $\kappa_1 = \kappa(1-d)$, $\alpha_2 = \tilde{\alpha}_s c$, $\kappa_2 = \kappa d$, and the replacement $l(l+1) \to (l+\frac{1}{2})^2$ has been made in accordance with the WKB method.

The WKB quantization condition appropriate to (10) gives:

$$\int_{r_1}^{r_2} \sqrt{\frac{1}{4}\left(E + \frac{\alpha_1}{r} - \kappa_1 r\right)^2 - \left(m - \frac{\alpha_2}{r} + \kappa_2 r\right)^2 - \frac{(l+\frac{1}{2})^2}{r^2}}\, dr = \pi(n' + \tfrac{1}{2}), \tag{11}$$

where $n' = 0, 1, 2, \ldots$, r_1 and r_2 are the classical turning points. At $l \gg 1$ the first turning point, r_1, is determined mainly by the term $\sim r^{-2}$ and the second one, r_2, by the quadratic term $\sim r^2$. Therefore, in this approximation we have for integral (11) [21]:

$$\frac{\pi}{2}\,\frac{E^2/4 - m^2 - (\alpha_1\kappa_1 - 4\alpha_2\kappa_2)/2}{\sqrt{4\kappa_2^2 - \kappa_1^2}} - \frac{\pi}{2}\sqrt{\alpha_2^2 - \frac{\alpha_1^2}{4} + \left(l + \frac{1}{2}\right)^2} = \pi(n' + \tfrac{1}{2}), \tag{12}$$

and for the quarkonium squared mass, E_n^2, at $l \gg 1$ this gives

$$E_n^2 = 8\kappa\left[a\left(2n' + l + \frac{3}{2}\right) - b\tilde{\alpha}_s\right] + 4m^2, \tag{13}$$

where $a = \sqrt{d^2 - (1-d)^2/4}$, $b = cd - (1-c)(1-d)/4$. Note, that the value a is real if $\frac{1}{3} < d < -1$.

The Regge trajectories given by (13) is very similar to that of a harmonic oscillator-type hamiltonian and good to describe the light-flavor mesons with $\kappa \simeq 0.15$ GeV^2, $a \simeq 1$, $b \simeq 1$.

4 An interpolating mass formula

Thus we have two analytic expressions (7) and (13). The first of them (7) is good to describe the lower-state masses of heavy quarkonia and another (13), on the contrary, the masses of light-flavour mesons. Besides, the energy spectrum in (7) lies in the region $E_n < 2m$ and the energy spectrum of the second one (13) lies in the region $E_n > 2m$.

It is known that an angular momentum in the Regge phenomenology is considered as an analytic function in the complex plane. We choose the leading Regge trajectories associated to $S = 1$ $q\bar{q}$ states, with total angular momentum $J = l + 1$ and $n' = 0$, and consider the square of the momentum as the function of continuous variable $z = l + 1$:

$$p^2(z) = E_n^2 - 4m(m + V_0). \tag{14}$$

Now let us suppose that (7) gives the asymptotic for the function $p^2(z)$ at small z, that is,

$$p^2(z) \simeq -\frac{\tilde{\alpha}_s^2 m^2}{z^2}, \quad z \to 0. \tag{15}$$

Then (13) gives another asymptotic, but for large z:

$$p^2(z) \simeq 8\kappa a z, \quad z \to \infty. \tag{16}$$

The question arises: what is the possibility to construct an approximate formula for the $p^2(z)$ which satisfies both of these constraints?

For this let us consider the two-point Padé approximant [22]

$$[K/N]_f(z) = \frac{\sum_{i=0}^{K} a_i z^i}{\sum_{j=0}^{N} b_j z^j}. \tag{17}$$

Table 1. The leading state masses of light and heavy quarkonia

state	m_{theor} (GeV/c²)	m_{exp} (GeV/c²)	parameters in (18)
Light mesons			
1^3S_1	0.764	0.768	$\alpha_s = 0.91 \pm 0.03$
1^3P_2	1.328	1.318	$\kappa = 0.13 \pm 0.01$ GeV²
1^3D_3	1.687	1.691	$m = 0.33$ GeV-fixed
1^3F_4	1.978	–	
1^3G_5	2.231	–	
2^3S_1	1.680	1.700	
3^3S_1	2.228	–	
$s\bar{s}$ states			
1^3S_1	1.019	1.019	$\alpha_s = 0.84 \pm 0.19$
1^3P_2	1.525	1.525	$\kappa = 0.13 \pm 0.02$ GeV²
1^3D_3	1.854	1.854	$m = 0.48 \pm 0.17$ GeV
1^3F_4	2.126	–	
1^3G_5	2.365	–	
2^3S_1	1.843	–	
3^3S_1	2.360	–	
$c\bar{c}$ states			
1^3S_1	3.097	3.097	$\alpha_s = 0.73 \pm 0.03$
1^3P_2	3.556	3.556	$\kappa = 0.12 \pm 0.02$ GeV²
1^3D_3	3.739	–	$m = 1.73 \pm 0.03$ GeV
1^3F_4	3.880	–	
1^3G_5	4.008	–	
2^3S_1	3.686	3.686	
3^3S_1	3.983	–	
4^3S_1	4.229	–	
5^3S_1	4.453	–	
$b\bar{b}$ states			
1^3S_1	9.460	9.460	$\alpha_s = 0.45 \pm 0.01$
1^3P_2	9.914	9.913	$\kappa = 0.30 \pm 0.01$ GeV²
1^3D_3	10.095	–	$m = 4.89 \pm 0.01$ GeV
1^3F_4	10.235	–	
1^3G_5	10.362	–	
2^3S_1	10.036	10.023	
3^3S_1	10.333	10.355	
4^3S_1	10.585	10.580	

To obtain the asymptotics (15), (16) let us choose $K = 3$, $N = 2$ in (17). Now it is easily to see that the Padé approximant (17) satisfy to two asymptotics (15), (16) if: $a_0 = -\tilde{\alpha}_s^2 m^2$, $a_1 = 0$, $a_2 = 0$, $a_3 = 8\kappa a$, $b_0 = 0$, $b_1 = 0$, $b_2 = 1$. Therefore, with the help of relation (14) we obtain the following interpolating formula for E_n^2:

$$E_n^2 = 8\kappa[a(2n' + l + \tfrac{3}{2}) - b\tilde{\alpha}_s] - \frac{\tilde{\alpha}_s^2 m^2}{(n' + l + 1)^2} + 4m^2. \tag{18}$$

Formula (18) reproduces both the mass spectra of light and heavy quarkonia well if $c \simeq 1$, $d \simeq 1$ ($a \simeq 1$, $b \simeq 1$) (see Table 1).

This means that the potential is a Lorentz scalar ($V(r) \simeq 0$, see (8)) in an extreme relativistic limit for higher excited states. Consequently one can write (18) in the following simplest form:

$$E_n^2 = 8\kappa(2n' + l + \tfrac{3}{2} - \tilde{\alpha}_s) - \frac{\tilde{\alpha}_s^2 m^2}{(n' + l + 1)^2} + 4m^2. \tag{19}$$

Formula (19) reflects in some sense the structure of two asymptotics (3) and (4) of the potential in E^2 representation: the term $-\tilde{\alpha}_s^2 m^2/(n' + l + 1)^2$ corresponds to the one-gluon approximation (function (3)), and the second one, $8\kappa(2n' + l + \frac{3}{2} - \tilde{\alpha}_s)$, corresponds to the confining potential (function (4)).

5 The Regge trajectories

The interpolating mass formula (19) has several interesting consequences. First of all it allows one to reproduce an analytic expression for the quarkonium Regge trajectories. For this let us transform (19) into an equation for the angular momentum l:

$$l^3 + c_1(E^2)l^2 + c_2(E^2)l + c_3(E^2) = 0, \tag{20}$$

where

$$c_1(E^2) = 4n' + 7/2 - \tilde{\alpha}_s + \lambda(E^2),$$

$$c_2(E^2) = (n' + 1)^2 + 2(n' + 1)[2n' + 3/2 - \tilde{\alpha}_s + \lambda(E^2)],$$

$$c_3(E^2) = (n' + 1)^2[2n' + 3/2 - \tilde{\alpha}_s + \lambda(E^2)] - \tilde{\alpha}_s^2 m^2/(8\kappa),$$

$$\lambda(E^2) = (4m^2 - E^2)/(8\kappa).$$

Replacing E^2 by new variable t consider the cubic equation (20) in the whole region $-\infty < t < \infty$. The investigation of solutions of this equation show that at $Q(t) > 0$ there is one real solution of the form:

$$\alpha(t) = \sqrt[3]{-q(t)/2 + \sqrt{Q(t)}} + \sqrt[3]{-q(t)/2 - \sqrt{Q(t)}} - c_1(t)/3, \quad Q(t) > 0, \tag{21}$$

where

$$Q(t) = p^3(t)/27 + q^2(t)/4,$$

$$p(t) = -c_1^2(t)/3 + c_2(t),$$

$$q(t) = 2c_1^3(t)/27 - c_1(t)c_2(t)/3 + c_3(t).$$

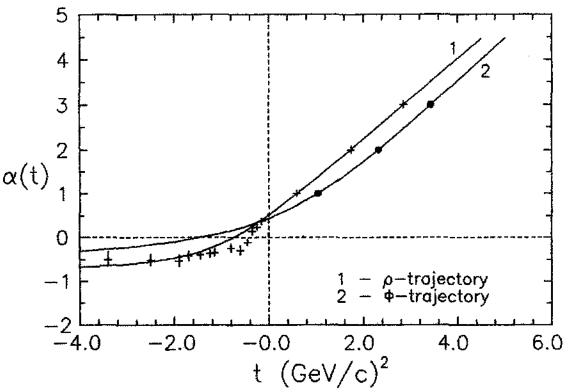

Fig. 1. The leading ρ and Φ Regge trajectories calculated according to (19): $1-\rho$ trajectory giving the triple Regge description of π^0 production in reactions (1), the experimental data at $t<0$ are taken from [1] (trajectory fit region $0.81 < x < 0.98$); $2-\Phi$ trajectory ($s\bar{s}$ states); At $t>0$ are given the integer numbers $J=1,2,3$ of the total angular momentum versus the mass of resonances

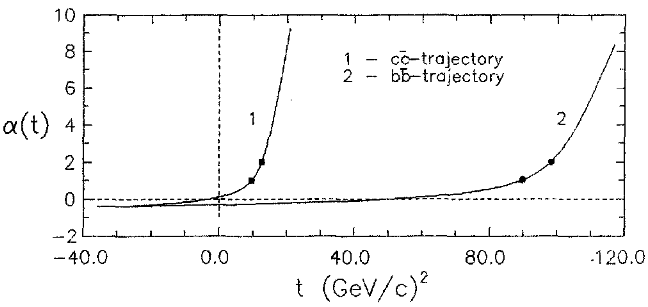

Fig. 2. The leading Ψ and Υ Regge trajectories calculated according to (19). The symbols denote the empirical masses of the $^3S_1$ and $^3P_2$ excited states of $c\bar{c}$ and $b\bar{b}$ systems

From other side at $Q(t) \le 0$ there are three real solutions. Only first of this solutions,

$$\alpha(t) = 2\sqrt{-p(t)/3}\cos[\beta(t)/3] - c_1(t)/3, \quad Q(t) < 0, \quad (22)$$

where $\beta(t) = \arccos[-q(t)/(2\sqrt{-p^3(t)/27})]$, smoothly go over at $Q(t)=0$ into solution (21). Therefore two functions (21), (22) produce the Regge trajectory in the whole region $-\infty < t < \infty$.

The Regge trajectories (21), (22) are linear at $t \to \infty$ with the universal slope $\alpha' = (8\kappa)^{-1}$ and flatten off at -1 for $t \to -\infty$ (see Figs. 1, 2). The first derivative, $\alpha'(t)$, is positive in the whole region $-\infty < t < \infty$. The intercepts $\alpha(0)$, and slopes α' of quarkonium Regge trajectories calculated are:

$$\begin{aligned}
&\alpha_\rho(0) \simeq 0.52, && \alpha'_\rho \simeq 0.79\ \text{GeV}^{-2};\\
&\alpha_{s\bar{s}}(0) \simeq 0.43, && \alpha'_{s\bar{s}} \simeq 0.43\ \text{GeV}^{-2};\\
&\alpha_{c\bar{c}}(0) \simeq 0.10, && \alpha'_{c\bar{c}} \simeq 0.04\ \text{GeV}^{-2};\\
&\alpha_{b\bar{b}}(0) \simeq -0.31, && \alpha'_{b\bar{b}} \simeq 0.003\ \text{GeV}^{-2}. \qquad (23)
\end{aligned}$$

6 Spin corrections

In order to demonstrate the correctness and plausibility of the interpolating mass formula (19) consider its with the account of the spin corrections. The spin-independent potential (which may be velocity dependent) essentially yield the center of gravity of levels, whereas the spin-dependent term, V_{SD}, gives the splitting both of the $^3S_1$ and $^1S_0$ states and of each $L \ge 1$ level into the four states $^3L_{L-1}$, ^3L_L, $^3L_{L+1}$ and $^1L_1$.

A simple interpolating formula (19) describes equally well the center of gravity of energy levels of all $q\bar{q}$ mesons ranging from the $u\bar{d}(d\bar{d}, u\bar{u}, s\bar{s})$ states up to the heaviest known $b\bar{b}$ systems (see Table 1). The relativistic structures which occur in QCD have similarities to as well as important differences from relativistic corrections in QED. Not all spin-dependent forces are short range in QCD. This result follows only from QCD and the existence of the static limit is independent of any detailed mechanism for confinement or of perturbation theory.

The total spin-dependent potential $V_{\text{SD}}(r)$ given by [20, 23]:

$$\begin{aligned}
V_{\text{SD}} = &\frac{1}{2}\left(\frac{\mathbf{s}_1}{m_1^2} + \frac{\mathbf{s}_2}{m_2^2}\right)\cdot\mathbf{L}\left(\frac{1}{r}\frac{dW}{dr} + \frac{2}{r}\frac{dV_1}{dr}\right)\\
&+ \frac{(\mathbf{s}_1+\mathbf{s}_2)\cdot\mathbf{L}}{m_1 m_2}\frac{1}{r}\frac{dV_2}{dr} + \frac{2}{3}\frac{(\mathbf{s}_1\cdot\mathbf{s}_2)}{m_1 m_2}V_3(r)\\
&+ \frac{(3\mathbf{s}_1\cdot\hat{r}\mathbf{s}_2\cdot\hat{r} - \mathbf{s}_1\cdot\mathbf{s}_2)}{3m_1 m_2}V_4(r), \qquad (24)
\end{aligned}$$

where $\mathbf{s}_1$, $\mathbf{s}_2$ are the quark spins, $\mathbf{L}$ is the relative orbital angular momentum, $V_i(r)$ $(i=1,2,3,4)$ are the unknowing functions in general case, $W(r)$ is the static potential. Equation (24) is the complete spin-dependent potential in QCD through order $1/m^2$ with $V_i(r)$ as expectation values of the appropriate electric and magnetic fields; it is valid both in QED and QCD, independent of perturbation theory, and will serve as the basis of the examination of spin-dependent forces.

The general form for the spin-dependent forces in QCD (24) may be directly applied to meson system involving two sufficiently heavy quarks. However, if m_i is not large, the Dirac structure obtained for heavy-quark systems may still be valid but now with various $V_i(r)$ completely unknown functions of r and m_i. However, there exists an important relation between the three functions $W(r)$, $V_1(r)$ and $V_2(r)$ which follows from the Lorentz invariance of the theory [20]

$$W(r) + V_1(r) - V_2(r) = 0. \qquad (25)$$

Let us choose the potentials V_1, V_2 in the form:

$$V_1(r) = -\eta\kappa r, \quad V_2(r) = -\frac{\tilde{\alpha}_s}{r} + (1-\eta)\kappa r, \qquad (26)$$

where η is some parameter. Then for the spin-orbit interaction potential, $V_{ls}(r)$, we obtain:

$$V_{ls}(r) = \frac{3}{2}\frac{\tilde{\alpha}_s}{m^2 r^3}(\mathbf{S}\cdot\mathbf{L}) + \frac{3}{2}\frac{\kappa}{m^2 r}(\mathbf{S}\cdot\mathbf{L})\left(1 - \frac{4}{3}\eta\right), \qquad (27)$$

where $\mathbf{S} = \mathbf{s}_1 + \mathbf{s}_2$. Using for calculation of the expectation values $\langle r^{-1}\rangle$ and $\langle r^{-3}\rangle$ the Coulomb wave functions, we obtain for the energy of spin-orbit interaction between two heavy quarks (the results obtained below related to bound states of two heavy quarks):

$$E_{ls} = \frac{3}{32}\tilde{\alpha}_s^4 m \frac{J(J+1) - l(l+1) - S(S+1)}{l(l+1)(l+\frac{1}{2})(n'+l+1)^3} + \frac{3}{8}\frac{\tilde{\alpha}_s \kappa}{m}\frac{J(J+1) - l(l+1) - S(S+1)}{(n'+l+1)^2}\left(1 - \frac{4}{3}\eta\right). \quad (28)$$

In our approach we have neglected by the tensor term which proportional to the $V_4(r)$ in (24).

As for the spin-spin interaction, empirically, the differences of the squared masses of corresponding spin-singlet and spin-triplet partners, which contain at least one light quark (u, d, s) are constant. However, because a Coulomb-like contribution $\sim 1/r$ this causes some problems. For instance, due to the relation $\Delta(1/r) = -4\pi\delta^{(3)}(\mathbf{r})$, the spin-spin interaction $V_{ss}(r) = \frac{2}{3}(\mathbf{s}_1 \cdot \mathbf{s}_2)/(m_1 m_2)\Delta W(r)$ derived from a pure Coulomb potential involves a δ function,

$$H_{ss} = \frac{32\pi\alpha_s}{9m^2}(\mathbf{s}_1 \cdot \mathbf{s}_2)\delta^3(\mathbf{r}), \quad (29)$$

and the Breit-Fermi Hamiltonian unbounded from below. These problems are not present in the completely relativistic form of the bound-state wave function [14, 15]. To account the spin-spin interaction we have used the results obtained for the singlet-triplet mass-squared differences in [15]:

$$\Delta M_{ss}^2 = M_{s=1}^2 - M_{s=0}^2 = \frac{32}{9}\alpha_s \kappa \quad (30)$$

for light $q\bar{q}$ systems (in the instantaneous-limit approximation [15]), and

$$\Delta M_{ss}^2 = M_{s=1}^2 - M_{s=0}^2 = \frac{256}{3\pi^2}\alpha_s \kappa \quad (31)$$

for heavy quarkonia (hydrogen-like trial functions) [15]. All these predictions for the mass-squared differences are independent of the mass of the particles which constitute the bound state.

Now using the results obtained above we can write an interpolating mass formula taking into account (perturbatively) fine and hyperfine splittings of levels. Let us write such a formula in the form:

$$M_n^2 = (E_n + E_{ls} + E_{ss})^2 \simeq E_n^2 + 2E_n E_{ls} + 2E_n E_{ss}, \quad (32)$$

where E_{ls} is given by (28). In order to estimate the energy of spin-spin interaction, E_{ss}, we use (30), (31), namely we suppose that $\Delta M_{ss}^2 \simeq 2E_0 \Delta E_{ss}$ (in accordance with formula $\Delta E^2 = 2E\Delta E$). For ground state mass of bound state, E_0, we have taken approximately $E_0 \simeq m_1 + m_2 = 2m$, where m is the constituent quark mass. Then for the energy difference, ΔE_{ss}, we have approximately:

$$\Delta E_{ss} = \frac{1}{4m}\Delta M_{ss}^2 \Delta S, \quad (33)$$

where $\Delta S = \langle \mathbf{s}_1 \cdot \mathbf{s}_2 \rangle_{S=1} - \langle \mathbf{s}_1 \cdot \mathbf{s}_2 \rangle_{S=0} = 1/4 - (-3/4) \equiv 1$ is the difference of the expectation values for $S = 1$ and $S = 0$ spin states of quarks. Therefore for the energy of spin-spin interaction we have approximately:

$$E_{ss} = \frac{1}{4m}\Delta M_{ss}^2 \langle \mathbf{s}_1 \cdot \mathbf{s}_2 \rangle. \quad (34)$$

Substituting (28) and (34) in (32) we obtain:

$$M_n^2 = E_n^2 + 2E_n E_{ls} + \frac{E_n}{2m}\Delta M_{ss}^2 \langle \mathbf{s}_1 \cdot \mathbf{s}_2 \rangle, \quad (35)$$

where E_n^2 (and E_n) is given by (19). The formula (35) reproduces both the mass spectra of light and heavy quarkonia well. The results of the fits, the predicted energy levels and the corresponding sets of parameters are summarized in Tables 2–5, where the symbol* denotes the masses of the fitted states.

The fitting results give for parameter η, $\eta \simeq 0$ (see (26–28)), for all quarkonia except lightest ones (see Table 2) for which $\eta = 0.065$. This means that the function $V_1(r) \simeq 0$ in the (25). The additive constant V_0 in the interquark potential have determined, in fact, by both the slope κ and Coulomb-like coupling constant α_s. The average relative errors, $\langle (E^{\text{th}} - E^{\text{exp}})/E^{\text{exp}} \rangle$, of the fitting results are listened in the Tables 2–5. Note also that calculated in Table 2 the mass of 1^1S_0 state is very sensitive to light quark mass m. Generally (35) is useful (as mentioned above) for approximate description of mass spectra of heavy quarkonia.

Table 2. Light mesons

state	m_{theor} (GeV/c^2)	m_{exp} (GeV/c^2)	parameters in (35)
1^1S_0	0.139	0.139	$\alpha_s = 0.892 \pm 0.023$
1^3S_1	0.770	0.768*	$\kappa = 0.125 \pm 0.004$ GeV^2
1^3P_2	1.334	1.318*	$m = 0.231$ GeV – fixed
1^3P_0	0.972	0.983	$\left\langle \frac{\Delta E}{E} \right\rangle \simeq 4.7\%$
1^1P_1	1.225	1.233*	
1^3D_3	1.678	1.691*	
1^3F_4	1.953	–	
1^3G_5	2.191	–	
2^1S_0	1.217	1.300	
2^3S_1	1.683	1.700*	

Table 3. $s\bar{s}$ states

state	m_{theor} (GeV/c^2)	m_{exp} (GeV/c^2)	parameters in (35)
1^1S_0	0.780	–	$\alpha_s = 0.82 \pm 0.17$
1^3S_1	1.019	1.019*	$\kappa = 0.13 \pm 0.02$ GeV^2
1^3P_2	1.527	1.525*	$m = 0.43 \pm 0.17$ GeV
1^3P_1	1.392	–	$\left\langle \frac{\Delta E}{E} \right\rangle \simeq 2.5\%$
1^1P_0	1.319	–	
1^1P_1	1.461	–	
1^3D_3	1.845	1.854*	
1^3F_4	2.109	–	
1^3G_5	2.341	–	
2^1S_0	1.606	–	
2^3S_1	1.836	–	

Table 4. $c\bar{c}$ states

state	m_{theor} (GeV/c^2)	m_{exp} (GeV/c^2)	parameters in (35)
1^1S_0	2.986	2.984	$\alpha_s = 0.73 \pm 0.03$
1^3S_1	3.096	3.097*	$\kappa = 0.12 \pm 0.02\,\text{GeV}^2$
1^3P_2	3.554	3.556*	$m = 1.71 \pm 0.02\,\text{GeV}$
1^3P_1	3.504	3.511*	$\left\langle \frac{\Delta E}{E} \right\rangle \simeq 0.1\%$
1^1P_0	3.479	3.415	
1^1P_1	3.529	3.525	
1^3D_3	3.725	–	
1^3F_4	3.864	–	
1^3G_5	3.990	–	
2^1S_0	3.577	3.592*	
2^3S_1	3.687	3.686*	

Table 5. $b\bar{b}$ states

state	m_{theor} (GeV/c^2)	m_{exp} (GeV/c^2)	parameters in (35)
1^1S_0	9.398	–	$\alpha_s = 0.45 \pm 0.01$
1^3S_1	9.459	9.460*	$\kappa = 0.30 \pm 0.01\,\text{GeV}^2$
1^3P_2	9.920	9.913*	$m = 4.89 \pm 0.01\,\text{GeV}$
1^3P_1	9.896	9.891	$\left\langle \frac{\Delta E}{E} \right\rangle \simeq 0.2\%$
1^3P_0	9.884	9.870	
1^1P_1	9.908	–	
1^3D_3	10.096	–	
1^3F_4	10.234	–	
1^3G_5	10.359	–	
2^1S_0	9.983	–	
2^3S_1	10.044	10.023*	
3^3S_1	10.340	10.355*	
4^3S_1	10.590	10.580*	

7 Conclusion

The model was motivated, as noted previously, by our desire to construct an analytic expression for the quarkonium Regge trajectories which could be combined both heavy and light quarkonia. We have obtained the interpolating formula for the square of the quarkonium mass and (by product) analytic expression for the Regge trajectories $\alpha(t)$. This simple mass formula and following from it the Regge trajectories have, as shown above, several interesting consequences.

Our trajectory functions level off at -1 for $-t \to \infty$ and we see this same universal behaviour for all leading trajectories from π to Υ. This value $\alpha(t)$, $\alpha(t) \simeq -1$, implies that the cross section $(1-x)^{1-2\alpha(t)}$, which is predicted by the triple Regge model, behaves like $(1-x)^3$. Is this to be attributed to the behaviour of the vector meson exchange or is it some hard scattering contribution swamping the Regge theory for $x > 0.7$. It seems likely that in this x region, at least, triple Regge theory is the correct description of both of pion [1] and proton [24] induced reactions. We see similar behaviour in the all neutral and full inclusive data. It is quite possible that QCD can explain these observation.

Quark-quark and quark-gluon scattering is a fashionable model for high $p_\perp$ processes near $x = 0$ [3]. Naturally, this will also contribute to the high-x region. However, one must mention the quark-recombination [4] and fragmentation [5, 6] models which have been applied to meson production from a proton beam at high x [25]. These models are normally applied at low $p_\perp$ but clearly the data [1, 24] show that this region is dominated by triple Regge terms. May be these models can provide a description of high $-t$ data [1] where $\alpha(t)$ has settled down to its "universal" value of -1 with the corresponding $(1-x)^3$ behaviour of the cross section at large $-t$.

We have calculated the intercepts and slopes of the leading Regge trajectories, $\alpha_\rho(t)$, $\alpha_\Phi(t)$, $\alpha_\Psi(t)$ and $\alpha_\Upsilon(t)$, which are used as free parameters in many models [4–6, 9–11].

In order to demonstrate the correctness of our basic formula (19) we have included (by standard fashion) the spin-dependent corrections. The obtained mass formula (35) include fine and hyperfine splitting of the energy levels. We see that this interpolating mass formula is able to describe with good accuracy the spectra of all quark-antiquark bound states. The relative magnitude of the Coulomb-like parameter α_s is in accordance with the ideas of asymptotic freedom as is expected for the strong gauge coupling constant of QCD, that is, $\alpha_{u\bar{d}} > \alpha_{s\bar{s}} > \alpha_{c\bar{c}} > \alpha_{b\bar{b}}$. Of course a complete analysis would require ingredients that we have not all considered here but discussed in the literature.

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