New results on the conformal anomaly and anomaly-induced effective action of gravity

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Contents

Some examples of $4D$ conformal theories

- **General scalar action with conformal nonminimal term**

$$S_{scal} = \frac{1}{2}\int d^4x\sqrt{-g}\,\left\{g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{6}R\phi^2\right\}\,,$$

is invariant under local conformal transformation,

$$g_{\mu\nu} \to g'_{\mu\nu} = g_{\mu\nu}\,e^{2\sigma}\,, \quad \phi \to \phi' = \phi\,e^{-\sigma}\,, \qquad \sigma = \sigma(x)\,.$$

R. Penrose, (Les Houches lectures, Gordon and Breach, 1964); N.A. Chernikov & E.A. Tagirov (Ann. Inst. H. Poincare, 1968).

- - **Massless spinor, maybe also coupled to vector or axial vector (antisymmetric torsion)**

$$S_{1/2} = \frac{i}{2}\int d^4x\sqrt{-g}\,\{\,\bar\psi\,\gamma^\mu\,\nabla_\mu\psi - \nabla_\mu\bar\psi\,\gamma^\mu\psi\,\}$$

The transformation rules are

$$\psi \to \psi' = \psi\,e^{k_f\sigma}\,, \quad \bar\psi \to \bar\psi' = \bar\psi\,e^{k_f\sigma}\,, \quad k_f = -\frac{3}{2}$$

- **Electromagnetic field**

$$S_{vec} = \frac{1}{4}\int d^4x\sqrt{-g}\,F_{\mu\nu}\,F^{\mu\nu}\,, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

- **The conformal (Weyl) gravity in the dimension $D = 4$.**

$$S_W = \int d^4x\sqrt{-g}\,C^2(4)\,,$$

$$C^2 = C^{\mu\nu\alpha\beta}\,C_{\mu\nu\alpha\beta} = R^2_{\mu\nu\alpha\beta} - 2R^2_{\mu\nu} + \frac{1}{3}\,R^2\,.$$

In both cases there are several ways to have a generalization to arbitrary dimensional spaces, see, e.g., *arXiv:2107.13125 (EPJP).*

- **Fourth derivative scalar**

$$S_4 = \int d^4x\sqrt{-g}\;\varphi\,\Delta_4\,\varphi\,,$$

where $$\Delta_4 = \Box^2 + 2R^{\mu\nu}\nabla_\mu\nabla_\nu - \frac{2}{3}\,R\Box + \frac{1}{3}\,R_{;\mu}\,\nabla^\mu\,.$$

The transformation law is $\varphi \to \varphi'\,.$

E.S. Fradkin & A.A. Tseytlin, PLB,NPB - 1982.

S.M. Paneitz, MIT preprint - 1983; SIGMA - 2008

- **Antisymmetric tensor field coupled to Dirac fermion**

$$S_B = \int d^4x\sqrt{-g}\,\Big\{\frac{1}{2}\big(W_4 + \lambda W_1\big) - \frac{1}{2}M^2 B_{\mu\nu} - \frac{1}{4!}\big(f_2 W_2 + f_3 W_3\big)\Big\}$$
$$+\, i\int d^4x\sqrt{-g}\,\bar{\psi}\{\gamma^\mu\nabla_\mu - \Sigma^{\mu\nu}B_{\mu\nu} - im\}\psi, \qquad m = M = 0.$$

where $\gamma^\mu = e^\mu_a\gamma^a$, $\Sigma^{\mu\nu} = \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$ **and** λ **is an arbitrary nonminimal parameter and** $f_{2,3}$ **are the quartic self-couplings,**

$$W_1 = \sqrt{-g}\,B^{\mu\nu}B^{\alpha\beta}C_{\alpha\beta\mu\nu},$$
$$W_2 = \sqrt{-g}\,(B_{\mu\nu}B^{\mu\nu})^2, \qquad W_3 = \sqrt{-g}\,B_{\mu\nu}B^{\nu\alpha}B_{\alpha\beta}B^{\beta\mu},$$
$$W_4 = \sqrt{-g}\,\big\{(\nabla_\alpha B_{\mu\nu})(\nabla^\alpha B^{\mu\nu}) - 4(\nabla_\mu B^{\mu\nu})(\nabla^\alpha B_{\alpha\nu})$$
$$+\, 2B^{\mu\nu}R^\alpha_\nu B_{\mu\alpha} - \frac{1}{6}RB_{\mu\nu}B^{\mu\nu}\big\}. \qquad (1)$$

The conformal transformation is $B_{\mu\nu} \to B'_{\mu\nu} = B_{\mu\nu}\,e^\sigma$.

- *T.P. Branson, Com.Part.Diff.Eqs.* ***7*** *(1982) 393;*
- *J. Erdmenger, hep-th/9704108, CQG.*
- *E.S. Fradkin, and A.A. Tseytlin, NPB 203 (1982) 157.*
- *I.Sh., ... Cheshire Cat Smile ..., arXiv:2310.04131*

Quantum (Semiclassical) Theory

Introduction: *Birrell & Davies (1980);*
Buchbinder & I.Sh. (Oxford Un. Press, 2021).

Review (including applications): *I.Sh. gr-qc/0801.0216, CQG.*

The two remarkable statements at the quantum level are

1. The conformal invariance holds in the one-loop divergences.

I.L. Buchbinder, Theor. Math. Phys. **61** *(1984) 393.*

2. Classical conformal invariance is broken by trace anomaly.

D. M. Capper, M. J. Duff and L. Halpern, Ph.R.D 10 (1974) 461;
S. Deser, M.J. Duff and C. Isham, NPB 111 (1976) 45;
M.J. Duff, hep-th/9308075, CQG - review.

The pertinent question is that how these two things can be combined and how these two things can be used together.

Is it possible to have renormalizable, albeit anomalous, theory?

Let us consider what we know about anomaly-induced action.

What could be the building blocks for $\delta\Gamma^{(1)}_{div}$? We can distinguish two types of local conformal symmetry in the vacuum sector:

- **C-type local conformal symmetry:**

$$g_{\mu\nu} = \bar{g}_{\mu\nu}\, e^{2\sigma(x)} \quad \Longrightarrow \quad S_C(g_{\mu\nu}) = S_C(\bar{g}_{\mu\nu})\,. \qquad (C)$$

Strong condition, as it requires cancelation in all orders in σ!

For D-dimensional space this implies cancelation of $1, 2, \dots, D$ powers of σ and its derivatives. Example: $\int C^2$-term in $4D$.

- **N-type local conformal symmetry:**

$$S_N(g_{\mu\nu}) \neq S_N(\bar{g}_{\mu\nu})\,, \qquad \textbf{but} \qquad \frac{1}{\sqrt{-g}}\, g_{\mu\nu}\, \frac{\delta S_N}{\delta g_{\mu\nu}} = 0\,. \qquad (N)$$

Compared to the C-type, here only linear in σ terms should cancel. It is the same as for the global symmetry with $\sigma = const$.

Examples of N-type invariants in $4D$, pure metric case

Vacuum action in semiclassical gravity has the form

$$S_{vac} = S_{EH} + S_{HD}$$

where

$$S_{EH} = \frac{1}{16\pi G} \int d^4x \sqrt{-g}\, \{ R + 2\Lambda \} \,.$$

is the Einstein-Hilbert action with a CC.

S_{HD} **includes higher derivative terms**

$$S_{HD} = \int d^4x \sqrt{-g}\, \{ a_1 C^2 + a_2 E + a_3 \Box R + a_4 R^2 \} \,,$$

1. Topological term $E = R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - 4\, R_{\alpha\beta} R^{\alpha\beta} + R^2$

is the integrand of the 4D **Gauss-Bonnet topological invariant.**

2. Surface term $a_3 \Box R$.

Conformal anomaly

k_Φ **is the conformal weight of the field** Φ.
The Noether identity for the local conformal symmetry

$$\Big(-\, 2\, g_{\mu\nu}\, \frac{\delta}{\delta g_{\mu\nu}} + k_\Phi\, \Phi\, \frac{\delta}{\delta \Phi} \Big)\, S(g_{\mu\nu},\, \Phi) \,=\, 0$$

produces on shell $\quad -\,\frac{2}{\sqrt{g}}\, g_{\mu\nu}\, \frac{\delta S_{vac}(g_{\mu\nu})}{\delta g_{\mu\nu}} \,=\, T_{(vac)}{}_\mu^{\ \mu} \,=\, T_\mu^\mu \,=\, 0\,.$

At quantum level $S_{vac}(g_{\mu\nu})$ **is replaced by the EA** $\Gamma_{vac}(g_{\mu\nu})\,.$

For free fields only 1-loop order is relevant

$$\Gamma_{div} \,=\, -\,\frac{1}{n-4}\, \int d^4x \sqrt{g}\, \big\{ \beta_1 C^2 + \beta_2 E + \beta_3 \Box R \big\}\,.$$

For the global conf. symmetry the renormalization group tells us

$$\langle T_\mu^\mu \rangle \,=\, \big\{ \beta_1 C^2 + \beta_2 E + a' \Box R \big\}\,,$$

where $a' = \beta_3$. **In the local case** a' is **ambiguous**.

The simplest way to derive the conformal anomaly is using dimensional regularization (Duff, 1977).

The expression for divergences

$$\bar{\Gamma}_{div} = \frac{1}{n-4} \int d^4x\sqrt{g}\,\{\beta_1 C^2 + \beta_2 E + \beta_3 \Box R\} \ .$$

where

$$\begin{pmatrix} \beta_1 \\ -\beta_2 \\ \beta_3 \end{pmatrix} = \frac{1}{360(4\pi)^2} \begin{pmatrix} 3N_0 + 18N_{1/2} + 36N_1 \\ N_0 + 11N_{1/2} + 62N_1 \\ 2N_0 + 12N_{1/2} - 36N_1 \end{pmatrix}$$

The renormalized one-loop effective action has the form

$$\Gamma_R = S + \bar{\Gamma} + \Delta S \,,$$

where $\bar{\Gamma} = \bar{\Gamma}_{div} + \bar{\Gamma}_{fin}$ **is the naive quantum correction to the classical action and** ΔS **is a counterterm.**

ΔS **is an infinite local counterterm which is called to cancel the divergence. It is the only source of non-invariance.**

The anomalous trace is

$$T = \langle T^\mu_\mu \rangle = -\frac{2}{\sqrt{-g}}\, g_{\mu\nu}\, \frac{\delta \Gamma_R}{\delta g_{\mu\nu}}\bigg|_{n=4} = -\frac{2}{\sqrt{-g}}\, g_{\mu\nu}\, \frac{\delta \Delta S}{\delta g_{\mu\nu}}\bigg|_{n=4} .$$

Conformal parametrization of the metric:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} \cdot e^{2\sigma} , \qquad \sigma = \sigma(x)$$

where $\bar{g}_{\mu\nu}$ is the fiducial metric with fixed determinant.

There is a useful relation

$$\frac{2}{\sqrt{-g}}\, g_{\mu\nu} \frac{\delta A[g_{\mu\nu}]}{\delta g_{\mu\nu}} = \frac{1}{\sqrt{-\bar{g}}}\, \frac{\delta A[\bar{g}_{\mu\nu}\, e^{2\sigma}]}{\delta\sigma}\bigg|_{\bar{g}_{\mu\nu}\to g_{\mu\nu},\, \sigma\to 0,\, n\to 4} \qquad (*)$$

$$\int d^n x \sqrt{-g}\, C^2(n) = \int d^n x \sqrt{-\bar{g}}\, e^{(n-4)\sigma}\, \bar{C}^2(n) .$$

Then $$\frac{\delta}{\delta\sigma} \int \frac{d^4 x \sqrt{-\bar{g}}}{n-4}\, e^{(n-4)\sigma}\, \bar{C}^2(n)\bigg|_{n\to 4} = \sqrt{-g}\, C^2 .$$

The terms with derivatives of $\sigma(x)$ are irrelevant.

In the simplest case $\sigma = \lambda = const$, we immediately arrive at the expression for T with $a' = \beta_3$.

For global conformal transform this procedure always works,

$$\langle T^\mu_\mu \rangle = \frac{1}{(4\pi)^2} \left(\omega C^2 + bE + c\Box R \right) .$$

However the local case $\sigma(x)$ is more complicated, e.g.,

$$\frac{\delta}{\delta g_{\mu\nu}} \int \sqrt{-g}\Box R \equiv 0 \, .$$

We have a conflict between global and local conf. anomalies.

Or a conflict between formulas and intuitive expectations.
M.J. Duff, Class. Quantum. Grav. (1994)

Problem resolved:
M. Asorey, E. Gorbar & I.Sh., CQG 21 (2003).

Classification of terms in anomaly is based on C/N separation: conformal terms vs topological term vs surface terms.

S. Deser, M.J. Duff and C.J. Isham, NPB (1976).

S. Deser, and A. Schwimmer, PLB (1993), hep-th/9302047.

S. Deser, PLB (2000), hep-th/9911129.

There is universality of signs of the coefficients of Weyl invariant (C-type) and topological terms.

The related property of the renormalization group flows probably holds beyond the one-loop level, at least for some QFT models.

This opened a new area which is known as c- and a-theorems,

Z. Komargodski, A. Schwimmer, JHEP (2011), arXiv:1107.3987.

Z. Komargodski, JHEP (2012), arXiv:1112.4538.

M.A. Luty, J. Polchinski, R. Rattazzi, JHEP (2013), arXiv:1204.5221.

Anomaly-induced Effective Action (EA) of vacuum.

One can use the trace (conformal) anomaly $\langle T^\mu_\mu \rangle$ to derive the finite part of the one-loop EA

$$\frac{2}{\sqrt{-g}}\, g_{\mu\nu} \frac{\delta\, \bar{\Gamma}_{ind}}{\delta g_{\mu\nu}} \;=\; \langle T^\mu_\mu \rangle\,.$$

In $2D$ the equation to solve is

$$\frac{2}{\sqrt{-g}}\, g_{\mu\nu} \frac{\delta\, \bar{\Gamma}_{ind}}{\delta g_{\mu\nu}} \;=\; \langle T^\mu_\mu \rangle \;=\; \frac{1}{(4\pi)}\, aR\,.$$

In $4D$ we meet the equation

$$\frac{2}{\sqrt{-g}}\, g_{\mu\nu} \frac{\delta\, \bar{\Gamma}_{ind}}{\delta g_{\mu\nu}} \;=\; \langle T^\mu_\mu \rangle \;=\; \frac{1}{(4\pi)^2}\, \big(\, \omega C^2 + bE + c\Box R \,\big)\,.$$

R.J. Riegert, Phys. Lett. **B134** *(1984) 56.*
E.S. Fradkin and A.A. Tseytlin, Phys. Lett. **B134** *(1984) 187.*

Particular dimensions

$\underline{2D}$. **There are no C-type invariants and only one N-type invariant,**

$$S_2(g_{\mu\nu}) \,=\, \int d^2x\sqrt{-g}\,R \quad\Longrightarrow\quad \langle T^\mu_\mu\rangle \,=\, aR\,,$$

Integration of anomaly is simple and yields the Polyakov action

$$\Gamma_{ind}(g_{\mu\nu}) \,=\, \frac{a}{4}\int d^2x\sqrt{-g}\,R\,\frac{1}{\Box}\,R\,,$$

A.M. Polyakov, Phys. Lett. B (1981).

$\underline{4D}$. **There is one C-type invariant, and two N-type invariants,**

$$\langle T^\mu_\mu\rangle \,=\, -\omega C^2 - bE - c\Box R\,,$$

where

$$\begin{pmatrix}\omega\\ b\\ c\end{pmatrix} = \frac{1}{360(4\pi)^2}\begin{pmatrix}+3N_0 + 18N_{1/2} + 36N_1\\ -N_0 - 11N_{1/2} - 62N_1\\ +2N_0 + 12N_{1/2} - 36N_1\end{pmatrix}.$$

In $4D$ we can obtain the non-local covariant solution for the anomaly-induced effective action of vacuum.

First one has to establish the relations

$$\sqrt{-g}C^2 = \sqrt{-\bar{g}}\bar{C}^2\,, \qquad \sqrt{-\bar{g}}\,\bar{\Delta}_4 = \sqrt{-g}\,\Delta_4\,,$$

$$\sqrt{-g}\Big(E - \frac{2}{3}\Box R\Big) \;=\; \sqrt{-\bar{g}}\Big(\bar{E} - \frac{2}{3}\bar{\Box}\bar{R} + 4\bar{\Delta}_4\sigma\Big) \qquad !!$$

and also introduce the Green function

$$\sqrt{-g}\Delta_4\, G(x,y) = \delta(x,y)\,.$$

Using these formulas we find, for a functional $A(g_{\mu\nu}) = A(\bar{g}_{\mu\nu})$,

$$\frac{\delta}{\delta\sigma}\int_x A\,\left(E - \frac{2}{3}\Box R\right)\Big| \;=\; 4\sqrt{-g}\Delta_4\,A\,,$$

where $\displaystyle\int_x = \int d^4x\sqrt{-g(x)}\,, \qquad \Big| = \Big|_{\bar{g}_{\mu\nu}\to g_{\mu\nu}}\,.$

As a consequence, we obtain

$$\frac{\delta}{\delta\sigma(y)} \iint\limits_{x\,z} \frac{1}{4}\, C^2(x)\, G(x,z)\, \Big(E - \frac{2}{3}\Box R\Big)_z \Big|$$

$$= \int d^4x \sqrt{-\bar{g}(x)}\, \bar{\Delta}_4(x)\, \bar{G}(x,y)\, \bar{C}^2(x)\Big| \;=\; \sqrt{-g(y)}\, C^2(y)\,.$$

Hence, the part of Γ_{ind} which is responsible for $T_\omega = -\omega C^2$, is

$$\Gamma_\omega = \frac{\omega}{4} \iint\limits_{x\,y} C^2(x)\, G(x,y)\, \Big(E - \frac{2}{3}\Box R\Big)_y\,.$$

Similarly, one can check that the variation $T_b = -b\,(E - \frac{2}{3}\Box R)$ is produced by the term

$$\Gamma_b = \frac{b}{8} \iint\limits_{x\,y} \Big(E - \frac{2}{3}\Box R\Big)_x\, G(x,y)\, \Big(E - \frac{2}{3}\Box R\Big)_y\,.$$

Finally, we can use the simple relation

$$- \frac{2}{\sqrt{-g}}\, g_{\mu\nu}\, \frac{\delta}{\delta g_{\mu\nu}} \int_x R^2(x) \,=\, 12\,\Box R\,.$$

to establish the remaining local constituent of Γ_{ind}:

$$\Gamma_c = -\frac{3c+2b}{36(4\pi)^2} \int_x R^2(x)\,.$$

The general covariant solution for Γ_{ind} is the sum,

$$\Gamma_{ind} = S_c[g_{\mu\nu}] \,-\, \frac{3c+2b}{36(4\pi)^2} \int_x R^2(x)$$

$$+ \frac{\omega}{4} \int_x \int_y C^2(x)\, G(x,y) \left(E - \frac{2}{3}\Box R\right)_y$$

$$+ \frac{b}{8} \int_x \int_y \left(E - \frac{2}{3}\Box R\right)_x G(x,y) \left(E - \frac{2}{3}\Box R\right)_y\,.$$

One can rewrite this expression using auxiliary scalars.

The nonlocal terms can be recast into symmetric form

$$\left(E - \frac{2}{3}\Box R\right)_x G(x,y)\left[\frac{\omega}{4}C^2 - \frac{b}{8}(E - \frac{2}{3}\Box R)\right]_y$$

$$= \frac{b}{8}\iint_{x\,y}\left(E - \frac{2}{3}\Box R - \frac{\omega}{b}C^2\right)_x G(x,y)\left(E - \frac{2}{3}\Box R - \frac{\omega}{b}C^2\right)_y$$

$$- \frac{\omega^2}{8b}\iint_{x\,y} C_x^2\, G(x,y)\, C_y^2\,.$$

Thus we arrive at the local covariant expression for EA

$$\Gamma_{ind} = S_c[g_{\mu\nu}] - \frac{3c+2b}{36(4\pi)^2}\int_x R^2(x) + \int_x \Big\{\frac{1}{2}\,\varphi\Delta_4\varphi - \frac{1}{2}\,\psi\Delta_4\psi$$

$$+ \frac{\omega}{8\pi\sqrt{-b}}\,\psi\, C^2 + \varphi\left[\,\frac{\sqrt{-b}}{8\pi}\left(E - \frac{2}{3}\,\Box R\right) - \frac{\omega}{8\pi\sqrt{-b}}\,C^2\right]\Big\}\,.$$

The above form of EA is the best one for Γ_{ind}.
I.Sh. and A.Jacksenaev, Phys. Lett. B (1994)

Similar expression has been independently introduced by
P. Mazur & E. Mottola, 1997-1998-2001.

Comments:

1) Imposing boundary conditions on the two auxiliary fields φ and ψ is equivalent to defining boundary conditions for the Green functions $G(x,y)$.

2) Introducing the new term $\int C_x^2 G(x,y) C_y^2$ into the action may be viewed as redefinition of the conformal functional $S_c[g_{\mu\nu}]$.

However, writing the non-conformal terms in the symmetric form, essentially modifies the four-point function.
Using ψ we restore the structure generated by anomaly.

General perspective: a few mysteries remains

- **Why the beta-functions for conformal and topological terms, which come from fields of all spins, have the same sign?**

- **And why exists the "main formula"**

$$\sqrt{-g}\Big(E - \frac{2}{3}\Box R\Big) = \sqrt{-\bar{g}}\Big(\bar{E} - \frac{2}{3}\bar{\Box}\bar{R} + 4\bar{\Delta}_4\sigma\Big) \quad \textbf{??}$$

And why the "magic" $2/3$ coefficient? It there some general mathematical rule behind this occurrence?

- **Since our knowledge is restricted, we need further examples.**
- **But before that let us consider some general features.**

$\forall D$. **Several C-type invariants, and two kinds of N-type invariants,**

$$T = \langle T^\mu_\mu \rangle = c_r\, W^r_D + a\, E_D + \Xi_D\,,$$

with the sum over r. Here W^r_D are conformal invariant terms and

$$E_D = D^{-1}\, \varepsilon^{\rho_1 \cdots \rho_D}\, \varepsilon^{\sigma_1 \dots \sigma_D}\, R_{\rho_1\sigma_1\rho_2\sigma_2} \cdots R_{\rho_{D-1}\sigma_{D-1}\rho_D\sigma_D}.$$

Example: $\underline{6D}$. **There are three C-type terms,**

$$W_6^1 = C_{\mu\nu\rho\sigma}\, C^{\mu\alpha\beta\nu}\, C_{\alpha\,\cdot\,\cdot\,\beta}^{\;\;\rho\sigma}\,,$$

$$W_6^2 = C_{\mu\nu\rho\sigma}\, C^{\rho\sigma\alpha\beta}\, C_{\alpha\beta\,\cdot\,\cdot}^{\;\;\;\;\mu\nu}\,,$$

$$W_6^3 = C_{\mu\rho\sigma\lambda}\Big(\delta_\nu^\mu + 4R_\nu^\mu - \frac{6}{5}\,R\delta_\nu^\mu\Big)\, C^{\nu\rho\sigma\lambda} \,+\, \nabla_\mu J^\mu\,,$$

where $\quad J_\mu \,=\, \big(4R_\mu^{\,\cdot\lambda\rho\sigma}\nabla^\nu + 3R^{\nu\lambda\rho\sigma}\nabla_\mu\big)R_{\nu\lambda\rho\sigma}$

$$+\Big(\frac{1}{2}R\nabla_\mu - R_\mu^\nu\nabla_\nu\Big)R + R^{\nu\lambda}\big(\nabla_\nu R_{\lambda\mu} - 5\nabla_\mu R_{\nu\lambda}\big).$$

This is a more complicated case. There are also several N-type terms, including one topological and others total derivatives.

F. Bastianelli, S. Frolov, & A. Tseytlin, JHEP (2000), hep-th/0001041.

For a general dimension, Ξ_D **is a linear combination of surface terms,** $\Xi_D = \sum \gamma_k\chi_k$. **The numerical coefficients** a, c, γ_k **depend on the number of massless conformal fields of different spins.**

Our purpose is to find the anomaly-induced EA Γ_{ind} , such that

$$- \frac{2}{\sqrt{-g}}\, g_{\mu\nu}\, \frac{\delta\Gamma_{ind}}{\delta g_{\mu\nu}} \;=\; T\,.$$

Integration of anomaly requires modified topological invariant

$$\tilde{E}_D = E_D + \sum \alpha_k \chi_k\,, \qquad\qquad (*)$$

where the values of α_k are chosen to provide the special conformal property of the "improved" topological term.

Under the local conformal transformation $\quad g_{\mu\nu} = \bar{g}_{\mu\nu}\, e^{2\sigma(x)}$
there should be $\quad \sqrt{-g}\tilde{E}_D \;=\; \sqrt{-\bar{g}}\big(\bar{\tilde{E}}_D + \kappa\bar{\Delta}_D\sigma\big)$,

where κ is a constant and $\Delta_D = \Box^{D/2} + \dots$ is a conformal operator acting on a conformally inert scalar.

The existence of such a modified topological term (*) for the general D is an important unproved conjecture.

F. Ferreira, I.Sh, P. Teixeira, EPJ+ **131** *(2016) , 1507.03620.*

An explicit solution in $6D$

F. Ferreira, I.Sh, arXiv:1702.06892 (PLB); 1812.01140 (PRD).

The candidate terms to the total derivatives in $\Gamma^{(1)}_{div}$ and $\langle T \rangle$

$$\chi_1 = \Box^2 R\,, \qquad \chi_{2,3,4} = \Box\big(R^2_{\mu\nu\alpha\beta},\, R^2_{\mu\nu},\, R^2\big),$$

$$\chi_{5,6,7,8} = \nabla_\mu \nabla_\nu \big(R^\mu{}_{\lambda\alpha\beta} R^{\nu\lambda\alpha\beta},\; R_{\alpha\beta} R^{\mu\alpha\nu\beta},\; R^\mu_\alpha R^{\nu\alpha},\; R R^{\mu\nu}\big),$$

where the Noether identity

$$\chi_2 - 4\chi_3 + \chi_4 - 4\chi_5 + 8\chi_6 + 8\chi_7 - 4\chi_8 = 0$$

corresponds to both conformal and diff. symmetries.

The result for $D = 6$ is $\quad \tilde{E}_6 = E_6 + \sum_k \alpha_k \chi_k\,,$

with $\quad \alpha_1 = \frac{3}{5}\,, \quad \alpha_2 = -\frac{9}{10} - \frac{5}{4}\xi_1 + \frac{3}{8}\xi_2\,, \quad \alpha_3 = \xi_1\,, \quad \alpha_4 = 0\,,$

$$\alpha_5 = \frac{84}{5} + 3\xi_1 + \frac{11}{2}\xi_2\,, \quad \alpha_6 = -\frac{36}{5} - 2\xi_1 - 5\xi_2\,,$$

$\alpha_7 = -\frac{18}{5} - \xi_1 - \frac{7}{2}\xi_2\,, \quad \alpha_8 = \xi_2$ **and** $\forall\; \xi_1,\, \xi_2\,.$

The conformal operator is (coincides with *K. Hamada, Prog. Theor. Phys. (2001), hep-th/0012053*)

$$\Delta_6 = \Box^3 + 4R^{\mu\nu}\nabla_\mu\nabla_\nu\Box - R\Box^2 + 4(\nabla^\alpha R^{\mu\nu})\nabla_\alpha\nabla_\mu\nabla_\nu + V^{\mu\nu}\nabla_\mu\nabla_\nu + N^\lambda\nabla_\lambda\,,$$

where

$$V^{\mu\nu} = \Big(\frac{78}{5} + \xi_1 + \frac{3\xi_2}{2}\Big)\Big(R^{\mu\alpha}R^\nu_\alpha - \frac{1}{3}RR^{\mu\nu}\Big) + \Big[\Big(1 + \frac{\xi_2}{6}\Big)R^2 - \Big(\frac{29}{5} - \frac{\xi_1}{6} + \frac{17\xi_2}{12}\Big)R^2_{\rho\sigma} + \Big(\frac{16}{5} - \frac{\xi_1}{3} + \frac{7\xi_2}{6}\Big)R^2_{\rho\sigma\alpha\beta} - \frac{3}{5}(\Box R)\Big]g^{\mu\nu} + \Big(\frac{64}{5} + \frac{4\xi_1}{3} + 2\xi_2\Big)\Big(R_{\alpha\beta}R^{\mu\alpha\nu\beta} - R^\mu{}_{\alpha\beta\gamma}R^{\nu\alpha\beta\gamma}\Big)$$

and

$$N^\lambda = \frac{2}{5}(\nabla^\lambda\Box R) + \frac{8}{3}(\xi_1 - \xi_2)R_{\alpha\beta\rho\sigma}(\nabla^\rho R^{\alpha\beta\sigma\lambda}) - \Big(\frac{3}{5} + \frac{\xi_1}{6} - \frac{\xi_2}{12}\Big)R(\nabla^\lambda R) + \Big(\frac{14}{5} - \frac{\xi_1}{3} - \frac{\xi_2}{2}\Big)R_{\rho\sigma}(\nabla^\rho R^{\sigma\lambda}) + \Big(\frac{6}{5} + \frac{5\xi_1}{3} - \frac{5\xi_2}{6}\Big)R_{\rho\sigma}(\nabla^\lambda R^{\rho\sigma}) + \Big(\frac{13}{5} + \frac{\xi_1}{6} + \frac{\xi_2}{4}\Big)R^{\rho\lambda}(\nabla_\rho R) + \Big(\frac{64}{5} + \frac{4\xi_1}{3} + 2\xi_2\Big)R^{\rho\sigma\alpha\lambda}(\nabla_\rho R_{\sigma\alpha})\,.$$

To integrate anomaly, one has to find local metric-dependent Lagrangians $\mathcal{L}_i$ providing that, with some coefficients c_{ik} , there is an identity (always works, but explanations unknown)

$$- \frac{2}{\sqrt{-g}}\, g_{\mu\nu}\, \frac{\delta}{\delta g_{\mu\nu}}\, \sum_i c_{ik} \int_x \mathcal{L}_i \,=\, \chi_k \,.$$

After that, the problem reduces to integrating the first two terms in anomaly, using Green function

$$\sqrt{-g}\,\Delta_D^x\, G(x,x') \,=\, \delta^D(x,x')\,, \qquad G \,=\, \bar{G}\,.$$

The non-local solution for the anomaly-induced EA is

$$\Gamma_{ind} \,=\, S_c + \iint_{x\,y} \Big\{ \frac{1}{4}\, c_r\, W_D^r \,+\, \frac{b}{8}\, \tilde{E}_D(x) \Big\}\, G(x,y)\, \tilde{E}_D(y)$$

$$+ \sum_k \big(\gamma_k - \alpha_k\big) \sum_i c_{ik} \int_x \mathcal{L}_i \,.$$

The modification of the coefficients $\gamma_k \to \gamma_k - \alpha_k$ takes place since part of the surface terms were absorbed into $\tilde{E}_D$.

One can construct a local covariant presentation with the use of two auxiliary fields

$$\bar{\Gamma} = S_c + \sum_k (\gamma_k - \alpha_k) \sum_i c_{ik} \int_x \mathcal{L}_i$$

$$+ \frac{1}{2} \int_x \Big\{ \varphi \Delta_D \varphi - \psi \Delta_D \psi + \sqrt{-b}\, \varphi\, \tilde{E}_D + \frac{1}{\sqrt{-b}} (\psi - \varphi)\, c_r\, W_D^r(x) \Big\} .$$

Here we assumed $b < 0$, as in the $4D$ case.
For the opposite sign one can modify $\tilde{E}_D \to -\tilde{E}_D$.

It is remarkable that the leading UV part of the vacuum effective action in an arbitrary even dimension can be given in such an extremely simple and general form.

However, this universal representation exists only under the assumption of the universal conformal property of the topological term in an arbitrary dimension.

Does anomaly mean the conformal theories are necessary nonrenormalizable?

1. **Renormalizability of the theory depends on the form of the counterterms required to cancel divergences in the given order.**

2. **UV divergences are removed by adding local counterterms. The conformal models are not supposed to be different.**

3. **Anomaly-induced action includes nonlocal and local terms.**

4. **It is unlikely that nonlocal terms from the subdiagrams produce local divergent terms in the superficial integration. Thus, the nonlocal terms in the effective action may be not important for renormalizability.**

5. **The local terms may depend on the renormalization scheme. In principle, there may be a scheme providing the absence of nonconformal local terms, even in higher loops.**

6. **It is clearly important to understand the ambiguities in the local terms, starting from the one-loop approximation.**

Examples of the ambiguities of local terms

I. Dimensional regularization.

The source of arbitrariness is that the term C^2 can be extended for $n \neq 4$ as $C^2(d)$ where $d = n + \gamma(n-4)$ with arbitrary γ. E.g.,

$$\frac{2}{\sqrt{-g}}\, g_{\mu\nu}\, \frac{\delta}{\delta g_{\mu\nu}}\, \frac{\mu^{n-4}}{n-4} \int d^n x \sqrt{-g}\, \beta_1\, C^2(4)\Big|_{n\to 4} = C^2 - \frac{2}{3}\,\Box R\,.$$

In general, $\Box R$ term in the anomaly and, consequently, the finite local $\int_x R^2$ term in the effective action, depend on γ.

M. Asorey, E. Gorbar, I.Sh., hep-th/0307187 (Class. Quant. Grav.)

The standard solution to this problem is to "add a finite conterterm" $\int_x R^2$ and eliminate this term once and forever.

This operation is possible because $\int_x R^2$ term belongs to the vacuum sector of the effective action and does not break the conformal symmetry in the sector of quantum fields.

Examples of the ambiguities of local terms

II. Covariant Pauli-Villars (PV) regularization.

PV regularization requires introducing scalar fields with different masses and nonminimal parameters

$$S_{\rm reg} = \sum_{i=0}^{N} \int d^4x\sqrt{-g}\,\Big\{ \frac{1}{2}\,g^{\mu\nu}\partial_\mu\,\varphi_i\partial_\nu\varphi_i + \frac{\xi_i}{2}\,R\,\varphi_i^2 + \frac{m_i^2}{2}\,\varphi_i^2 \Big\}$$

The PV fields φ_i **are massive** $m_i = \mu_i M \neq 0$ **and may have either bosonic or fermionic statistics.**

One can choose masses and statistics in such a way that all divergences in the theory are cancelled. This regularization implies that $M \to \infty$ **at the end.**

The anomaly can be obtained in this limit using the nonlocal form factors derived in *E.V. Gorbar, I.Sh., hep-ph/0210388, JHEP.*

It turns out that the result for the $\Box R$**- term in the anomaly depends on the choice of** ξ_i.

The effective action of metric and external scalar field. The non-local form:

$$\Gamma_{ind} = S_c - \int_x \Big(\frac{2b+3c}{36}R^2 + \frac{\beta_\tau}{6}R\Phi^2\Big) - \frac{1}{8b}\iint_{x,y} Y(x)G(x,y)Y(y)$$

$$+ \frac{b}{8}\iint_{x,y} \Big(E_4 - \frac{2}{3}\Box R + \frac{1}{b}Y\Big)_x G(x,y)\Big(E_4 - \frac{2}{3}\Box R + \frac{1}{b}Y\Big)_y$$

with $Y(g_{\mu\nu},\Phi) = wC^2 - \gamma_\Phi\Big[(\nabla\Phi)^2 + \frac{1}{6}R\Phi^2\Big] + \frac{1}{4!}\widetilde{\beta}_\lambda\Phi^4.$

Here the local terms have the mentioned ambiguity.

M. Asorey, W.C. Silva, I.Sh., P. do Vale, arXiv:2202.00154 (EPJP).

There is also local representation with two auxiliary fields

$$\Gamma_{ind} = S_c[g_{\mu\nu},\Phi] - \int_x \Big\{\frac{2b+3c}{36}R^2 + \frac{\beta_\tau}{6}R\Phi^2\Big\} + \int_x \Big\{\frac{1}{2}\varphi\Delta_4\varphi$$

$$- \frac{1}{2}\psi\Delta_4\psi + \frac{\sqrt{-b}}{2}\varphi\Big(E_4 - \frac{2}{3}\Box R + \frac{1}{b}Y\Big) + \frac{1}{2\sqrt{-b}}\psi Y\Big\}.$$

Different from the vacuum sector, the ambiguities in the $R\Phi^2$- term affect the invariance of the quantum fields and produce nonconformal divergences in higher loops, see e.g.

S.J. Hathrell, Ann. Phys. 139 (1982) 136.

Thus, the hope to have conformal theory renormalizable is essentially based on the ambiguity in these terms.

One has to elaborate a detailed scheme of regularization, then define the scheme of renormalization and achieve fixing the ambiguities, such that in any order of loop expansion there would be no nonconformal local terms.

Such a scheme would certainly include a special regularization and a lot of fine-tunings for the relevant coefficients.

Only in this special scheme one can think on nonperturbative universality properties of the anomaly coefficients.

Anomaly-induced effective action in the IR

To derive the IR limit from the anomaly we need a set of additional assumptions

M. Giannotti and E. Mottola, arXiv:0812.0351 (PRD).
M. Asorey, W.C. Silva, I.Sh., P. do Vale, arXiv:2202.00154 (EPJP).

i) All matter fields are approximately massless and conformal.

ii) Scalar terms Φ^4, X_c are dominating over the curvature terms

$$\left|\Phi^2\right| \gg \left|R_{....}\right| \qquad \text{and} \qquad \left|(\nabla\Phi)^2\right| \gg \left|R^2_{....}\right|$$

for all components of the curvature tensor.

iii) As always in GR, the IR limit means the gravitational field is weak, including the linear order in the metric perturbations over the flat background dominating.

This implies $|\Box R| \gg |R^2_{....}|$ **for all curvature contractions.**

Anomaly-induced effective action in the IR

The non-local structures in the induced action reduce to

$$G = \Delta_4^{-1} = \Big(\Box^2 + 2R^{\mu\nu}\nabla_\mu\nabla_\nu - \frac{2}{3}R\Box + \frac{1}{3}R^{;\mu}\nabla_\mu\Big)^{-1} \approx \frac{1}{\Box^2}.$$

The leading terms are those with Φ and $\Box R$, hence

$$E_4 - \frac{2}{3}\Box R + \frac{1}{b}Y \approx -\frac{2}{3}\Box R - \frac{1}{b}\Big(\gamma_\Phi X_c - \frac{1}{4!}\widetilde{\beta}_\lambda \Phi^4\Big),$$

where $X_c \equiv \frac{1}{2}\Big[(\nabla\Phi)^2 + \frac{1}{6}R\Phi^2\Big]$.

Finally,

$$\Gamma_{ind,IR} \approx \frac{b}{18}\int_x R^2 - \frac{1}{6}\int_x\int_y \Big(\frac{1}{4!}\widetilde{\beta}_\lambda \Phi^4 - \gamma_\Phi X_c\Big)_x \Big(\frac{1}{\Box}\Big)_{x,y} R(y)$$

which is potentially interesting for various applications.

Anomaly-induced effective action in the IR

In the symmetry-breaking regime $\Phi \approx v$. Then we get, in the IR induced action,

$$-\frac{1}{6}\iint_{xy}\Big(\frac{1}{4!}\,\widetilde{\beta}_\lambda v^4 \,-\, \frac{1}{6}\,\gamma_\Phi R v^2\Big)_x\Big(\frac{1}{\Box}\Big)_{x,y}R(y) \;\sim\; \int R\frac{1}{\Box}R.$$

This is interesting because this kind of the action was extensively discussed in cosmology and could be never derived as part of low-energy induced action or alike.
Finally, assuming conformal transformation

$$g_{\mu\nu} = \bar{g}_{\mu\nu}e^{2\sigma}, \qquad \Phi = \bar{\Phi}e^{-\sigma}$$

and a weak gravitational field, in the linear in σ approximation, we get one-loop effective potential

$$V_{eff}^{(1)} \;=\; \frac{1}{4!}\Big(\lambda + \frac{1}{2}\,\widetilde{\beta}_\lambda \ln\frac{\Phi^2}{\mu^2}\Big)\Phi^4 \;-\; \frac{1}{12}\Big(1 + \gamma_\Phi \ln\frac{\Phi^2}{\mu^2}\Big)R\Phi^2,$$

$$\widetilde{\beta}_\lambda = \beta_\lambda + 4\lambda\gamma_\Phi .$$

Conclusions

- **Integrating conformal anomaly is very efficient and economic way to derive the effective action of vacuum.**

- **There are many generalizations of the original method that concerns the background metric. We can use it for external scalars, vectors or torsion.**

- **In the case of background metric and scalar, we can obtain the covariant expression for the effective action of vacuum. This expression has non-local and local parts. The local part is typically ambiguous.**

- **Another new development is the connection between anomaly and the IR limit of the effective action (e.g., effective potential), with potentially interesting applications.**