The state-of-the-art covariant tetraquark equations

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Introduction

- History of the covariant equations for tetraquarks:

  (i) 1992, Kvi., Khvedelidze, Theor. Math. Phys. **90**, 62,
  4-body covariant equations without annihilation (like a 4 quark system, no transition to 2q)

  (ii) 2012, Heupel, Eichmann, Fischer, Phys.Lett. **B718**, 545,
  numerical solution of the 4-body covariant equations

  (iii) 2014, Kvi., Blankleider, Phys.Rev. **D90**, 045042,
  inclusion of the effect of quark-antiquark annihilation, coupling to the 2-body, quark-antiquark, channel

  (iv) 2020, Santowsky, Eichmann, Fischer, Wallbott, Williams, Phys.Rev.**D102**, 5,
  another, "phenomenological", attempt to include the coupling to the 2-body, $q\bar{q}$, channel

  (v) 2022, Kvi., Blankleider, Phys.Rev. **D106**, 5,
  the general exact equations for tetraquarks

1992, Kvi., Khvedelidze, Theor. Math. Phys. **90**, 62, 4-body covariant equations

- 4-body covariant equations like a 4 quark system, $\Psi = G_0^{(4)} K^{(4)} \Psi$
- Challenges in derivation:
  (i) The problem of overcounting even for no coupling to two-body channel -sum of pair interaction kernels overcounts therefore subtractions are needed
  $K^{(4)} = \sum K_{ij}^{(2)} - K_{12}^{(2)} K_{34}^{(2)} - K_{13}^{(2)} K_{24}^{(2)} - K_{14}^{(2)} K_{23}^{(2)}$,
  *Such subtractions are not canceled within $K^{(4)}$, cancelation happens between different iterations of $K^{(4)}$.
  $K^{(4)} + K^{(4)} K^{(4)} = \ldots - K_{12}^{(2)} K_{34}^{(2)} + K_{12}^{(2)} K_{34}^{(2)} + K_{34}^{(2)} K_{12}^{(2)} = \ldots + K_{12}^{(2)} K_{34}^{(2)}$ *Unusual structure of $K^{(4)}$ for using Faddeev-Yakubovsky rearrangement
  (ii) In the TOPT approach 4q kernels even cannot be expressed in terms of 2q kernels $K_{ij}^{(2)}$

1992, Kvi., Khvedelidze, Theor. Math. Phys. **90**, 62, 4-body covariant equations

- To rearrange the equation $\Psi = G_0^{(4)} K^{(4)} \Psi$ break the kernel into three parts, $K^{(4)} = K_1 + K_2 + K_3$, where
  $K_3 = K_{12}^{(2)} + K_{34}^{(2)} - K_{12}^{(2)} K_{34}^{(2)}$,
  $K_1 = K_{13}^{(2)} + K_{24}^{(2)} - K_{13}^{(2)} K_{24}^{(2)}$, ($K_2$ similarly),
  and define T_i, $T_i = K_i + K_i G_0^{(4)} T_i$ with the solutions,
  $T_3 = T_{12}^{(2)} + T_{34}^{(2)} + T_{12}^{(2)} T_{34}^{(2)}$,
  $T_1 = T_{13}^{(2)} + T_{24}^{(2)} + T_{13}^{(2)} T_{24}^{(2)}$, ($T_2$ similarly),
  where $T_{ij}^{(2)}$ are two-body off-shell scattering amplitudes,
  $T_{ij}^{(2)} = K_{ij}^{(2)} + K_{ij}^{(2)} G_0^{(2)} T_{ij}^{(2)}$.
  Then the rearranged equation is
  $\Psi = \sum_i \Psi_i$. where $\Psi_i = G_0^{(4)} T_i \sum_{j \neq i} \Psi_j$

1992, Kvi., Khvedelidze, Theor. Math. Phys. **90**, 62, 4-body covariant equations

- Three goals achieved. No analog in QM for two of them:
  (i) kernels T_i are expressed in terms of two-body t-matrices $T_{ij}^{(2)}$ (usual result of Faddeev rearrangement in Quantum Mechanics)
  (ii) got rid of subtraction terms present before rearrangement (subtractions do not exist in the 4-body potential of QM)
  (iii) In the 3D quasipotential and TOPT approaches the rearrangement is necessary at least for the kernels to be expressed in terms of functions defined in the two-body subsystems $T_1 + T_2 + \int dz T_1(E - z) T_2(z)$ (such a problem does not exist in Quantum Mechanics, equations' kernels are clear before rearrangement as well, the sum of pair interaction potentials)

1992, Kvi., Khvedelidze, Theor. Math. Phys. **90**, 62, 4-body covariant equations

- That time (in 1992) we even did not dream that this sort of equations could be solved numerically, we have been attracted only by the mathematical challenges they pose. But, alas, in 20 years since then they are being solved! (see next slides)
- (i)This fact,
  (ii)the advances in the methods of numerical solutions,
  (iii)the well elaborated input functions offered on the market
  (iv) calculations carried out by now
  revived our interest

2012, Heupel, Eichmann, Fischer, Phys.Lett. **B718**, 545, numerical solution of the 4-body covariant equations

- The first results for tetraquarks in a covariant continuum approach based on our equation of 1992 for four quarks, but applied to a system of 2 quarks+2 antiquarks, $2q2\bar{q}$,
  $\Psi_i = G_0^{(4)} T_i \sum_{j\neq i} \Psi_j$,
  where
  $T_3 = T_{12}^{(2)} + T_{34}^{(2)} + T_{12}^{(2)} T_{34}^{(2)}$,
  $T_1 = T_{13}^{(2)} + T_{24}^{(2)} + T_{13}^{(2)} T_{24}^{(2)}$, ($T_2$ similarly),
- This approach is "similar in spirit to the quark-diquark model of the nucleon" elaborated in G. Eichmann, I R. Alkofer, et al., Phys. Rev. C 79 (2009) 012202,

2012, Heupel, Eichmann, Fischer, Phys.Lett. **B718**, 545, numerical solution of the 4-body covariant equations

- Using 4q eqs for the system of 2q+2$\bar{q}$ implies that in the $2q2\bar{q}$ Green function $G^{(4)}$ $q\bar{q}$ annihilation diagrams are neglected, i.e. its $q\bar{q}$-irreducible part $G_{ir}^{(4)}$ is considered, see below
- Further approximations are made to reduce $2q2\bar{q}$ space to $MM + D\bar{D}$, i.e. to come to a picture of tetraquark as composed of two mesons, MM, or of diquark-antidiquark, $D\bar{D}$: (i) $T_3 \sim T_{12}^{(2)} T_{34}^{(2)}$, $T_1 \sim T_{13}^{(2)} T_{24}^{(2)}$, ($T_2$ similarly),

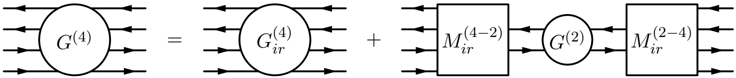
2012, Heupel, Eichmann, Fischer, Phys.Lett. **B718**, 545, numerical solution of the 4-body covariant equations

(ii) reduction to a two-body problem proceeds by assuming that the two-body T-matrices $T_{ij}^{(2)}$ are dominated by meson and diquark pole contributions, $T_{ij}^{(2)} = i\Gamma_{ij} D_{ij} \bar{\Gamma}_{ij}$,
where $D_{ij}(P_{ij}) = 1/(P_{ij}^2 - m_{ij}^2)$ is the propagator for the bound particle (diquark, antidiquark, or meson) in the two-body channel ij

- Physical results in Phys.Lett. **B718**, 545 (2012):
  For the lightest scalar tetraquark a mass of the order of 400 MeV and a wave function dominated by the pion-pion constituents is found.
  Both results are in agreement with a meson molecule picture for the $f_0(600)$.
  The results suggest the presence of a potentially narrow all-charm tetraquark in the mass region $5 - 6$ GeV

2014, Kvi., Blankleider, Phys.Rev. **D90**, 045042, simplest coupling to 2-body, quark-antiquark, channel

- So far mathematically pure four-quark system was considered.
- But in Quantum Field Theory (QFT) the number of particles is not conserved.
- The $2q2\bar{q}$ Green function $G^{(4)}$ in terms of its $q\bar{q}$-irreducible part $G_{ir}^{(4)}$ and its $q\bar{q}$-reducible part $M_{ir}^{(4-2)} G^{(2)} M_{ir}^{(2-4)}$.
The exact relation:



- First attempt to account for coupling to two-body $q\bar{q}$ state 2014, Kvi., Blankleider, Phys.Rev. **D90**, 045042

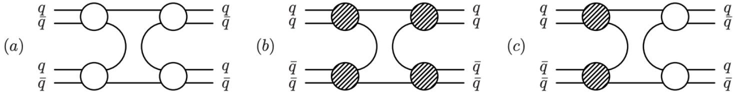
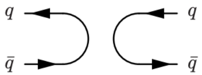


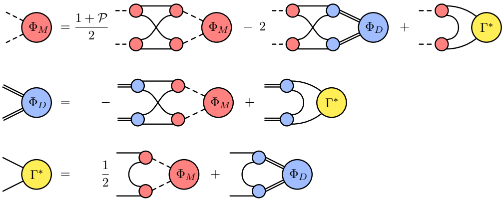
FIG. 1. Examples of terms involving $q\bar{q}$ annihilation contributing to the tetraquark amplitude within a model involving only meson, diquark and antidiquark constituents: (a) MM scattering, (b) $D\bar{D}$ scattering, (c) $D\bar{D} \leftarrow MM$ transition. Such terms are not taken into account when the covariant four-body equations of KK [3] are used to describe the $2q2\bar{q}$ system.



- this disconnected diagrams are used as (unusual!) parts of the input $2q$ T-matrices, but in a sophisticated way to avoid non-physical/inexistent contributions to the four-body eqs

2014, Kvi., Blankleider, Phys.Rev. **D90**, 045042, simplest coupling to 2-body, quark-antiquark, channel

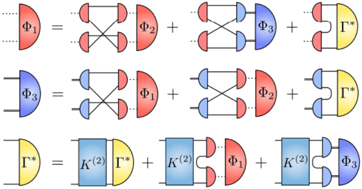
- Equations with minimal coupling to $q\bar{q}$ channel were obtained



Confusing "disconnected part" of the input $q\bar{q}$ T-matrix with δ-function.
No problem since the kernels of the final eqs do not involve δ-function, although featuring in the derivation.

2020, Santowsky, Eichmann, Fischer, Wallbott, Williams, Phys.Rev.**D102**, 5, "phenomenological" attempt to include coupling to $q\bar{q}$ channel

- Phenomenological motivated equations



where $K^{(2)}$ is one-gluon-exchange $q\bar{q}$ kernel.
Making an educated ("phenomenologically motivated") guess one must be prepared to face the possibility that these guessed equations may be inconsistent with the rules of QFT?

2022, Kvi., Blankleider, Phys.Rev. **D106**, 5, the general exact equations for tetraquarks

- We start with the two-body eq although for the tetraquark,
  $$\Gamma^* = K^{(2)} G_0^{(2)} \Gamma^*$$
  where the kernel $K^{(2)} = \Delta + \bar{N} G_{ir}^{(4)} N$ is complicated by the infinite series for $G_{ir}^{(4)}$,
  $$G_{ir}^{(4)} = G_0^{(4)} + G_0^{(4)} K^{(4)} G_0^{(4)} + G_0^{(4)} K^{(4)} G_0^{(4)} K^{(4)} G_0^{(4)} + \ldots \text{ (B)}$$

- This complication is handled by using the equation for $G_{ir}^{(4)}$,
  $$G_{ir}^{(4)} = G_0^{(4)} + G_0^{(4)} K^{(4)} G_{ir}^{(4)}$$
  to derive coupled channel set of 2q-4q equations with a simpler kernel $K^{(4)}$.

2022, Kvi., Blankleider, Phys.Rev. **D106**, 5, the general exact equations for tetraquarks

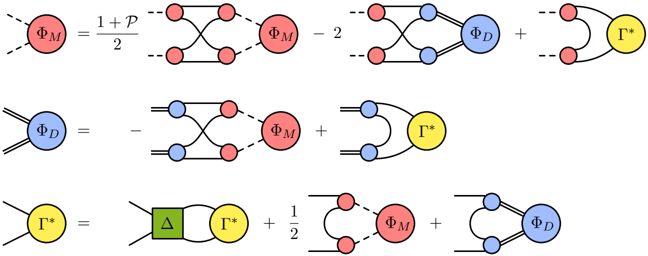
- Indeed the 2q eq $\Gamma^* = \left[\Delta + \bar{N} G_{ir}^{(4)} N\right] G_0^{(2)} \Gamma^*$ can be written as $\Gamma^* = \Delta G_0^{(2)} \Gamma^* + \bar{N} G_0^{(4)} \Phi$, where $\Phi = G_0^{(4)-1} G_{ir}^{(4)} N G_0^{(2)} \Gamma^*$ This relation defines Φ as the 4q component of the tetraquark Using the eq for $G_{ir}^{(4)}$ we get the eq $\Phi = K^{(4)} G_0^{(4)} \Phi + N G_0^{(2)} \Gamma^*$, which in combination with $\Gamma^* = \Delta G_0^{(2)} \Gamma^* + \bar{N} G_0^{(4)} \Phi$, represents the set of eqs with "simple" kernels $K^{(4)}, N, \Delta, \bar{N}$.

2022, Kvi., Blankleider, Phys.Rev. **D106**, 5, the general exact equations for tetraquarks

- In the above mentioned pole approximation for input 2q t-matrices these 4q eqs become mathematically 2-body eqs in $q\bar{q}$-MM space,
  $\Phi = VG_0^M\Phi + NG_0^{(2)}\Gamma^*.$
  $\Gamma^* = \Delta G_0^{(2)}\Gamma^* + \bar{N}G_0^M\Phi$
  where G_0^M are the two-body 2M and $D\bar{D}$ propagators;
  V is a 2×2 matrix whose elements are the kernels for the transitions between 2M and $D\bar{D}$ states.

2022, Kvi., Blankleider, Phys.Rev. **D106**, 5, the general exact equations for tetraquarks

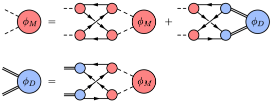
- The final equations in diagrams:



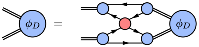
These are general exact four-body eqs (even if $N, K^{(4)}$ are approximate in the case where 4q-2q transitions are allowed. Here is important that Δ (and N) are clearly specified: Δ is the sum of all $q\bar{q}$ irreducible diagrams except those included in $\bar{N}G_{ir}^{(4)}N$.

KB, Unified tetraquark equations, Phys.Rev.D 107 (2023) 9, 094014

- **Currently there are two different seemingly unrelated eqs:**
  (i) The Giessen Group tetraquark eq in the double scattering approximation, $T_3 \sim T_{12}^{(2)} T_{34}^{(2)}$, W. Heupel, G. Eichmann, C. S. Fischer, Phys.Lett.B 718, 545 (2012)



(ii) The Moscow Group tetraquark eq, D. Ebert, R. N. Faustov, and V. O. Galkin, E. M. Savchenko, Phys. Lett. B 634, 214 (2006), Phys. Rev. D 102, 114030 (2020),



KB, Unified tetraquark equations, Phys.Rev.D 107 (2023) 9, 094014

- Universe 7 (2021) 4, 94:
  "there are significant disagreements between different theoretical approaches. Indeed, Refs. [...] predict heavy tetraquark masses **below or slightly above the thresholds** of the decays to two quarkonia and, thus, stable or significantly suppressed against fall-apart decays with a very narrow decay width. ... other approaches predict such tetraquark masses **significantly above** these thresholds and, thus, they can be observed only as broad resonances."
  Therefore it is important to compare theoretical basis of these approaches.
- A universal set of equations is derived which produces all of these approaches in different approximations. Exposing three body $Mq\bar{q}$, $\bar{D}qq$, $D\bar{q}\bar{q}$ states is inevitable to compare them on the theoretical foundation level.

KB, Unified tetraquark equations, Phys.Rev.D 107 (2023) 9, 094014

- Giessen Group eq, in the approximation of only double scattering, $T_3 \sim T_{12}^{(2)} T_{34}^{(2)}$,
  $\phi = V_G \phi$
- If we take into account the single scatterings $T_{12}^{(2)} + T_{34}^{(2)}$ in $T_3 = T_{12}^{(2)} T_{34}^{(2)} + T_{12}^{(2)} + T_{34}^{(2)}$, the Giessen Group eq modifies to the unified eq,
  $\phi = (V_G + V_M)\phi$,
  where V_M is the Moscow Group kernel.

Tetraquark currents

- The structure of tetraquarks can be studied by calculating tetraquark currents?
- The complexity of the tetraquark eqs would make the construction of the gauge invariant currents impossible if not for the "gauging equations method" [KB, PRC60, 044004 (1999)]. It makes the construction extremely simple just as, for example, in the case of the quark-diquark model for baryons.

Thanks

THANK YOU!