Equation of State and Multiple Particle Production

Roman N. Rogalyov

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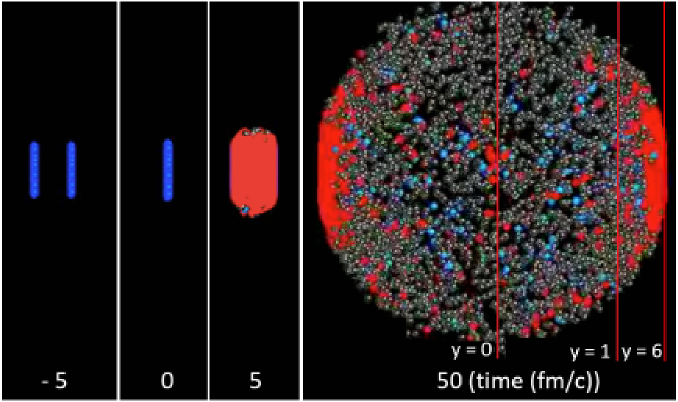
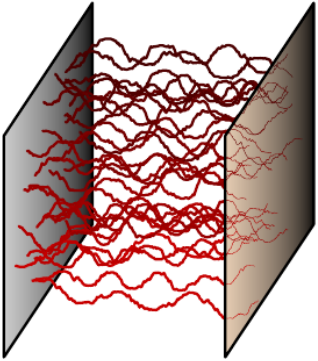
Outline

1. Sketch of QGP formation
2. Quantities under study:
   - net-baryon probability distributions $\mathcal{P}_n$ and $\mathbf{P}_n$;
   - their moments and cumulants;
   - their relation to the pressure and grand canonical partition function.
3. Equation of State (EoS) $p = f(\rho)$ at $T > T_{RW}$ and $T < T_c$.
4. Asymptotic behavior of $\mathbf{P}_n$ at $n \to \infty$.
5. Phenomenological issues.

In collaboration aith V.A.Goy

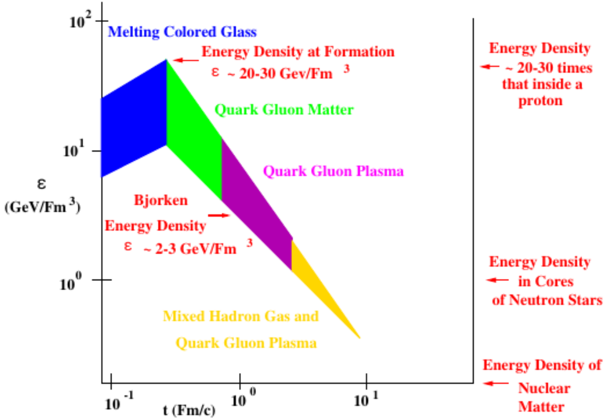


Bird's-eye view of heavy-nuclei collision

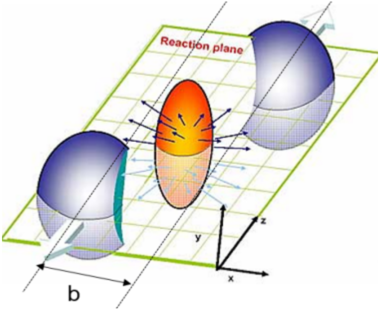


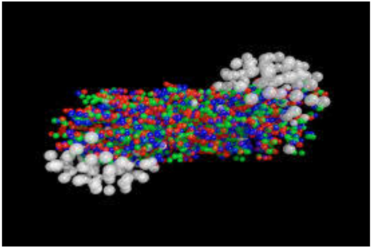
Glazma formation

- Colliding Pb nuclei are $\sim \frac{1}{200}$ fm thick
- Glazma stage
- QGP stage (fireball)
- Freezeout (QGP $\Longrightarrow$ hadrons)



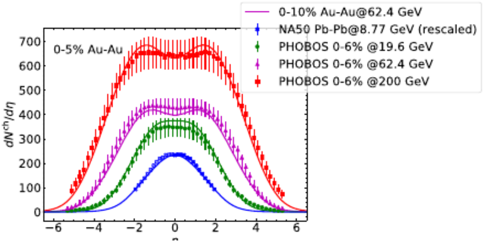
Larry McLerran, hep-ph/0202025

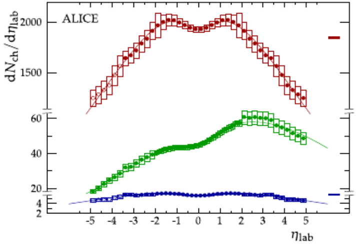

Reaction plane
y
z
x
b



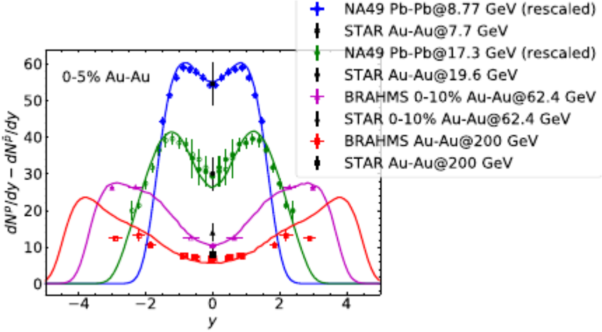
Distribution of charged particles over rapidity

from Lipei Du at al., 2211.16408



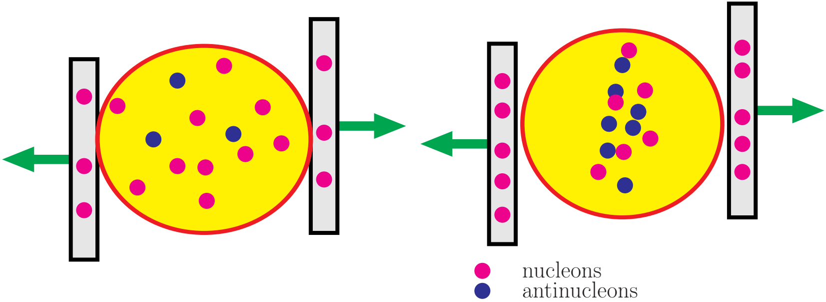


Net-Baryon Number Distribution over rapidity



from Lipei Du at al., 2211.16408

At very high energies,
the net-baryon number seems to be produced
at an initial stage of evolution



$$\sqrt{s_{NN}} < 200 \text{ GeV}$$

$$\sqrt{s_{NN}} > 200 \text{ GeV}$$

We study production of the net-baryon number n:

$$n = N_b - N_{\bar{b}}$$

N_b – number of baryons
$N_{\bar{b}}$ – number of anti-baryons

$$N_b = N_p + N_n + N_{\Xi} + N_{\Lambda} + ... + 3N_{(^3\mathrm{He})} + ...$$

It is generally accepted that

$$\boxed{N_p = 0.4 N_b}$$

The quantities under study:

- probability $\mathcal{P}_n$ that the net baryon charge| of the fireball at a given μ_B equals n (RHIC)
- probability $\mathbf{P}_n$ that the net baryon charge| of the fireball at $\mu_B = 0$ equals n (LHC)
- and the corresponding moments $\mu_k = \sum_{n=-\infty}^{+\infty} \mathcal{P}_n n^k$

$$\mathbf{P}_n = \frac{Z_C(n)}{Z_{GC}(0)}$$

The probabilities $\mathcal{P}_n$ can be determined from

- experimental data

$$N_{events}\,(\text{Net-Baryon Number} = n) \; = \; N_{events}\,(\text{Net-Proton Number} = 0.4n)$$

- lattice simulations
  (of the net-baryon density at imaginary μ_B)
- models of strong-interactiong matter
  - Hadron Resonance Gas (HRG) model
  - Cluster Expansion Model (CEM)
  - Nambu–Jona-Lasinio (NJL or PNJL) model

Grand canonical partition function

$$Z_{GC}(\theta, T, V) \equiv Z_{GC}(\theta) = \sum_j \langle j| \exp\left(\frac{-\hat{H} + \mu\hat{B}}{T}\right)|j\rangle$$

can be expanded as follows:

$$Z_{GC}(\theta) \ = \ \exp\left(\frac{p(\theta)V}{T}\right) \ = \ \sum_{n=-\infty}^{\infty} Z_C(n) e^{n\theta},$$

$$\theta = \frac{\mu_B}{T}$$

The inverse transform

$$Z_C(n) = \int_{-\pi}^{\pi} \frac{d\theta_I}{2\pi} e^{-in\theta_I} Z_{GC}(\theta) \Big|_{\theta_R=0} \, .$$

can be used to determine $Z_C(n)$ and $\mathcal{P}_n$.

Pressure and baryon density are

$$p(\theta) = \frac{T}{V} \ln Z_{GC}(\theta) \qquad \rho(\theta) = \frac{1}{T} \frac{\partial p}{\partial \theta}$$

$\theta = \mu_B/T = \theta_R + \imath\theta_I$, If $\boldsymbol{p}(\theta)$ and $\rho(\theta)$ are known,

they determine the Equation of State (EoS)

$$\boldsymbol{p} = \boldsymbol{p}(\rho)$$

in parametric form

Thus the EoS of fireball matter
is connected with the distribution
of collision events
in the net-baryon number

In lattice QCD at $\mathrm{Re}\mu_B = 0$, $\mathrm{Im}\mu_B \neq 0$
we employ the formula

$$Z_{GC}(\theta) = \int \mathbf{D}U e^{-S_G} (\det \mathcal{D}(\mu_B))^{N_f}$$

to find the net baryon number density ρ
and $\Longrightarrow$ the grand canonical partition function

$$\rho(\theta) = \frac{1}{V} \frac{\partial (T \ln Z_{GC})}{\partial \mu_B} \Longrightarrow$$

$$Z_{GC}(\theta_I)|_{\theta_R = 0} = \exp\left(V \int_0^{\theta_I} \rho(x)\, dx \right)$$

Roberge-Weiss approach in QCD at $\mu_B \neq 0$:

Fock space includes only colorless states at all T and μ_B.

$$\theta \equiv \frac{\mu_B}{T} = \theta_R + \imath\theta_I$$

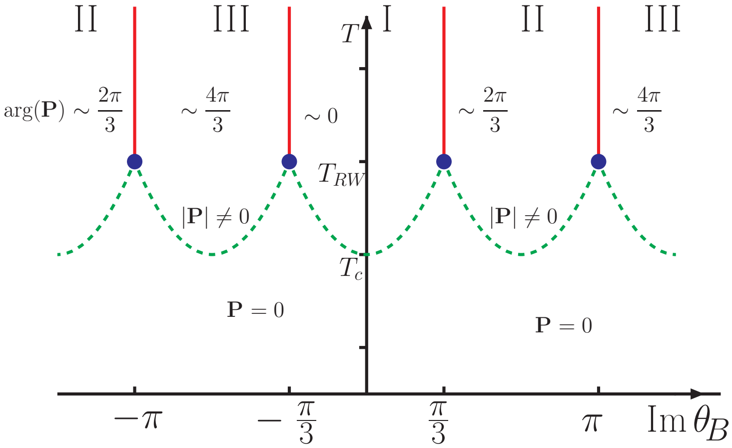
$$Z_{GC}(\theta_I) = Z_{GC}(\theta_I + 2\pi/N_c)$$

$\Downarrow$ $\Downarrow$

Quark number $\mathcal{Q}$ is a multiple of N_c

Grand canonical partition function

$$Z_{GC}(\theta, T, V) = \sum_j \langle j| \exp\left(\frac{-\hat{H} + \mu\hat{\mathcal{Q}}}{T}\right)|j\rangle$$


II
III
I
II
III
T
$\arg(\mathbf{P}) \sim \frac{2\pi}{3}$
$\sim \frac{4\pi}{3}$
~ 0
$\sim \frac{2\pi}{3}$
$\sim \frac{4\pi}{3}$
T_{RW}
$|\mathbf{P}| \neq 0$
$|\mathbf{P}| \neq 0$
T_c
$\mathbf{P} = 0$
$\mathbf{P} = 0$
$-\pi$
$-\frac{\pi}{3}$
$\frac{\pi}{3}$
π
$\mathrm{Im}\,\theta_B$

Results of lattice simulations, general situation

$T > T_{RW}$: $\mathrm{Im}\rho(\theta_I)$ is a periodic function fitted by the polynomial of the type

$$\mathrm{Im}\rho(\theta_I) \simeq a_1\theta_I - a_3\theta_I^3 + ... + \simeq a_n\theta_I^n$$

over each segment $\theta_I^{(n-1)} < \theta_I < \theta_I^{(n)}$,

where $\theta_I^{(n)} = \dfrac{(2n+1)\pi}{3}$;

$T \sim T_c$: $\mathrm{Im}\rho(\theta_I)$ should be fitted by

$$\mathrm{Im}\rho(\theta_I) \simeq f_1 \sin(\theta_I) + f_2 \sin(2\theta) + ... + f_n \sin(n\theta) + ...$$

where $\{f_n\}$ rapidly decreases with n.

Our results of lattice simulations

$T = 1.35T_c > T_{RW}$: $\text{Im}\rho(\theta_I)$ is $\frac{2\pi}{3}$-periodic function with discontinuities at $\theta_I = \frac{(2n+1)\pi}{3}$;

at $|\theta_I| < \frac{\pi}{3}$ is well fitted by the polynomial

$$\text{Im}\rho(\theta_I) \simeq a_1\theta_I - a_3\theta_I^3$$

$T = 0.93T_c$: $\text{Im}\rho(\theta_I)$ is well fitted by the sine

$$\text{Im}\rho(\theta_I) \simeq f_1 \sin(\theta_I)$$

Equation of State

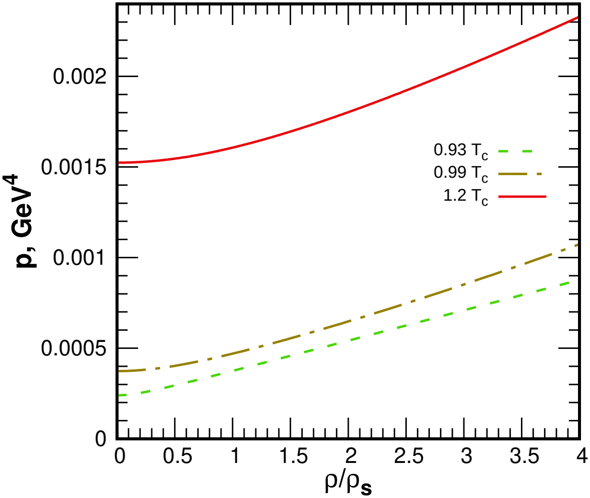
$T = 1.35 T_c > T_{RW}$:

$$\frac{\rho}{T^3} = a_1\theta + a_3\theta^3$$
$$\frac{p}{T^4} = \frac{a_1}{2}\theta^2 + \frac{a_3}{4}\theta^4 + \hat{p}_0,$$

$T = 0.93 T_c$:

$$\frac{\rho}{T^3} = f_1 \,\mathrm{sh}\theta$$
$$\frac{p}{T^4} = f_1 \left(\mathrm{ch}\theta - 1\right) + \hat{p}_0,$$

here $\hat{p}_0 = \left(\text{the pressure}/T^4\right)$ at $\theta = 0$.



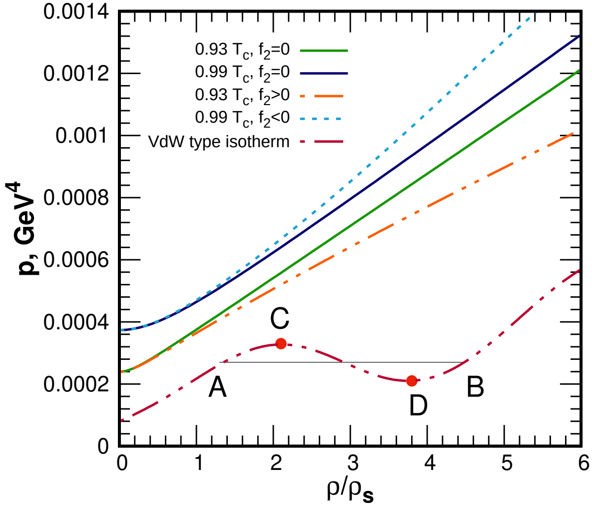
$\rho_s = \mathbf{0.153}$ fm^3; data for $\hat{p}_0$ are taken from HotQCD Collab.

Equation of State at $T \sim T_c$

$T = 0.99T_c$: $\mathrm{Im}\rho(\theta_I) \simeq f_1 \sin(\theta_I) + f_2 \sin(2\theta)$
$f_1 = 0.2541(8)\,, \quad f_2 = -0.0053(7)$

$$\begin{aligned} \hat{\rho} &= f_1\, s + 2 f_2\, s\sqrt{s^2+1}; \\ \hat{p} &= f_1(\sqrt{s^2+1} - 1) + f_2\, s^2 + \hat{p}_0. \end{aligned}$$

here $\hat{p}_0 = p/T^4$ at $\theta = 0$; $s = \sinh(\theta)$; $f_2 < 0$.



Van der Waals isotherm here is hypothetical

CD - is nonphysical part

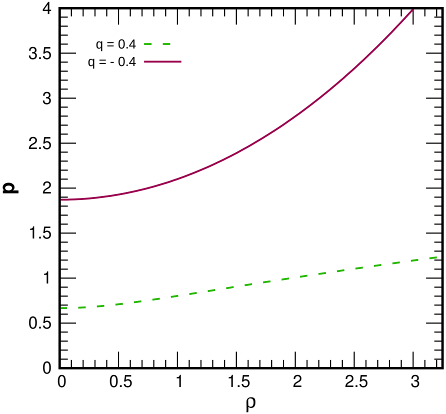
$p(\rho)$ increases at $\theta > 0$!!!

Models for $\rho(\theta)$

simplified versions

- CEM: $T_c < T < T_{RW}$:
  $\rho(\theta) = b\sum_{n=1}^{\infty} q^n \mathrm{sh}(n\theta), \qquad q < 0$
  $T = T_{RW} \implies q = -1, \qquad T \sim T_c \implies q = 0.$

- NJL: $T < T_c$:
  $\rho(\theta) = b\sum_{n=1}^{\infty} q^n \mathrm{sh}(n\theta), \qquad q > 0$



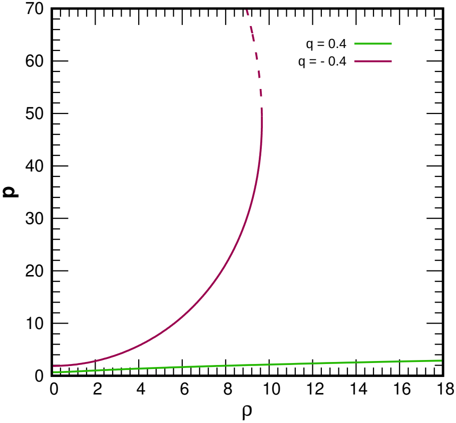
$$\mathbf{p} = \frac{p}{T^4} \text{ versus } \rho = \frac{\varrho}{T^3}$$

Isotherms in

- CEM (q=-0.4)
- NJL (q=0.4)
  (“physical” values $q \sim 10^{-3}$)

$$\mathbf{p} = C + \frac{1 - q \mathrm{ch}\,\theta}{1 + q^2 - 2q \mathrm{ch}\,\theta}$$

$$\rho = \frac{|q|(1-q^2)\mathrm{sh}\,\theta}{(1 + q^2 - 2q \mathrm{ch}\,\theta)^2}$$

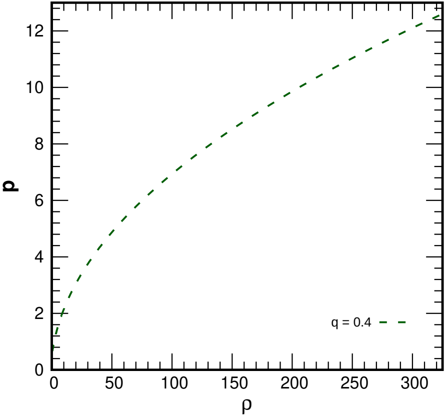


At $q < 0$, the values $\theta > \theta_c$ are unphysical,

where

$$\mathrm{ch}^2\theta_c + \frac{q^2+1}{2q}\mathrm{ch}\theta_c = 2$$

$$\mathbf{p} = \frac{p}{T^4} \text{ versus } \rho = \frac{\varrho}{T^3}$$

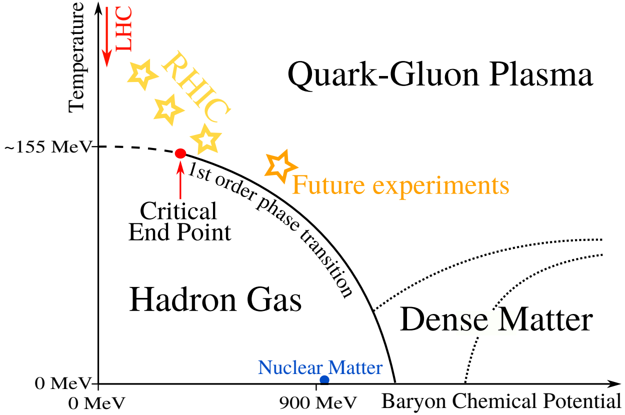


$$\theta \to \left(\ln \frac{1}{q}\right) - 0$$

$$\mathbf{p} \sim \sqrt{\frac{(1-q^2)\rho}{2q}}$$

Asymptotic behavior of the isoterm in NJL

Hypothetical QCD phase diagram



Fireball evolution: $T_{ini}, \mu_B^{(ini)} \longrightarrow T_F, \mu_B^{(F)}$

In addition to the
net-baryon number probability distribution $\mathcal{P}_n$
and its momenta μ_k,
primary attention is focused on

- The moments generating function

$$M(t) = 1 + \sum_{k=1}^{\infty} \frac{\mu_k}{k!}\, t^k$$

- and the cumulant generating function

$$K(t) = \ln M(t) = \sum_{k=1}^{\infty} \frac{\varkappa_k}{k!}\, t^k$$

Net-baryon probability distribution at $\mu_B = 0$

$\mathbf{P}_n \equiv \mathcal{P}_n(\theta = 0)$ involve all info on θ-dependence:

$$\mathcal{P}_n(\theta) = \frac{Z_C(n)e^{n\theta}}{Z_{GC}(\theta)} = \mathbf{P}_n e^{n\theta}\,\frac{Z_{GC}(0)}{Z_{GC}(\theta)}$$

$$M_\theta(t) = \frac{Z_{GC}(t+\theta)}{Z_{GC}(\theta)} \;\longrightarrow\; \mathfrak{M}(t) = \frac{Z_{GC}(t)}{Z_{GC}(0)}$$

$$K_\theta(t) = \;\longrightarrow\; \mathfrak{K}(t) = \frac{\big(p(t) - p(0)\big)V}{T}$$

$$\mathcal{P}_n(\theta) = \frac{Z_C(n)e^{n\theta}}{Z_{GC}(\theta)}$$ - is the probability that

the baryon charge at the given T and μ_B equals n.

C-parity conservation implies $Z_C(n) = Z_C(-n)$

$$\Longrightarrow \qquad \frac{\mathcal{P}_n}{\mathcal{P}_{-n}} = \xi^{2n} \qquad \Longrightarrow \qquad \mu_B = \frac{T}{2n} \ln\left(\frac{\mathcal{P}_n}{\mathcal{P}_{-n}}\right)$$

- possible procedure of measurement of μ_B [A.Nakamura, K.Nagato 2013]
- criterion of thermodynamical equilibrium: μ_B measured for different n coincide

$$\mathbf{P}_n = \frac{Z_C(n)}{Z_{GC}(0)}$$

and the respective cumulants

in contrast to θ-dependent cumulants $\varkappa_n(\theta)$

coincide with the coefficients

of the Taylor expansion of the pressure in θ:

$$p(\theta) = p(0) + \sum_{n=1}^{\infty} \frac{\kappa_{2n}}{(2n)!}\theta^{2n}$$

Main attention is focused on

EXP.: $\varkappa_n(\theta)$ at small n instead of $\mathcal{P}_n(\theta)$

THEOR.: κ_n at small n instead of $\mathbf{P}_n(\theta) \quad \forall n$

because $\kappa_n = \varkappa_n(0)$
are related to the Taylor expansion of the pressure.

We argue that

Asymptotic behavior of $\mathbf{P}_n$ at $n \to \infty$
may become an indicator of the chiral phase transition

Problem: Given κ_n find $\mathbf{P}_n$

$T > T_{RW}$: $\quad \mathbf{P}_n \simeq \exp\left(-\frac{n^2}{2a_1 VT^3}\right), \quad n \ll VT^3$

$$\mathbf{P}_n \simeq \exp\left(-\frac{3}{4}\sqrt[3]{\frac{3}{a_3}}\left(\frac{n}{VT^3}\right)^{4/3}\right), \quad \text{when } n \gg VT^3$$

$T < T_c$: coincidence with the HRG,

$$\mathbf{P}_n \simeq e^{-A} I_n(A)^{\dagger} \quad \Longrightarrow \quad A = 2\sqrt{b\bar{b}}$$

$(\bar{b})b$ is the average number of the (anti)baryons in the fireball

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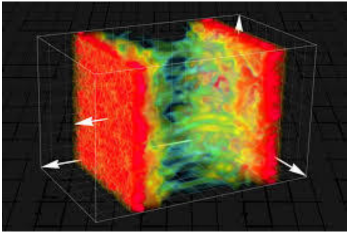
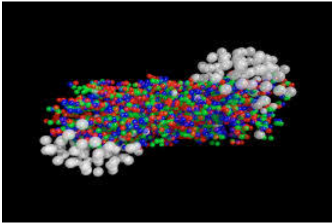
† [Bornyakov et al., 1611.04229]

Two scenarios of thermalization



1. The fireball after formation at an early stage is isolated from the remnants of colliding nuclei.

Evolution starts with the $Z_{GC}(\mu_{ini}, T, V)$ and proceeds with $Z_C(n, T, V)$.



2. Exchange of conserved charges (B, Q, S) proceeds during the fireball expansion.

Grand canonical approach works down to $T_{freezeout}$

Experimental data prefer the latter scenario

ALICE 2019:

$$0.8 < \frac{\kappa_4}{\kappa_2} < 1.0$$

In agreement with the HRG model

Gas of massless fermions (at reasonable values of VT^3):

$$\frac{\kappa_4}{\kappa_2} < 0.2$$

However: the ALICE 2017 result $\mu_B = 0 @ \sqrt{s_{NN}} = 5$ TeV indicates that the former scenario is not completely excluded.

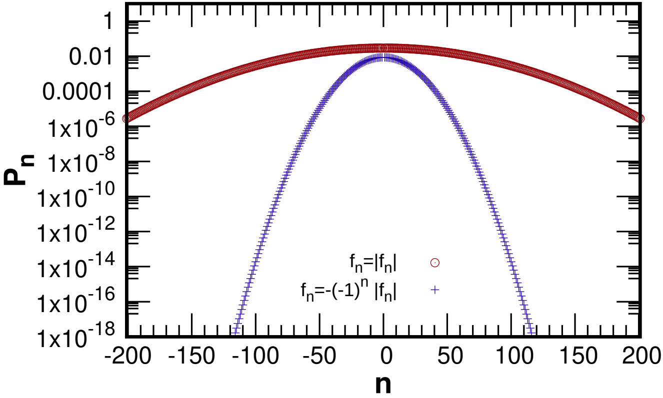
$$\mathrm{Im}\rho(\theta_I) \simeq a_1\theta_I + ... + a_{2J+1}\theta_I^{2J+1},$$

$$\text{sign } a_{2J+1} = (-1)^J$$

$$\mathbf{P}_n \sim \exp\left(-\frac{J}{J+1}\sqrt[J]{\frac{n^{J+1}}{\nu a_J}}\right) \qquad \nu = VT^3 \,.$$

$$\mathrm{Im}\rho(\theta_I) \simeq f_1 \sin(\theta_I) ... + f_J \sin(J\theta), \quad f_J > 0 \ \forall J$$

$$\mathbf{P}_n \sim \frac{(\nu f_J)^{n/J}}{\Gamma\left(\frac{n}{J}+1\right)}, \qquad \nu = VT^3$$



Net-baryon number probability distribution at $\mu_B = 0$ in CEM (blue) and NJL (brown)

The Krein criterion states that the problem of moments becomes indeterminate when

$$\int dx \frac{\ln \varphi(x)}{(1+x^2)} > -\infty, \qquad (1)$$

where $\varphi(x)$ is the probability density function.

The rate of decrease in $\mathbf{P}_n$ at low temperatures is very close to the line of demarcation between probability mass functions generating determinate and indeterminate moment problems

Conclusions:

- Net-baryon number distribution $\mathbf{P}_n$ is evaluated on a lattice at $T > T_{RW}$ (it is similar to but doesn't coincide with the free theory) and at $T < T_c$ (coincides with the HRG predictions).
- The probabilities $\mathbf{P}_n$ can in principle be reconstructed either from the cumulants of the net-baryon number probabilty distribution or from the EoS of strong-interacting matter. Relations between them can shed a new light on fireball evolution.
- The dependence of the EoS on T and fit parameters has been used to formulate a possible scenario of emergence of the van der Waals isotherms corresponding to the first-order chiral phase transition.