Generalized metrics in the presence of a scalar field

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A. Azizallahi, B. M, A. Hajibarat, H. Anjomshoa, arXiv: 2307.09328

B. M, P.K Kangazi, F. Sadeghi, arXiv: 2307.13588

Action:

$$S = \int d^4x \left(R - \partial_\mu \varphi(r)\, \partial^\mu \varphi(r) \right).$$

Equations of motion:

$$G_{\mu\nu} = \partial_\mu \varphi(r)\, \partial_\nu \varphi(r) - \frac{1}{2}\, g_{\mu\nu}\, \partial_\beta \varphi(r)\, \partial^\beta \varphi(r),$$

$$R_{\mu\nu} = \partial_\mu \varphi(r)\, \partial_\nu \varphi(r),$$

$$\nabla_\mu \nabla^\mu \varphi(r) = 0.$$

Introduction

FJNW metric:

$$ds^2 = -f^{\gamma}\, dt^2 + f^{-\gamma}\, dr^2 + f^{1-\gamma}\, r^2 \,(d\theta^2 + \sin^2\theta\, d\phi^2),$$

Metric function:

$$f(r) = 1 - \frac{2\,m}{r}.$$

This metric is not the most general solution for the action.

The metric has singularities at the following points:

$$r_{singularity} = \begin{cases} 0 \\ 2m \end{cases}$$

The generalised three parameter metrics:

$$ds^2 = -f^{\gamma}\, dt^2 + f^{\mu}\, k^{\nu}\, (\frac{dr^2}{f} + r^2\, d\theta^2) + f^{\beta}\, r^2\, d\phi^2,$$

Metrics functions:

$$f(r) = 1 - \frac{2\,m}{r},$$

$$k(r,\,\theta) = 1 - \frac{2\,m}{r} + \frac{m^2}{r^2}\,\mathcal{N}^2(\theta).$$

Generalised three parameter metrics

From $R_{\theta\theta} = 0$, **to the first order of m** $\Rightarrow$ $\beta = 1 - \gamma.$

From $R_{r\theta} = 0$, **to the first order of m** $\Rightarrow$ $\mu + \nu = \beta.$

$$\boldsymbol{\mu + \nu = 1 - \gamma}.$$

From $R_{r\theta} = 0$, **to the second order of m** $\Rightarrow$ $\mathcal{N}(\theta) = \sin\theta.$

$$R_{rr} = \partial_r\varphi(r)\,\partial_r\varphi(r), \quad \Rightarrow \quad \varphi(r) = \sqrt{\frac{1-\gamma^2-\nu}{2}}\,\ln\left(1 - \frac{2\,m}{r}\right).$$

Generalised three parameter metrics

Metrics:

$$ds^2 = -f^{\gamma}\, dt^2 + f^{\mu}\, k^{\nu}\, (\frac{dr^2}{f} + r^2\, d\theta^2) + f^{1-\gamma}\, r^2\, \sin^2\theta\, d\phi^2,$$

$$\mu + \nu = 1 - \gamma,$$

Metrics functions:

$$f(r) = 1 - \frac{2\,m}{r}, \qquad k(r,\,\theta) = 1 - \frac{2\,m}{r} + \frac{m^2\,\sin^2\theta}{r^2}.$$

parameters limits:

$\nu = 0, \quad \mu = 1 - \gamma$ $\Rightarrow$ FJNW metric

$\nu = 1 - \gamma^2, \quad \mu = \gamma^2 - \gamma, \quad \varphi(r) = 0$ $\Rightarrow$ γ-metric

Generalised three parameter metrics

The four-dimensional γ-metric in isotropic form:

$$ds^2 = -f^\gamma\, dt^2 + (1 + \frac{m}{2\,\rho})^4 \left[f^\mu\, k^\nu\, (d\rho^2 + \rho^2\, d\theta^2) + \rho^2\, f^\beta \sin^2\theta\, d\phi^2\right],$$

$$r = \rho\,(1 + \frac{m}{2\,\rho})^2,$$

Metrics parameters:

$$\mu + \nu = 1 - \gamma,$$

Metrics functions:

$$f = (\frac{m - 2\,\rho}{m + 2\,\rho})^2, \qquad k = \frac{m^4 + 16\,\rho^4 - 8\,m^2\,\rho^2\,\cos(2\,\theta)}{(m + 2\,\rho)^4}.$$

Generalised three parameter metrics

Ricci scalar:

$$R = \frac{2\,(\gamma^2 + \nu - 1)\, m^2}{r^{2+\gamma-\nu}\,(r - 2\,m)^{\,2-\gamma-\nu}\,(r^2 - 2\,m\,r + m^2 - m^2\, \cos^2\theta)^{\nu}}$$

Singularities points:

$$r_{singularity} = \begin{cases} 0 & \text{for } \ \nu - \gamma < 2 \\ 2\,m & \text{for } \ \nu + \gamma < 2 \\ m\,(1 \pm \cos\theta) & \text{for } \ \nu > 0 \end{cases}$$

Generalised three parameter metrics

Five dimensional metrics:

$$ds^2 = -f^\gamma\, dt^2 + f^\mu\, k^\nu\, (\frac{dr^2}{f} + r^2\, d\theta^2) + r^2\, f^\beta\, (\sin^2\theta\, d\phi^2 + \cos^2\theta\, d\psi^2),$$

$$\mu + \nu = \beta, \qquad \beta = \frac{1-\gamma}{2}.$$

Higher dimensions?

Metrics functions:

$$f(r) = 1 - \frac{2\,m}{r^2}, \qquad k(r,\,\theta) = 1 - \frac{2\,m}{r^2} + \frac{m^2\,\sin^2(2\,\theta)}{r^4},$$

$$R_{rr} = \partial_r\varphi(r)\,\partial_r\varphi(r) \quad \Rightarrow \quad \varphi(r) = \sqrt{\frac{\frac{3}{4}\,(1-\gamma^2) - \nu}{2}}\,\ln(1 - \frac{2\,m}{r^2}).$$

Generalised three parameter metrics

Metrics limits:

five-dimensional JNW metric:

$$\nu = 0, \quad \mu = \frac{1-\gamma}{2}$$

five-dimensional γ-metric:

$$\nu = \frac{3}{4}(1-\gamma^2), \quad \mu = \frac{1}{4}(3\gamma^2 - 2\gamma - 1), \quad \varphi(r) = 0$$

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A. Hajibarat, B. M, A. Azizallahi, arXiv: 2110.06667

Ricci scalar in five dimensions:

$$R = \frac{2\,m^2\,(2\,\gamma + 4\,\mu + 1 - 3\,\gamma^2)}{r^{2+\gamma-2\,\nu+3}\,(r^2 - 2\,m)^{\frac{3-\gamma-2\,\nu}{2}}\,(r^4 - 2\,m\,r^2 + m^2\,\sin^2(2\,\theta))^{\nu}}$$

Singularities points:

$$r_{singularity} = \begin{cases} 0 & \text{for } \; 2\nu - \gamma < 3 \\ \sqrt{2\,m} & \text{for } \; \gamma + 2\,\nu < 3 \\ \sqrt{m\,(1 \pm \cos(2\,\theta))} & \text{for } \; \nu > 0 \end{cases}$$

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A. Azizallahi, B. M, A. Hajibarat, H. Anjomshoa, arXiv:2307.09328

Generalised three parameter metrics

The five-dimensional γ-metric in isotropic form:

$$ds^2 = -f^\gamma\, dt^2 + (1 + \frac{m}{2\,\rho})^2 \left[f^\mu\, k^\nu\, (d\rho^2 + \rho^2\, d\theta^2) + \rho^2\, f^\beta (\sin^2\theta\, d\phi^2 + \cos^2\theta\, d\psi^2)\right],$$

$$r = \rho\,(1 + \frac{m}{2\,\rho^2}),$$

Metrics parameters:

$$\mu + \nu = \frac{1-\gamma}{2}$$

Metrics functions:

$$f = (\frac{m - 2\,\rho^2}{m + 2\,\rho^2})^2, \qquad k = \frac{m^4 + 16\,\rho^8 - 8\,m^2\,\rho^4\,\cos(4\,\theta)}{(m + 2\,\rho^2)^4}.$$

Generalised three parameter metrics

Three parameter metrics in toroidal coordinates:

$$ds^2 = -f^{\gamma} dt^2 + \frac{f^{\mu}}{f} dr^2 + r^2 f^{\mu} d\theta^2 + r^2 f^{\frac{1-\gamma}{d-3}} \sum_{i=1}^{d-3} d\phi_i^2,$$

Metrics function:

$$f(r) = \frac{2\,m}{r^{d-3}}.$$

Scalar field:

$$\varphi(r) = \sqrt{\frac{d - 2\,\gamma - (d-2)\,\gamma^2}{4\,(d-3)} - \frac{\mu}{2}}\ \ln\left(\frac{2\,m}{r^{d-3}}\right).$$

Generalised three parameter metrics

Three parameter metrics in flat coordinates:

$$ds^2 = -f^{\gamma} dt^2 + \frac{f^{\mu} k^{\nu}}{f} dr^2 + r^2 f^{\mu} k^{\nu} d\theta^2 + \theta^2 r^2 f^{\frac{1-\gamma}{d-3}} d\phi^2$$

$$+ r^2 f^{\frac{1-\gamma}{d-3}} \sum_{i=1}^{d-4} d\psi_i^2, \qquad \mu + \nu = \frac{1-\gamma}{d-3},$$

Metrics functions:

$$f(r) = \frac{2m}{r^{d-3}}, \qquad k(r,\theta) = \frac{2m}{r^{d-3}} + \frac{(d-3)^2 m^2 \theta^2}{r^{2(d-3)}}.$$

Scalar field:

$$\varphi(r) = \sqrt{\frac{(d-2)(1-\gamma^2)}{4(d-3)} - \frac{\nu}{2}} \ \ln\left(\frac{2m}{r^{d-3}}\right).$$

Rotating three parameter metrics

Lewis-Weyl-Papapetrou (LWP) metric:

$$ds^2 = -f(dt - \omega\, d\phi)^2 + f^{-1}\left[e^{2\eta}\,(d\rho^2 + dz^2) + \rho^2\, d\phi^2\right].$$

Coordinate transformations:

$$\rho = (x^2-1)^{\frac{1}{2}}\,(1-y^2)^{\frac{1}{2}},$$

$$z = x\,y.$$

LWP metric in x and y coordinates:

$$ds^2 = -f(dt - \omega\, d\phi)^2 + \frac{\sigma^2}{f}\left[e^{2\eta}\,(x^2-y^2)(\frac{dx^2}{x^2-1} + \frac{dy^2}{1-y^2}) + (x^2-1)\,(1-y^2)\, d\phi^2\right].$$

Rotating three parameter metrics

From R_{tt} and $R_{t\phi}$:

$$\partial_x \omega = \sigma\,(1-y^2)\,f^{-2}\,\partial_y u, \quad \partial_y \omega = \sigma\,(1-x^2)\,f^{-2}\,\partial_x u.$$

From R_{xx}, R_{yy} and R_{xy}:

$$\partial_x \eta = \cdots + 4\,\pi\,\frac{(1-y^2)}{(x^2-y^2)}\,x\,(x^2-1)\,(\partial_x \varphi)^2,$$

$$\partial_y \eta = \cdots + 4\,\pi\,\frac{(x^2-1)^2\,y}{(x^2-y^2)}\,(\partial_x \varphi)^2.$$

From $\partial_y \partial_x \eta = \partial_x \partial_y \eta$:

$$\varphi'' + \frac{2\,x}{x^2-1}\,\varphi' = 0, \quad \Rightarrow \quad \varphi = c_1\,\ln\left(\frac{x-1}{x+1}\right) + c_2.$$

Rotating three parameter metrics

Ernst equations:

$$(\varepsilon + \varepsilon^*)\,\Delta\varepsilon = 2\,(\nabla\varepsilon)^2,$$

$$\Delta \equiv \sigma^{-2}\,(x^2 - y^2)^{-1}\,\partial_x\left[(x^2-1)\,\partial_x\right] + \partial_x\left[(x^2-1)\,\partial_x\right],$$

$$\nabla \equiv \sigma^{-1}\,(x^2 - y^2)^{-\frac{1}{2}}\,\hat{i}\left[(x^2-1)^{\frac{1}{2}}\,\partial_x\right] + \hat{j}\left[(1-y^2)^{\frac{1}{2}}\,\partial_x\right].$$

Ernst potential:

$$\varepsilon = f + i\,u,$$

$$\sigma = \sqrt{m^2 - a^2},$$

$a =$ rotation parameter.

Rotating three parameter metrics

Ernst potential:

$$\varepsilon = \Big(\frac{x-1}{x+1}\Big)^q \; \frac{x-1+(x^2-1)^{-q}\; d_+}{x+1+(x^2-1)^{-q}\; d_-},$$

$$\begin{aligned}
d_\pm &= -\alpha^2\,(x\pm 1)\,h_+\,h_-(x^2-1)^{-q} \\
&\quad + i\,\alpha\,\big[y\,(h_+ + h_-)\pm(h_+ - h_-)\big], \\
h_\pm &= (x\pm y)^{2\,q}, \\
\alpha &= -\frac{a}{m+\sigma}, \\
q &= 1-\gamma.
\end{aligned}$$

Rotating three parameter metrics

Metrics functions :

$$f = \frac{A}{B},$$

$$\omega = -2\left(a + \sigma\,(\frac{C}{A})\right),$$

$$e^{2\eta} = \frac{1}{4}\left(1 + \frac{m}{\sigma}\right)^2 \frac{A}{(x^2-1)^{1+q}}\,\left(\frac{x^2-1}{x^2-y^2}\right)^{1-\nu},$$

$$A = \; a_+\,a_- + b_+\,b_-, \qquad B = \; a_+^2 + b_+^2,$$

$$C = \;(x+1)^q\left[x(1-y^2)\,(\lambda+\xi)\,a_+ + y\,(x^2-1)\,(1-\lambda\,\xi)\,b_+\right],$$

$$a_\pm = (x \pm 1)^q\,\left[x(1-\lambda\,\xi) \pm (1+\lambda\,\xi)\right],$$

$$b_\pm = (x \pm 1)^q\,\left[y(\lambda+\xi) \mp (\lambda-\xi)\right],$$

$$\lambda = \alpha\left(x^2-1\right)^{-q}(x+y)^{2\,q}, \qquad \xi = \alpha\left(x^2-1\right)^{-q}(x-y)^{2\,q}.$$

Rotating three parameter metrics

Coordinate conversion:

$$x = \frac{r - m}{\sqrt{m^2 - a^2}}, \qquad y = \cos\theta.$$

Scalar field:

$$\Box\varphi = 0, \quad \Rightarrow \quad \varphi(r) = \sqrt{\frac{1 - \gamma^2 - \nu}{2}}\, \ln\left(\frac{r - m - \sqrt{m^2 - a^2}}{r - m + \sqrt{m^2 - a^2}}\right).$$

Ricci tensor:

$$R_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \partial_r\varphi\,\partial_r\varphi & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Rotating three parameter metrics

Limits of metrics:

$\nu = 1 - \gamma^2, \quad \mu = \gamma^2 - \gamma$ $\Rightarrow$ rotating $\boldsymbol{\gamma}$-metric

$\nu = -\mu, \quad \gamma = 1\,(q = 0)$ $\Rightarrow$ Bogush-Gal'tsov (BG) metric

$\nu = 1 - \gamma = -q, \quad \mu = 0$ $\Rightarrow$ rotating metric at $\nu = -q$

$\nu = 0, \quad \mu = 1 - \gamma$ $\Rightarrow$ rotating FJNW metric

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I. Bogush, D. Gal'tsov, arXiv: 2001.02936

B. M, P.K Kangazian Kangazi, F. Sadeghi, arXiv: 2307.13588

Rotating three parameter metrics

Quasi Normal Modes (QNMs) for rotating JNW metric by using light-ring method $\nu = 0,\ \mu = 1-\gamma$**:**

$$
\begin{aligned}
&Q_{JNW} = (\Omega + i\,\Gamma)_{JNW} \\
&= j\,\Big[\frac{1}{3\sqrt{3}\,m} \pm \frac{1}{3\sqrt{3}\,m}\,\Big(\frac{11}{54}\,(\frac{a}{m})^2 + \frac{2\sqrt{3}\,a}{9\,m} \\
&- q\,\ln 3\Big)\Big] + i\,\Big(n + \frac{1}{2}\Big)\,\Big\{\frac{1}{3\sqrt{3}\,m}\Big[1 - \frac{2\,a^2}{27\,m^2} \\
&- q\,\ln 3\Big]\Big\}.
\end{aligned}
$$

QNMs for BG metric $\gamma = 0,\ \mu = -\nu$:

$$Q_{BG} = (\Omega + i\,\Gamma)_{BG}$$
$$= j\,\Big[\frac{1}{3\,\sqrt{3}\,m} \pm \frac{1}{3\,\sqrt{3}\,m}\big(\frac{11}{54}\,(\frac{a}{m})^2 + \frac{2\,\sqrt{3}\,a}{9\,m}\big)\Big]$$
$$+ i\,\big(n + \frac{1}{2}\big)\,\Big\{\frac{1}{3\,\sqrt{3}\,m}\Big[1 - \frac{2\,a^2}{27\,m^2} - \nu\,(\ln 2$$
$$- \frac{\ln 3}{2})\Big]\Big\}.$$

Rotating three parameter metrics

Comparison between stability of rotating JNW metric: and BG metric:

$|q_{JNW}| \geq |\nu_{BG}| \qquad \Rightarrow \qquad |\Gamma_{JNW}| \geq |\Gamma_{BG}|$

If the q parameter in the rotating JNW metric is greater than or equal to the ν parameter in the BG metric, the BG metric is more stable compared to the rotating JNW metric.

Charged three parameter metrics

Einstein-Hilbert action in the presence of an electromagnetic field and a massless scalar field:

$$S = \int d^4x \sqrt{-g} \, [R - F_{\mu\nu} \, F^{\mu\nu} + g^{\mu\nu} \, \partial_\mu \varphi \, \partial_\nu \varphi].$$

Field equations:

$$R_{\mu\nu} = \partial_\mu \varphi(r) \, \partial_\nu \varphi(r) + T_{\mu\nu}{}^{EM},$$

$$\Box \varphi(r) = 0,$$

$$T^{EM}_{\mu\nu} = 2 \, F_\mu{}^{\alpha} \, F_{\nu\alpha} - \frac{1}{2} \, g_{\mu\nu} \, F_{\gamma\sigma} \, F^{\gamma\sigma}.$$

Charged three parameter metrics

Ernst equations in the presence of the electromagnetic field:

$$\big(Re(\varepsilon) + \Phi\,\Phi^\star\big)\,\nabla^2\,\varepsilon = \nabla\,\varepsilon\,.\,(\nabla\,\varepsilon + 2\,\Phi^\star\,\nabla\,\Phi),$$
$$\big(Re(\varepsilon) + \Phi\,\Phi^\star\big)\,\nabla^2\,\Phi = \nabla\,\Phi\,.\,(\nabla\,\varepsilon + 2\,\Phi^\star\,\nabla\,\Phi).$$

Ernst potential:

$$\varepsilon := f - |\Phi|^2 + \imath\,\chi$$

Φ and χ functions:

$$\Phi := A_t + \imath\,\tilde{A}_\phi,$$
$$\hat{\phi}\times\nabla\,\tilde{A}_\phi := -\rho^{-1}\,f(\nabla\,A_\phi + \omega\,\nabla\,A_t).$$
$$\hat{\phi}\times\nabla\,\chi := -\rho^{-1}\,f^2\,\nabla\,\omega - 2\,\hat{\phi}\times Im(\Phi^\star\,\nabla\,\Phi).$$

Charged three parameter metrics

LWP metric in $(t,\ \rho,\ z,\ \phi)$ coordinate:

$$ds^2 = -f(dt - \omega\, d\phi)^2 + f^{-1}\left[e^{2\,\eta}\,(d\rho^2 + dz^2) + \rho^2\, d\phi^2\right].$$

three parameter metric in $(t,\ r,\ \theta,\ \phi)$ coordinate:

$$ds^2 = -f^{\gamma}\, dt^2 + f^{\mu}\, k^{\nu}\,(\frac{dr^2}{f} + r^2\, d\theta^2) + f^{1-\gamma}\, r^2\, d\phi^2.$$

Charged three parameter metrics

Harrison transformation:

$$\varepsilon' = \frac{\varepsilon}{1 - 2\,\alpha^\star\,\Phi - |\alpha|^2\,\epsilon}, \qquad \Phi' = \frac{\alpha\,\varepsilon + \Phi}{1 - 2\,\alpha^\star\,\Phi - |\alpha|^2\,\epsilon}.$$

Metrics functions:

$$\varepsilon = f(r) = (1 - \frac{2\,m}{r})^\gamma,$$

$$\omega = \Phi = 0,$$

$$e^{2\,\eta} = \left[1 + \frac{m^2\,\sin^2\theta}{r\,(r - 2\,m)}\right]^\nu\,\left[\frac{2\,r\,(r - 2\,m)}{m^2\,[1 - \cos(2\,\theta)] + 2\,r\,[r - 2\,M]}\right].$$

Charged three parameter metrics

Harrison transformation:

$$\varepsilon' = \frac{1}{r^\gamma\,(r-2\,m)^{-\gamma} - |\alpha|^2}, \qquad \Phi' = \frac{\alpha}{r^\gamma\,(r-2\,m)^{-\gamma} - |\alpha|^2}.$$

Metrics functions:

$$f' = \varepsilon' + |\Phi'|^2 = \frac{\left[r\,(r-2\,m)\right]^\gamma}{\left[r^\gamma - (r-2\,m)^\gamma - |\alpha|^2\right]^2},$$

$$\chi' = 0.$$

Vector potential:

$$A' = \frac{\alpha_R\,(r-2\,m)^\gamma}{r^\gamma - |\alpha|^2\,(r-2\,m)^\gamma}\,dt + 2\,\gamma\,m\,\alpha_I\,\cos\theta\,d\phi.$$

Charged three parameter metrics

Transformations:

$$R = r(1-|\alpha|^2) + 2\,m\,|\alpha|^2, \qquad T = \Big(\frac{t}{1-|\alpha|^2}\Big)^{\gamma},$$

$$m = \frac{M}{1+|\alpha|^2}, \qquad q_m = 2\,m\,\alpha_I, \qquad q_e = -2\,m\,\alpha_R.$$

$$e^2 := q_e^2 + q_m^2, \qquad \xi := \frac{e}{M}, \qquad p =: \sqrt{1-\xi^2}.$$

Charged three parameter metrics

Final charged metrics:

$$ds^2 = -\frac{\Delta^\gamma}{K^2}\, dt^2 + K^2\, \Delta^\mu\, \Sigma^\nu \left[\frac{1}{\Delta}\, dR^2 + R^2\, d\theta^2\right]$$
$$+ R^2\, K^2\, \Delta^{1-\gamma}\, \sin^2\theta\, d\phi^2,$$

Metrics Functions:

$$\Delta = 1 - \frac{2\,M}{R} + \frac{q_e^2 + q_m^2}{R^2},$$
$$\Sigma = 1 - \frac{2\,M}{R} + \frac{q_e^2 + q_m^2}{R^2} + \frac{(M^2 - (q_e^2 + q_m^2))\, \sin^2\theta}{R^2},$$
$$K = \left(\frac{1}{2\,p}\right)^\gamma (1+p)^{\gamma-1} \Big[(1+p)\big(1 - \frac{M\,(1-p)}{R}\big)^\gamma$$
$$- (1-p)\,\big(1 - \frac{M\,(1+p)}{R}\big)^\gamma\Big].$$

Charged three parameter metrics

Vector potential:

$$A' = \frac{F(R)}{K}\, dt + q_m\, \gamma\, \cos\theta\, d\phi,$$

$$F(R) = \frac{q_e}{(1+p)\,M}\,\big(1 - \frac{(1+p)\,M}{R}\big)^{\gamma}$$

Scalar field:

$$\Box\varphi = 0, \quad \Rightarrow \quad \varphi(r) = \sqrt{\frac{1-\gamma^2-\nu}{2}}\, \ln\big(\frac{r - m - \sqrt{m^2 - q_e^2 - q_m^2}}{r - m + \sqrt{m^2 - q_e^2 - q_m^2}}\big).$$

Charged three parameter metrics

Ricci scalar:

$$R(r,\,\theta) = \frac{Y(r,\,\theta)}{r^2\,K^3\,\Delta^{1+\mu}\,\Sigma^{2+\nu}}$$

Singularities points:

$$r_{singularity} = \begin{cases} 0 & \text{for all } \mu \text{ and } \nu \\ m \pm \sqrt{m^2 - (q_e^2 + q_m^2)} & \text{for } \mu > -1 \\ m \pm \sqrt{(m^2 - (q_e^2 + q_m^2))\,\cos^2\theta} & \text{for } \nu > -2 \end{cases}$$

Charged three parameter metrics

Charged metric in the presence of a scalar field at $\gamma = 1$:

$$ds^2 = -(1 - \frac{2\,m}{r} + \frac{q^2}{r^2})\,dt^2 + \left[1 + \frac{(m^2 - q^2)\,\sin^2\theta}{r^2 - 2\,m + q^2}\right]^\nu$$

$$\times \left[\frac{dr^2}{1 - 2\,m/r + q^2/r^2} + r^2\,d\theta^2\right] + r^2\,\sin^2\theta\,d\phi^2.$$

In the absence of a scalar field ($\nu = 0$), the above metric becomes the Reissner–Nordström metric.

Conclusion:

✓The JNW metric is not the most general solution for the Einstein-Hilbert action in the presence of a scalar field.
We derived a more general three parameter solution.

✓The Janis-Newman method is not a suitable method for obtaining the rotating JNW metric.
We obtained a class of rotating metrics that include the rotating JNW metric using the Ernst's method.

✓ By using Ernst's equations in the presence of an electromagnetic field and also using Harrison's transformations, we were able to obtain the three-parameter metric charged form in the presence of a scalar field.

✓ A lot of new calculations can be done for the new three parameter rotating metrics in the future.

Thank You!