Conformal scalar and spinor fields in curved space

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Аннотация на русском языке. We have studied the conformal transformation of scalar and spinor field equations. These fields scale as, $\tilde{\phi}(\psi) = \Omega^{\xi}\,\phi(\psi)$ and the metric tensor as $\tilde{g}_{\mu\nu} = \Omega^2\, g_{\mu\nu}$, where Ω and ξ are called the conformal factor and wight, respectively. The conformal mass in Klein-Gordon mass $(\tilde{m})$ is related to original scalar mass, (m) by $\tilde{m} = \Omega^{-1}\, m$. Moreover, the Klein-Gordon equation in the conformal's frame reduces to the quantum Telegraph equation of a particle whose mass is given by, $M = (\xi + 1)\, m$, in Minkowski's frame. The conformal wave equation in 2 dimensions with $\xi = 1$ yields the quantum Telegraph equation with a mass. We have found that the conformal wave equation in 2 dimensions yields the Dirac equation for $\xi = \pm\, i$ in flat space. In 4 dimensions the mass of the conformal spinor field scales as $\tilde{m} = \Omega^{-2} m$. The spinor charge (q) is influenced by the conformal transformation and becomes $Q_c = q\xi/(\xi + \frac{3}{2})$. The conformal factor for a spinor field is found to be equal to the phase factor of the spinor field. Moreover, the conformal transformation preserves the probability of the spinor particle. There exists a certain conformal transformation that transforms the Klein-Gordon equation into the Dirac equation. An Aharonov- Bohm-like effect is found to occur due to a conformal transformation of the spinor field. Breaking of conformal invariance is found to give rise to a mass of the particle that is tantamount to the Higgs mechanism.

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1. Introduction

A scalar field is governed by the Klein-Gordon equation that is known to describe a particle with zero spins. A dissipative like-Klein-Gordon equation is recently derived which is found to reproduce the Schrodinger and Dirac equations under certain transformations [1]. In general relativity one is usually interested in a gravitational theory that is conformally invariant [2]. The general theory of relativity is however non-invariant under the conformal transformation. A scalar-tensor theory is found to be invariant under the conformal transformation that led physicists to explore the implications of such theories [3]. A famous theory among this plethora of models is the Brans-Dicke theory [3]. Even though Einstein's equations are not conformally invariant, it reveals that there exists an additional energy-momentum tensor associated with Einstein's tensor [4]. We have shown recently that such a tensor accounts for the matter aspect of the gravitational field [5]. If such a fluid is coupled to a scalar field it yields an energy density connected with the space curvature giving mass to the graviton. A theory that is not invariant under conformal transformation can also be interesting. Such an investigation may have not been tackled before.

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We would like here to study the Klein-Gordon and Dirac wave equations under conformal transformation. The breaking of the conformal transformation gives all fields present a mass. This is because the conformal transformation of Einstein's tensor yields an additional energy-momentum term that expresses the vacuum contribution to the curvature of space. This vacuum is found to couple to all fields present [5]. Recall that Dirac's equation describes particles with spin equal to $1/2$, while the Klein-Gordon equation describes spin-less scalar bosons. Such an investigation reveals that the Klein-Gordon equation in conformal's frame yields the dissipative Klein-Gordon equation, or the quantum Telegraph equation that is believed to reproduce the Schrodinger and Dirac equations [1]. Moreover, the conformal transformation of the spinor field preserves the probability, and the conformal mass scales inversely with the fermion mass. In addition, we found that an Aharonov-Bohm-like effect exists in curve space arising from the conformal transformation of the spinor field [6].

2. Conformal scalar field dynamics

The Lagrangian of the massive scalar field is given by

$$\mathcal{L}=\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi-V(\phi)\,, \tag{1}$$

where $V(\phi)=\frac{m^2}{2}\phi^2$ for free massive scalar field. The field equation associated with the scalar defined in eq.(1) is obtained from the Lagrange equation

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}=\frac{\partial\mathcal{L}}{\partial\phi}\,, \tag{2}$$

that yields the Klein-Gordon equation,

$$\eta^{\mu\nu}\,\partial_\mu\partial_\nu\phi+\left(\frac{mc}{\hbar}\right)^2\phi=0\,. \tag{3}$$

where $\eta_{\mu\nu}$ is the Minkowski metric tensor. In curved space-time, the partial derivatives are replaced by the covariant derivatives, so that eq.(1) becomes

$$\mathcal{L}=\frac{1}{2}\,g^{\mu\nu}\,(\nabla_\mu\phi)(\nabla_\nu\phi)-V(\phi)\,, \tag{4}$$

where $g_{\mu\nu}$ is the metric tensor describing the curved space-time. In the curved space-time, eq.(3) becomes

$$\Box\phi+\left(\frac{mc}{\hbar}\right)^2\phi=0\,,\qquad\qquad \Box=g^{\mu\nu}\nabla_\mu\nabla_\nu\,. \tag{5}$$

Under the conformal transformation [2, 7],

$$\tilde{g}_{\mu\nu}=\Omega^2(x)\,g_{\mu\nu}\,,\qquad\qquad \tilde{\phi}=\Omega^\xi\,\phi\,, \tag{6}$$

where ξ is some real number, the conformal Klein-Gordon in curved space-time will be [2]

$$\tilde{\Box}\tilde{\phi}+\left(\frac{\tilde{m}c}{\hbar}\right)^2\tilde{\phi}=0\,, \tag{7}$$

which upon applying eq.(6) and (5) becomes

$$\Box\phi+\frac{2\xi+2}{\Omega}g^{\mu\nu}(\nabla_\mu\Omega)(\nabla_\nu\phi)+\left(\frac{\xi}{\Omega}\,g^{\mu\nu}\nabla_\mu\nabla_\nu\Omega+\frac{\xi(\xi+1)}{\Omega^2}\,g^{\mu\nu}(\nabla_\mu\Omega)\,(\nabla_\nu\Omega)+\frac{c^2}{\hbar^2}\tilde{m}^2\Omega^2\right)\phi=0. \tag{8}$$

It is remarked that the conformal wave operator (d'Alembertian) is conformally invariant in 2-dimensions for $\xi=0$ [2]. The conformal factor, Ω, is found to represent the vacuum (background) field if one considers the conformal transformation of Einstein's equations. This energy-momentum tensor of the vacuum field is shown to have a perfect fluid distribution [5]. This vacuum contribution is decoupled from the curvature term when the conformal transformation is broken by Einstein's equations.

In D-dimensional space, one has [2]

$$\begin{aligned}\tilde{\Box}\tilde{\phi}=\Omega^{\xi-2}\left[\Box\phi+(2\xi+D-2)\,\Omega^{-1}g^{\mu\nu}\,(\nabla_\mu\Omega)\,(\nabla_\nu\,\phi)\right]+\\ \Omega^{\xi-2}\left[\xi\Omega^{-1}g^{\mu\nu}\nabla_\mu\nabla_\nu\Omega+\xi(D+\xi-3)\,\Omega^{-2}g^{\mu\nu}\,(\nabla_\mu\Omega)\,(\nabla_\nu\Omega)\right]\phi.\end{aligned} \tag{8a}$$

Applying the case $D=2$ in the conformal wave equation $\tilde{\Box}\tilde{\phi}=0$, one finds

$$\Box\phi+\frac{2\xi}{c^2}\,\frac{\dot{\Omega}}{\Omega}\frac{\partial\phi}{\partial t}+\left(\xi\frac{\Box\Omega}{\Omega}+\frac{\xi(\xi-1)}{c^2}\frac{\dot{\Omega}^2}{\Omega^2}\right)\phi=0\,, \tag{9}$$

where we have assumed that $\Omega=\Omega(t)$. This particular case would break the conformal symmetry where $\Omega=\Omega(x)$. Interestingly, in 2-dimensions, when $\xi=1$, eq.(10) reduces to the quantum Telegraph equation. However, when $\xi=\pm i$, eq.(9) yields the Dirac equation in the quadratic form [1, 10]. It is thus very interesting that the wave equation and Dirac equation are related in 2 dimensions by the conformal transformation, where Ω is given in eq.(13).

Thus, if the conformal factor, Ω, depends on time only, then eq.(8) becomes

$$\Box\phi+\frac{2(\xi+1)}{c^2}\frac{\dot{\Omega}}{\Omega}\frac{\partial\phi}{\partial t}+\left(\xi\frac{\Box\Omega}{\Omega}+\frac{\xi(\xi+1)}{c^2}\frac{\dot{\Omega}^2}{\Omega^2}+\frac{c^2}{\hbar^2}\tilde{m}^2\Omega^2\right)\phi=0\,. \tag{10}$$

in Minkowski's frame. Equation (10) can be rewritten in the form

$$\Box\phi+\frac{2(\xi+1)}{c^2}\,\frac{\dot{\Omega}}{\Omega}\frac{\partial\phi}{\partial t}+\left(\xi\frac{\Box\Omega}{\Omega}+\frac{\xi(\xi+1)}{c^2}\frac{\dot{\Omega}^2}{\Omega^2}+\left(\frac{mc}{\hbar}\right)^2\right)\phi=0\,, \tag{11}$$

where

$$\tilde{m}=\Omega^{-1}\,m\,. \tag{12}$$

Notice that if $\xi = -1$, then eq.(11) reduces to the wave equation in Minkowski's frame if $\Omega \propto \exp(mc^2t/\hbar)$. However, a massless wave equation with $\xi = -1$ in the conformal's frame yields a Klein-Gordon equation with an imaginary mass in Minkowski's frame. But it yields a Klein-Gordon equation with a real mass if one uses $\Omega \propto \exp(imc^2t/\hbar)$, instead. Such an ansatz can be considered a way of giving mass to a massless field.

A quantum telegraph equation has recently been derived that is the quantum analog of the classical telegraph equation governing electric current and voltage in transmission lines [1]. It takes the form

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - \nabla^2\psi \pm \frac{2mi}{\hbar}\frac{\partial\psi}{\partial t} - \left(\frac{mc}{\hbar}\right)^2\psi = 0 \tag{13}$$

A second variant of it, that is a quadratic Dirac equation, is derived from eq.(13) by rotating space and time clockwise and counterclockwise by an angle of $\pi/2$, (*i.e.* $t \to \pm\, it$ and $r \to \pm\, ir$), takes the form

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - \nabla^2\psi \pm \frac{2mi}{\hbar}\frac{\partial\psi}{\partial t} - \left(\frac{mc}{\hbar}\right)^2\psi = 0 \tag{14}$$

Let us now consider the case [5]

$$\Omega = \Omega_0\, e^{mc^2t/\hbar}\,, \tag{15}$$

where Ω_0 is a constant. Substitute eq.(15) in eq.(11) to obtain

$$\Box\phi + \frac{2M}{\hbar}\frac{\partial\phi}{\partial t} + \left(\frac{Mc}{\hbar}\right)^2\phi = 0\,, \qquad M = (\xi + 1)\, m\,. \tag{16}$$

I interestingly, that the conformal Klein-Gordon equation yields the quantum Telegraph equation in Minkowski's frame. We have found earlier that Maxwell's equations in the conformal's frame yield the massive Maxwell's equations in the Minkowski's frame, where the photon mass is $\xi\, m$ and $\xi = \pm 1$ [8]. The Klein-Gordon equation is transformed into a massless wave equation if $\xi = -1$. This is appealing since the conformal transformation plays the role of the Higgs mechanism that is employed to give mass to elementary particles in the standard model [9].

An interesting case exists when $\xi + 1 = \pm\, i$ for which eq.(14) becomes

$$\Box\phi \pm \frac{2m\, i}{\hbar}\frac{\partial\phi}{\partial t} - \left(\frac{mc}{\hbar}\right)^2\phi = 0\,, \tag{17}$$

which is the Dirac equation in a quadratic form [1, 10]. Therefore, the conformal scalar field in eq.(6) becomes

$$\tilde{\phi} = \Omega^{\pm\, i - 1}\,\phi\,, \tag{18}$$

where Ω is given by eq.(15). Consequently, one can deal with the conformal transformation of Dirac's equation using the above recipe.

3. Conformal Spinor dynamics

Let us now consider the Dirac equation for spin-1/2 massless particles. This is expressed in a covariant form as [20]

$$i\hbar\gamma_\mu\partial^\mu\psi = 0\,, \tag{19}$$

where ψ is called the spinor. In curved space-time, the partial derivative is replaced by the covariant derivative, so that [11]

$$i\hbar\gamma_\mu\nabla^\mu\psi = 0\,. \tag{20}$$

Let us assume that the spinor transforms under the conformal transformation as

$$\tilde{\psi} = \Omega^\xi\,\psi\,. \tag{21}$$

where ξ is a real number (weight). The spinor covariant derivative is defined by [12]

$$\begin{aligned} \nabla_\mu &= \partial_\mu + \Omega_\mu\,, & [\gamma^a\,,\Omega_\nu] &= \omega^a_{b\nu}\,\gamma^b\,,\\ \Omega_\nu &= -\frac{i}{4}\omega_{ab\nu}\,\sigma^{ab}\,, & \sigma^{ab} &= \frac{i}{2}\left[\gamma^a\,,\gamma^b\right], \end{aligned} \tag{19a}$$

where $\omega^a_{b\nu}$ is the spin connection, and that $\gamma^\mu = e^\mu_a\gamma^a$, with e^μ_a is the vielbein. Note that the Latin indices refer to local inertial coordinates whereas Greek indices refer to general coordinates. When the spinor field is coupled to a gravitational field, spontaneous particle creation may occur.

Let us now consider the conformal transformation of eq.(20) which reads

$$i\hbar\tilde{\gamma}_\mu\tilde{g}^{\mu\nu}\tilde{\nabla}_\nu\tilde{\psi} = 0\,. \tag{22}$$

Apply now eqs.(6) and (20a) in eq.(22) to obtain [13]

$$i\hbar\gamma_\mu\nabla^\mu\psi + i\hbar(\xi+\frac{3}{2})\,\Omega^{-1}\gamma_\mu(\nabla^\mu\Omega)\psi = 0\,,\qquad \tilde{\gamma}^\mu = \Omega^{-1}\gamma^\mu\,. \tag{23}$$

Equation (21) reduces to

$$i\hbar\gamma_\mu\partial^\mu\psi + i\hbar(\xi+\frac{3}{2})\,\Omega^{-1}\gamma_\mu(\partial^\mu\Omega)\psi = 0\,, \tag{24}$$

in Minkowski's frame. If we assume that $\Omega = \Omega(t)$, then eq.(24) yields

$$i\hbar\,\gamma_\mu\partial^\mu\psi + \frac{i(\xi+\frac{3}{2})}{c}\,\hbar\,\gamma_0\frac{\dot{\Omega}}{\Omega}\psi = 0\,, \tag{25}$$

which can be compared with the massive Dirac's equation, *i.e.*,

$$i\hbar\,\gamma_\mu\partial^\mu\psi - mc\,\psi = 0\,. \tag{26}$$

It is now apparent that the conformal transformation of Dirac's equation, eq.(23), induces a mass term for the spinor (fermion) given by

$$m=-i\hbar\frac{(\xi+\frac{3}{2})}{c^2}\gamma_0\frac{\dot{\Omega}}{\Omega}\,. \tag{27}$$

It is known previously that it is the Higgs mechanism that offers a particle its mass [9]. Hence, a tantamount mechanism of giving mass to a massless particle now emerges. Interestingly, the above scenario reveals that the mass expresses the interaction of the particle with the outside world. Thus, if we demand that the massless Dirac's equation be conformally invariant then $\xi=-3/2$ so that $\tilde{\psi}=\Omega^{-3/2}\psi$. It seems breaking the conformal invariance gives rise to the mass generation of the particle.

Equation (27) can now be solved to give the conformal factor

$$\Omega=\Omega_0 e^{i\frac{m\gamma_0 c^2}{(\xi+\frac{3}{2})\hbar}\,t}\,. \tag{28}$$

Now the massive Dirac's equation, eq.(26), will transform in curved space, under the conformal transformation, as

$$i\hbar\,\gamma_\mu\nabla^\mu\psi-c\,\tilde{m}\Omega^2\,\psi+i\hbar(\xi+\frac{3}{2})\Omega^{-1}\gamma_\mu(\nabla^\mu\Omega)\psi=0\,, \tag{29}$$

where

$$\tilde{m}=\Omega^{-2}m\,. \tag{30}$$

If Ω if a function of time only, then eq.(29) reduces to

$$i\hbar\,\gamma_\mu\partial^\mu\psi-mc\,\psi+i(\xi+\frac{3}{2})\frac{\hbar}{c}\gamma_0\frac{\dot{\Omega}}{\Omega}\psi=0\,, \tag{31}$$

in Minkowski's frame.

Interestingly, an extra term in eq.(31) arises in the Dirac equation. This term could represent the magnetic moment of the spinor, which is described by [14–16]

$$i\hbar\gamma^\mu\partial_\mu\psi-mc\,\psi+a\,\sigma_{\mu\nu}\,F^{\mu\nu}\psi=0\,, \tag{32}$$

where a is some constant, $\sigma_{\mu\nu}$ and $F_{\mu\nu}$ are, respectively, the spin and the electromagnet field tensors.

Recall that when the spinor is coupled to the electromagnetic field, Dirac's equation becomes

$$i\hbar\gamma^\mu(\partial_\mu+iqA_\mu)\psi-mc\,\psi=0\,, \tag{32a}$$

which yields

$$i\hbar\,\gamma_\mu\partial^\mu\psi-mc\,\psi-\frac{q}{c}\gamma_0\varphi\,\psi=0\,, \tag{33}$$

where q is the particle's charge and $\vec{A}=0$. This shows that the conformal transformation appearing in eq.(29) is equivalent to coupling the spinor with an electromagnetic potential given by

$$\varphi=-i(\xi+\frac{3}{2})\frac{\hbar}{q}\frac{\dot{\Omega}}{\Omega}\,. \tag{34}$$

Hence, a free electron (q) would feel interaction with a virtual electric potential. Therefore, one can envisage the conformal factor as a background field permeating the whole space. It couples to the scalar field differently from that of the spinor field. One can also look at the additional term in eq.(31) as a mass term where the spinor effective mass is but

$$m_{eff.} = m - i(\xi + \frac{3}{2})\frac{\hbar}{c^2}\gamma_0\frac{\dot{\Omega}}{\Omega}\,, \tag{35}$$

that the spinor acquires when interacting with the conformal field, Ω.

Owing to eq.(34), the conformal factor is now connected with the electric potential by the relation

$$\Omega = \Omega_0 e^{\,i\int \frac{q\varphi}{\hbar(\xi+\frac{3}{2})}dt}\,, \tag{36}$$

so that eq.(19) yields

$$\tilde{\psi} = \psi_0 e^{\,i\int \frac{q\xi\varphi}{\hbar(\xi+\frac{3}{2})}dt}\psi\,, \qquad\qquad \xi \neq -3/2\,. \tag{37}$$

This result was also found by Schrodinger and London in the conformal Weyl theory that initiated the gauge invariance in quantum field theory [17–19]. London, however, stated that the re-scaling of the metric (Weyl) has to be replaced by a phase change of the wave function in the realm of quantum mechanics ($\psi' = e^{iQ\chi}\psi$ under $A'_\mu = A_\mu - \partial_\mu\chi$). When this is compared with the above transformation, we obtain

$$Q = \frac{\xi}{\xi + \frac{3}{2}}\,q\,, \qquad\qquad \xi \neq -3/2\,. \tag{38}$$

As a result of the conformal transformation of the Einstein tensor, an additional energy-momentum term arises, which owing to the Weyl theory should represent the electromagnetic field. Nonetheless, the additional term represents a material field (vacuum) in our present formalism.

Equation (34) can be rewritten in a form analogous to the second Josephson's relation as

$$\frac{2q\varphi}{\hbar} = \frac{\partial\eta}{\partial t}\,, \qquad\qquad \eta = \ln(\Omega)^{-2i\xi-3i}\,, \tag{39}$$

where η acts like a phase in the Josephson junction [21]. The conformal factor in curved space acts here like a phase in the quantum Josephson junction that leads to the Josephson effect. The connection between curved space and superconductivity was entertained by Horowitz *et al.* [22].

Interestingly, the conformal transformation of the spinor field preserves the probability, *i.e.*, $|\tilde{\psi}|^2 = |\psi|^2$. It is shown by Aharonov-Bohm (AB) that if a quantum particle passes through two different paths encapsulating a magnetic field, an interference pattern shows up where the difference in the phase angle is given by $\delta = q\varphi\, t/\hbar$ [6]. This case corresponds to

$$\delta = \frac{\xi}{\xi + \frac{3}{2}}\,, \qquad\qquad \xi \neq -3/2\,. \tag{40}$$

Equation (37) exhibits this result if the electric potential is constant. Here the origin of the electric potential is the conformal factor, as evident from eq.(34). Hence, eq.(40) suggests that in a conformal frame, this phenomenon exists. The sign of ξ indicates whether the path is clockwise or anticlockwise. According to eq.(38), one may define the charge $Q_c = q\xi/(\xi + \frac{3}{2})$ as the conformal charge.

4. Concluding remarks

We have found that the Klein-Gordon equation in the conformal's frame reduces to the quantum telegraph equation with mass given by $M = (\xi + 1)\, m$, where m is the mass of the scalar field. Moreover, the scalar field mass is transformed under conformal transformation by $\tilde{m} = \Omega^{-1} m$. However, if $\xi = -1$, we obtain the wave equation with zero mass. The conformal wave equation in 2 dimensions with $\xi = 1$ yields the quantum Telegraph equation in Minkowski's frame. The conformal mass of the spinor field is found to scale as $\tilde{m} = \Omega^{-2} m$. Additionally, the spinor field probability is preserved by the conformal transformation. An effect similar to the AB effect is found to occur due to a conformal transformation of the spinor field. The conformal transformation induces conformal mass and charge defined by $M_c = m/\xi$ and $Q_c = q\xi/(\xi + \frac{3}{2})$, respectively. There exists a certain conformal transformation that transforms the Klein-Gordon equation into the Dirac equation. The conformal transformation of massive Dirac's equation induces an extra mass that depends on the scale factor.

REFERENCES

1. Arbab, A. I., *Quantum Telegraph equation: New matter-wave equation*, Optik 140, 1010 (2017).

2. Wald, R. M., *General Relativity* , University of Chicago Press (1984).

3. Brans, C. H., Dicke, R. H., *Mach's Principle and a Relativistic Theory of Gravitation*, Phys. Rev. 124, 925 (1961).

4. Dabrowski, M. P., Garecki, J., Blaschke, D. B., *Conformal transformations and conformal invariance in gravitation*, Annalen Phys. 18, 13 (2009).

5. Arbab, A. I., Albughylil, R. S., Alfedeel, A. H. A., *Scalar field and particle dynamics in conformal frame*, Front. Phys. 10, 867766 (2022).

6. Bohm, D, Aharnnov, Y., *Significance of electromagnetic potentials in quantum theory*, Phys. Rev.115, 485 (1959).

7. Weyl, H., *Elektron and Gravitation I*, Zeitshrift fur Physik 56, 330 (1929).

8. Arbab, A. I., *Conformal electrodynamics in curved space*, Optik 241, 167009 (2021).

9. Higgs, P. W., *Broken symmetries and the masses of gauge bosons*, Phys. Rev. Lett. 13, 508 (1964).

10. Arbab, A. I., D*erivation of Dirac, Klein-Gordon, Schrodinger, diffusion and quantum heat transport equations from a universal quantum wave equation*, EPL 92, 40001 (2010).

11. Fock, V., *Geometrisierung der Diracschen Theorie des Elektrons,* Zeit. f. Phys. 57, 261 (1929).

12. Koke, C., Noh, C., Angelakis, D. G., *Dirac equation in 2-dimensional curved spacetime, particle creation, and coupled waveguide arrays*, https://arxiv.org/pdf/1607.04821.pdf .

13. Saharian, A. A., *Quantum field theory in curved spacetime*, http://training.hepi.tsu.ge/rtn/activities/sources/LectQFTrev.pdf , pg.65 .

14. Salamin, Y. I., *On the Dirac equation with anomalous magnetic moment term and a plane electromagnetic field*, J. Phys. A26, 6067 (1993).

15. Gupta, N. D. S., *On the Dirac equation with anomalous magnetic moment term in a beam of electromagnetic wave*, Pramana, 45(4), 327 (1995).

16. ltzykson, C. Zuber, J., *Qunnfm Field Theory*, McGraw-Hill, (1988).

17. Fock, V., Z. Phys. 39, 226 (1926).

18. Schrodinger, E., *Quantenmechanische Deutung der Theorie von Weyl*, Z. Physik 12, 13 (1922).

19. London, F., *Quantenmechanische Deutung der Theorie von Weyl*, Z. Physik 42, 375 (1927).

20. Bjorken, S. D., Drell, J. D., *Relativistic Quantum Mechanics*, (1964).

21. Josephson, B. D., *Possible new effects in superconductive tunneling*, Phys. Lett. 1, 251 (1962).

22. Horowitz, G. T., Santos, J. E., Way, B., *Holographic Josephson Junctions*, Phys. Rev. Lett. 106, 221601 (2011).