Conformal transformation of scalar and spinor fields

Arbab Ibrahim

Qassim University, Saudi Arabia

XXXV International Workshop on High Energy Physics "**From Quarks to Galaxies: Elucidating Dark Sides**"

November 30, 2023, Protvino, Russia

- Global scale transformations act on space-time coordinates via $x'^{\mu} = \Omega x^{\mu}$ such that the metric tensor transforms via $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ where Ω has the same value at each space-time point (global). A local transformation exists when Ω is a function of coordinates.
- However, it is now widely believed that all fundamental symmetries of nature must be local, as exemplified by the successes of gauge theories and general relativity.
- In gauge theory, the Dirac spinor scales globally or locally by $\psi(x) = e^{ie\chi(x)}\psi(x)$ under gauge transformation

$$A'_{\mu} = A_{\mu} - \partial_{\mu}\chi$$

Gauge fields

- It is shown by Noether that the invariance of a system under a given transformation will lead to a conserved quantity. The energy and momentum as manifestations of the invariance of a system (Lagrangian) under space and time translations. In field theory, a transformation pertains to the field (wavefunction) variables.

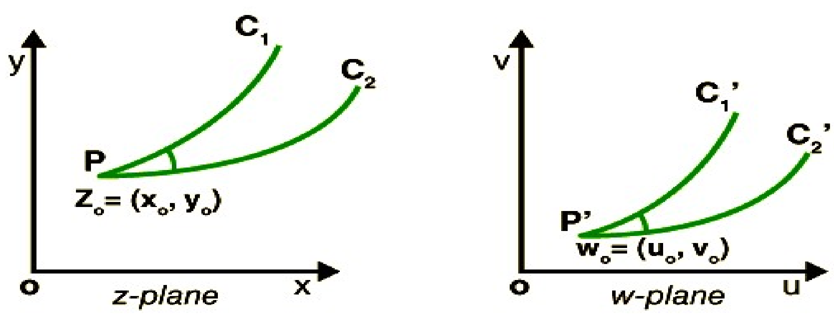
- Local transformations are more interesting and give rise to a subsequent gauge field. The gauge field is found to be responsible for mediating the interaction between particles. A symmetry is carried by a group having particular properties [U(n), SU(n), O(n), SO(n)].

- Symmetry can be broken implicitly or explicitly. The existence of photon mass will break the symmetry explicitly. This may not agree with our experimental experience with photons. The loss of symmetry makes calculations rather difficult. However, implicit (spontaneously) broken symmetry give mass to all massless particle (leptons and gauge boson except the photon).

Symmetry-Invariance

- What is the conserved quantity behind the conformal transformation?

- In complex number theory, a conformal transformation is related to the conservation of the angle between two curves when mapped from one coordinate to another.



Non-conformally invariant theories

- Theories under the general covariance principle can also exhibit invariance under the conformal transformation. Among these theories are Einstein's general relativity, Dirac equation, scalar field theories, and Maxwell electrodynamics. Ironically, only Einstein's theory is not conformally invariant. The conformal invariance (symmetry) sometimes is connected with zero mass or zero source. In general, physicists are interested in theories that are conformally invariant and study the consequences of this invariance.

- We would like here to study the consequences of a theory that is not conformally invariant and the physics behind it. In a previous study, we showed that the non-invariance terms in Einstein's general relativity are connected with the background (vacuum) in which masses exist. Graviton is considered to be massless. Breaking the conformal symmetry may give mass to the graviton. If we do a conformal transformation to Maxwell electrodynamics, a possibility exists when an additional term corresponds to the mass of the electromagnetic field.

- We will consider the scalar and spinor fields and the equations governing them. These are the Klein-Gordon and Dirac equations. We will consider the cases when a field is massless and when is massive. Our study shows that mass can be generated by breaking conformal transformation.

Conformal Einstein's equation of gravitation

- Einstein's equations of gravitation are

$$G_{\mu\nu} = \kappa\, T_{\mu\nu}\,, \qquad G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} R\,, \qquad \kappa = \frac{8\pi G}{c^4},$$

which under the conformal transformation, $\tilde{g}_{\mu\nu} = \Omega^2(x)\, g_{\mu\nu}$, and for D-dimensions become

$$\tilde{G}_{\mu\nu} = G_{\mu\nu} + \frac{D-2}{2\Omega^2}\left[4\Omega_{\mu}\Omega_{\nu} + (D-5)\Omega_{\lambda}\Omega^{\lambda} g_{\mu\nu}\right] - \frac{D-2}{\Omega}\left[\Omega_{;\mu\nu} - g_{\mu\nu}\Box\Omega\,\right].$$

In 4-diemsnions, one finds

$$\tilde{G}_{\mu\nu} = G_{\mu\nu} + \Lambda\, g_{\mu\nu} + 2\Lambda\, \delta^0_{\mu}\, \delta^0_{\nu}\,, \qquad \Lambda = \frac{m^2 c^2}{\hbar^2}$$

The additional terms can be attributed to the effect of the vacuum (or graviton mass).

Conformal scalar field dynamics

- In curved space-time the Klein-Gordon equation reads

$$\Box\phi + \left(\frac{mc}{\hbar}\right)^2 \phi = 0\,, \qquad \Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$$

which under the conformal transformation

$$\tilde{g}_{\mu\nu} = \Omega^2(x)\, g_{\mu\nu}\,, \qquad \tilde{\phi} = \Omega^{\xi}\,\phi$$

becomes

$$\tilde{\Box}\tilde{\phi} + \left(\frac{\tilde{m}c}{\hbar}\right)^2 \tilde{\phi} = 0$$

that reads

$$\Box\phi + \frac{2\xi+2}{\Omega} g^{\mu\nu}(\nabla_\mu\Omega)(\nabla_\nu\phi) + \left(\frac{\xi}{\Omega} g^{\mu\nu}\nabla_\mu\nabla_\nu\Omega + \frac{\xi(\xi+1)}{\Omega^2} g^{\mu\nu}(\nabla_\mu\Omega)(\nabla_\nu\Omega) + \frac{c^2}{\hbar^2}\tilde{m}^2\Omega^2\right)\phi = 0.$$

The quantum Telegraph equation

- Using quaternions, the momentum eigen-value equation yields an equation that is analogous to the telegraph equation governing electric signals in transmission lines. It is an undistorted wave that preserves the identity of a quantum particle. It is written as

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}-\nabla^2\psi+\frac{2m}{\hbar}\frac{\partial\psi}{\partial t}+\left(\frac{mc}{\hbar}\right)^2\psi=0$$

- Under the space and time rotation (it, ir), one obtains

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}-\nabla^2\psi\pm\frac{2mi}{\hbar}\frac{\partial\psi}{\partial t}-\left(\frac{mc}{\hbar}\right)^2\psi=0$$

- This equation can be derived by squaring the Dirac equation
- $(i\hbar\frac{\partial}{\partial t}-\beta mc^2)^2=(-\hbar c\vec{\alpha}\cdot\vec{\nabla})^2$ that yields $\Box\psi+\frac{2\beta m\,i}{\hbar}\frac{\partial\psi}{\partial t}-\left(\frac{mc}{\hbar}\right)^2\psi=0$..

Conformal equation of motion

For a time-dependent conformal factor, reads

$$\Box\phi + \frac{2(\xi+1)}{c^2}\frac{\dot{\Omega}}{\Omega}\frac{\partial\phi}{\partial t} + \left(\xi\,\frac{\Box\Omega}{\Omega} + \frac{\xi(\xi+1)}{c^2}\frac{\dot{\Omega}^2}{\Omega^2} + \frac{c^2}{\hbar^2}\tilde{m}^2\Omega^2\right)\phi = 0\,.$$

- In D-dimensional space-time, one has

$$\bar{\Box}\bar{\phi} = \Omega^{\xi-2}\left[\Box\phi + (2\xi + D - 2)\,\Omega^{-1}g^{\mu\nu}\,(\nabla_\mu\Omega)\,(\nabla_\nu\,\phi) + (\xi\Omega^{-1}g^{\mu\nu}\nabla_\mu\nabla_\nu\Omega + \xi(D+\xi-3)\,\Omega^{-2}g^{\mu\nu}\,(\nabla_\mu\Omega)\,(\nabla_\nu\Omega))\phi\right].$$

and for D=2, one has, for a massless wave equation, i.e., $\bar{\Box}\tilde{\phi} = 0$,

$$\Box\phi + \frac{2\xi}{c^2}\frac{\dot{\Omega}}{\Omega}\frac{\partial\phi}{\partial t} + \left(\xi\frac{\Box\Omega}{\Omega} + \frac{\xi(\xi-1)}{c^2}\frac{\dot{\Omega}^2}{\Omega^2}\right)\phi = 0$$

Let us consider a conformal factor of the form $\Omega = \Omega_0\, e^{mc^2 t/\hbar}$,

a) For $\xi = 0$, we restore our original wave equation.

b) For $\xi = 1$, one obtains an undistorted Telegraph equation.

c) For $\xi = \pm i$, one obtains a Dirac equation (quadratic form)

- The conformal massive Klein-Gordon equation (4-dimension)

$$\Box\phi + \frac{2(\xi+1)}{c^2}\frac{\dot\Omega}{\Omega}\frac{\partial\phi}{\partial t} + \left(\xi\frac{\Box\Omega}{\Omega} + \frac{\xi(\xi+1)}{c^2}\frac{\dot\Omega^2}{\Omega^2} + \left(\frac{mc}{\hbar}\right)^2\right)\phi = 0\,,$$

$$\tilde{m} = \Omega^{-1}\,m\,.$$

- For $\xi = 0$, we obtain the undistorted Telegraph equation ($\Omega = \Omega_0\, e^{mc^2 t/\hbar}$,).
- For $\xi = -1$., we obtain a massless Klein-Gordon equation.
- In general, the above equation becomes

$$\Box\phi + \frac{2M}{\hbar}\frac{\partial\phi}{\partial t} + \left(\frac{Mc}{\hbar}\right)^2\phi = 0\,, \qquad M = (\xi+1)\,m$$

- For $\xi + 1 = \pm i$, one finds the quadratic Dirac equation

$$\Box\phi \pm \frac{2m\,i}{\hbar}\frac{\partial\phi}{\partial t} - \left(\frac{mc}{\hbar}\right)^2\phi = 0$$

so that the scalar field scales as

$$\tilde{\phi} = \Omega^{\pm i - 1}\phi$$

Conformal transformation of spinor field

- Dirac equation in curved space for massless spinor is given by

$$i\hbar\gamma_\mu \nabla^\mu \psi = 0$$

where $\nabla_\mu = \partial_\mu + \Omega_\mu\,, \qquad [\gamma^a, \Omega_\nu] = \omega^a_{b\nu}\,\gamma^b\,, \qquad \Omega_\nu = -\frac{i}{4}\omega_{ab\nu}\,\sigma^{ab}\,, \qquad \sigma^{ab} = \frac{i}{2}[\gamma^a, \gamma^b]$

The conformal transformation of the Dirac equation above will be

$$i\hbar\tilde{\gamma}_\mu \tilde{g}^{\mu\nu}\tilde{\nabla}_\nu \tilde{\psi} = 0\,.$$

A conformal spinor field transforms as $\tilde{\psi} = \Omega^\xi\,\psi$. and $\tilde{\gamma}^\mu = \Omega^{-1}(x)\gamma^\mu$. Hence, one obtains

$$i\hbar\gamma_\mu \nabla^\mu \psi + i\hbar(\xi + \frac{3}{2})\,\Omega^{-1}\gamma_\mu(\nabla^\mu \Omega)\psi = 0\,.$$

This reduces, in Minkowski's space, to $i\hbar\gamma_\mu \partial^\mu \psi + i\hbar(\xi + \frac{3}{2})\,\Omega^{-1}\gamma_\mu(\partial^\mu \Omega)\psi = 0\,,$

Massive and massless spinors

- If Ω is time-dependent, then the above equation reduces to

$$i\hbar\,\gamma_\mu \partial^\mu \psi + \frac{i(\xi + \frac{3}{2})}{c}\,\hbar\,\gamma_0 \frac{\dot{\Omega}}{\Omega}\psi = 0\,,$$

which when compared with a massive spinor field

$$i\hbar\,\gamma_\mu \partial^\mu \psi - mc\,\psi = 0\,.$$

yields

$$m = -i\hbar\frac{(\xi + \frac{3}{2})}{c^2}\gamma_0\frac{\dot{\Omega}}{\Omega}$$

which for massless spinor, yields $\tilde{\psi} = \Omega^{-3/2}\psi$.

The massive Dirac equation in the conformal frame will be

$$i\hbar\,\gamma_\mu \nabla^\mu \psi - c\,\tilde{m}\Omega^2\,\psi + i\hbar(\xi + \frac{3}{2})\Omega^{-1}\gamma_\mu(\nabla^\mu\Omega)\psi = 0\,,$$

$$\tilde{m} = \Omega^{-2} m\,,$$

Gauge transform and phase angle

- If Ω is time-dependent, one obtains

$$i\hbar\gamma_\mu\partial^\mu\psi - mc\psi + i(\xi + \frac{3}{2})\frac{\hbar}{c}\gamma_0\frac{\dot{\Omega}}{\Omega}\psi = 0\,,$$

- Dirac equation when coupled to the e.m. field takes the form

$$(i\gamma^\mu(\partial_\mu + ieA_\mu) - m)\psi = 0.$$

- For special case, A=0, the above equation yields $\varphi = -i(\xi + \frac{3}{2})\frac{\hbar}{q}\frac{\dot{\Omega}}{\Omega}$.

which implies that $\Omega = \Omega_0 e^{i\int \frac{q\varphi}{\hbar(\xi+\frac{3}{2})}dt}$, and therefore the spinor field would transform as

- $\tilde{\psi} = \psi_0 e^{i\int \frac{q\xi\varphi}{\hbar(\xi+\frac{3}{2})}dt}\psi$. , $\psi(x) = e^{ie\chi(x)}\psi(x)$, $\bar{q} = \frac{\xi}{\xi + \frac{3}{2}}q$.

- This was obtained by Schrodinger and London in the conformal Weyl theory. The above scalar potential (φ) can be connected with the Josephson effect in a superconductor that relates it to a derivative of a phase angle (η) of electrons

$$\frac{2q\varphi}{\hbar} = \frac{\partial\eta}{\partial t}\,, \qquad \eta = \ln(\Omega)^{-2i\xi - 3i}$$

Aharonov-Bohm effect

- Aharonov-Bohm showed that if a quantum particle passes through two different paths encapsulating an isolated magnetic field, an interference pattern shows up when the difference in the phase angle is given by

$$\delta = q\varphi t/\hbar$$

- This phase difference can be connected to the conformal factor by the relation

$$\delta = \frac{\xi}{\xi + \frac{3}{2}}$$

Conclusions

- Conformal transformation (symmetry) opted for the simplest equations describing our nature.
- Breaking of gauge symmetry in a standard model permits massless particles and fields to gain mass via the Higgs mechanism. This mechanism allows a field (particle) to interact (coupled) with the Higgs (scalar) field.
- Breaking of conformal system also permit massless particles and fields to gain mass when these fields are coupled to the conformal function (field).
- Superconductivity may be connected with a special breaking of conformal transformation.
- An equation that is not conformally invariant adds a new understanding of the effect of the background (vacuum) on our system.
- The field's mass scales in a certain manner to the conformal function (field).

Thank you