Mutual dependence of a bosonic black hole with dark matter and possible explanation of asymptotically flat galaxy rotation curves

B. E. Meierovich
P. L. Kapitza institute for physical problems
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Longitudinal vector field

Dark matter makes itself felt only due to the gravitational interaction. Quanta of ordinary matter in flat space are described by vector fields. Let us assume that the wave function of dark matter quanta is also a vector field. Then it makes sense to find such a vector field in the general theory of relativity, which manifests itself exclusively in curved space-time.

Within the frames of minimal general relativity (field equations no higher than of the second order), the Lagrangian of a vector field φ_i is a scalar consisting of bilinear combinations of the covariant derivatives $\varphi_{i;k}$ and a scalar potential $V\left(\varphi^k\varphi_k\right)$. Bilinear combination of the covariant derivatives is a 4-index tensor

$$S_{iklm} = \varphi_{i;k}\varphi_{l;m}\,.$$

A general form of the Lagrangian *L*, formed via contractions of S_{iklm}, is

$$L = a\left(\varphi^m_{;m}\right)^2 + b\varphi^l_{;m}\varphi_l^{;m} + c\varphi^l_{;m}\varphi^m_{;l} - V\left(\varphi_m\varphi^m\right),$$

where *a*, *b*, and *c* are arbitrary constants. In general relativity the second covariant derivative of a vector is not invariant against the replacement of the order of differentiation:

$$\varphi^l_{;l;m} - \varphi^l_{;m;l} = R_{km}\varphi^k\,.$$

R_{km} is the Ricci tensor. In a curved space-time, all three kinetic terms in the Lagrangian *L* are equivalent.

The covariant derivative $\varphi_{i;k}$ can be presented as a sum of a symmetric G_{ik} and an antisymmetric F_{ik} parts: $\varphi_{i;k} = G_{ik} + F_{ik}$, $G_{ik} = \frac{1}{2}\left(\varphi_{i;k} + \varphi_{k;i}\right)$, $F_{ik} = \frac{1}{2}\left(\varphi_{i;k} - \varphi_{k;i}\right)$.

The scalar *S* can be presented in the form

$$S = a\left(G^k_k\right)^2 + (b+c)G^i_kG^k_i + (b-c)F^i_kF^k_i\,.$$

In a flat space-time $R_{km} = 0$, and therefore only two of the three kinetic terms are independent. As applied to ordinary matter, this gradient (gauge) invariance allows one to set $a = 0$. Then the covariant divergence $\varphi^l_{;l}$ becomes an arbitrary function that does not affect the action. In electrodynamics $\varphi^l_{;l} = 0$ is referred to as Lorentz gauge.

If we set $b = c = 0$ and $a \neq 0$, we get the Lagrangian

$$L = a\left(\varphi^m_{;m}\right)^2 - V\left(\varphi_m\varphi^m\right)$$

for a longitudinal vector field, which is not related to the ordinary matter. It turns out that in a curved space-time the longitudinal vector field φ_m adequately describes the observed properties of dark matter. The wave equation

$$a\varphi^m_{;m,k} = -V'\varphi_k\,,\quad V' = \frac{dV}{d\left(\varphi_m\varphi^m\right)}$$

follows from the Euler-Lagrange equation.

On the scale of a galaxy, the gravitational interaction is dominant. In the power series of the potential

$$V\left(\varphi_M\varphi^M\right) = V_0 + V_0'\varphi_M\varphi^M + \lambda\left(\varphi_M\varphi^M\right)^2 + \ldots \quad V_0' = \frac{dV(x)}{dx}\Big|_{x\to 0}$$

the first term V_0 is an addition to the cosmological constant. It affects the expansion of the Universe. On a galactic scale, the role of V_0 is negligible. The second term $V_0'\varphi_M\varphi^M$ is the source of gravitational interaction. The third term $\lambda\left(\varphi_M\varphi^M\right)^2$ is a correction for non-gravitational interactions of particles. It includes the elasticity of matter. Restricting to the term $V_0'\varphi_M\varphi^M$, we consider gravitating dark matter as an ideal gas. The mass of a quantum of a longitudinal vector field is denoted by μ, so as not to be confused with the mass m of a quantum of the bosonic scalar field of a black hole.

In the space-time with a static centrally symmetric metric

$$ds^2 = g_{ik}dx^i dx^k = e^{\nu(r)}\left(dx^0\right)^2 - e^{\lambda(r)}dr^2 - r^2 d\Omega^2$$

the energy-momentum tensor $T^k_{dark\,i}$ of a longitudinal vector field is

$$T^k_{dark\,i} = \delta^k_i \begin{cases} a\left(\varphi^m_{;m}\right)^2 - V_0' e^{\lambda}\left(\varphi^r\right)^2, & i = r, \\ a\left(\varphi^m_{;m}\right)^2 + V_0' e^{\lambda}\left(\varphi^r\right)^2, & i \neq r. \end{cases}$$

The gravitational interaction is described by Einstein equations

$$R_{ik} - \frac{1}{2}g_{ik}R = \kappa T_{ik}$$

According to the observed rotation curves of galaxies, the De Broglie wavelength $\lambda\!\!\!^{-} = \hbar/\mu c$ of the longitudinal vector field φ_m is about fifteen kiloparsec. This is many orders of magnitude larger than the size of a black hole. The surface radius r_h of the black hole in our Milky Way galaxy is less than 0.0002 light years. Therefore, the covariant divergence of the longitudinal field $\varphi^m_{;m}(r_h)$ practically is indistinguishable from $\varphi^m_{;m}(0)$. In the asymptotic region $r \sim \lambda\!\!\!^{-} >> r_h$ metric function $\lambda(r) << 1$. Linearized Einstein equations for metric functions $\nu(r)$, $\lambda(r)$

$$\nu' = \kappa r\left[\left(\frac{1}{\lambda\!\!\!^{-}}\varphi^r\right)^2 + \left(\varphi^m_{;m}\right)^2\right] + \frac{\lambda}{r}, \qquad \lambda' + \frac{\lambda}{r} = \kappa r\left[\left(\frac{1}{\lambda\!\!\!^{-}}\varphi^r\right)^2 - \left(\varphi^m_{;m}\right)^2\right]$$

together with the Klein-Gordon equation for the covariant divergence of a longitudinal vector field

$$\frac{1}{r^2}\frac{d}{dr}r^2\frac{d\varphi^m_{;m}}{dr} + \frac{1}{\lambda\!\!\!^{-2}}\varphi^m_{;m} = 0$$

make it possible to find the dependence on the radius of the speed of rotation of the star around the center $V(r)$ in the asymptotic region $r >> r_h$.

The solution of the Klein-Gordon equation, regular at the center, has the form

$$\varphi^m_{;m}(r) = \varphi^m_{;m}(0)\frac{\lambda\!\!\!^{-}}{r}\sin\left(\frac{r}{\lambda\!\!\!^{-}}\right).$$

From the wave equation $\varphi^m_{;m,k} = \frac{1}{\lambda\!\!\!^{-2}}\varphi_k$ we obtain

$$\varphi^r(r) = -\varphi^m_{;m}(0)\frac{\lambda\!\!\!^{-3}}{r^2}\left[\sin\left(\frac{r}{\lambda\!\!\!^{-}}\right) - \frac{r}{\lambda\!\!\!^{-}}\cos\left(\frac{r}{\lambda\!\!\!^{-}}\right)\right].$$

By substituting $\varphi^r(r)$ и $\varphi^m_{;m}(r)$ into the linearized Einstein equations ($\nu' = \ldots$ и $\lambda' = \ldots$) we fined:

$$r\frac{d\nu}{dr} = \kappa\left(\varphi^m_{;m}(0)\right)^2\lambda\!\!\!^{-2}\left[1 - \frac{\lambda\!\!\!^{-}}{r}\sin\left(\frac{2}{\lambda\!\!\!^{-}}r\right) + \left(\frac{\lambda\!\!\!^{-}}{r}\right)^2\sin^2\left(\frac{r}{\lambda\!\!\!^{-}}\right)\right] + \lambda,$$

$$\lambda(r) = \kappa\left(\varphi^m_{;m}(0)\right)^2\lambda\!\!\!^{-2}\left[\frac{\lambda\!\!\!^{-}}{2r}\sin\left(\frac{2}{\lambda\!\!\!^{-}}r\right) - \left(\frac{\lambda\!\!\!^{-}}{r}\right)^2\sin^2\left(\frac{r}{\lambda\!\!\!^{-}}\right)\right].$$

Deriving $\lambda(r)$, I used the identity

$$\left(\frac{\sin(ar)}{ar}\right)^2-\frac{\sin(2ar)}{ar}+\cos(2ar)=\frac{d}{dr}\left(\frac{\sin(2ar)}{2a}-\frac{\sin^2(ar)}{a^2r}\right).$$

When a star rotates around the center of the galaxy, the centripetal acceleration $\frac{c^2}{2}\frac{d\nu}{dr}$ is balanced by the centrifugal $\frac{V^2}{r}$. The speed of motion of the star $V(r)$ as a function of radius r asymptotically reaches a plateau with damped oscillations:

$$V(r)=c\sqrt{\frac{1}{2}r\frac{d\nu}{dr}}=V_{plat}\sqrt{1-\frac{\bar{\lambda}}{2r}\sin\left(\frac{2}{\bar{\lambda}}r\right)}, \qquad r>>r_h.$$

Graph of the function $\sqrt{1-\frac{\sin 2x}{2x}}$ is shown in Figure 1:

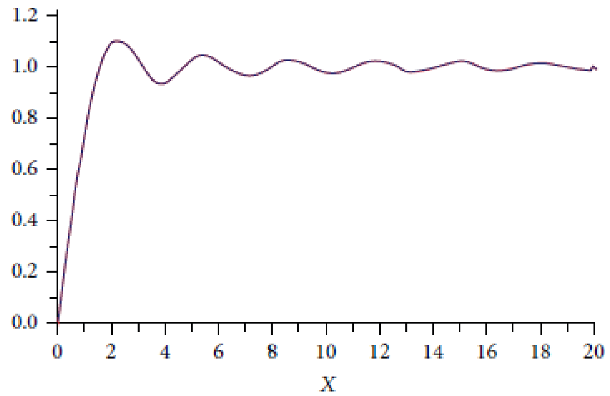


Figure 1.

The plateau velocity $V_{plat}=c\sqrt{\frac{\kappa}{2}\bar{\lambda}\varphi^m_{;m}(r_h)}$ is proportional to the covariate divergence $\varphi^m_{;m}(r_h)$, indistinguishable from the divergence in the center $\varphi^m_{;m}(0)$ because $\bar{\lambda}>>r_h$. For each galaxy, the value $\varphi^m_{;m}(r_h)$ depends on the interaction of the longitudinal vector field $\varphi^r(r)$ with the black hole located in the center of the galaxy.

On the static state of a black hole in the center of a galaxy

it is necessary to note the most important role of the dark sector. In a vacuum (without the stabilizing effect of dark matter), the equilibrium state of a super heavy black hole is impossible. According to NASA's "slicing the pie" chart, the universe contains only 4.6% ordinary matter.

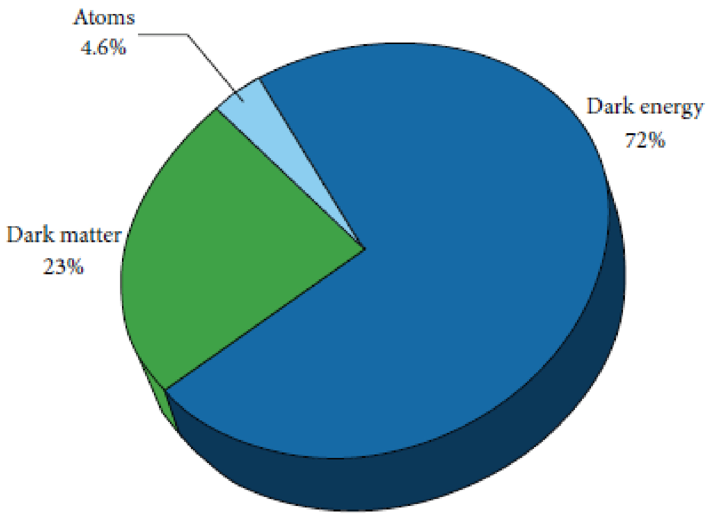


Figure 2. Composition of the Universe.
The remaining 95% is the so-called dark matter 23% and dark energy 72%.

It is believed that a black hole is a process of unlimited compression (collapse) of matter under the influence of the dominant force of its own gravitational field. Galaxies with black holes at their centers have existed for as long as the Universe. With such a slow evolution of a black hole, the locally equilibrium concentration of particles entering into chemical reactions transforming one into another depends on temperature and pressure, and does not depend on specific reaction channels. To find out a connection between dark matter and a black hole, it is necessary to show that there is an equilibrium state to which gravitational collapse can lead.

In the process of a collapse, with increasing pressure, elementary particles of the Standard Model

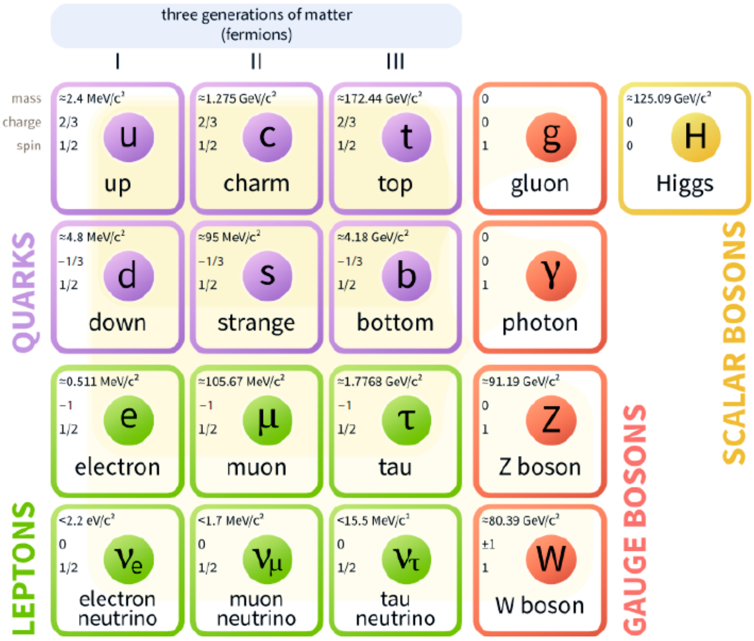


Figure 3. Standard Model of Elementary Particles

can become dominant at the next step after neutrons.

The energetically most favorable state of matter at low temperatures is the Bose-Einstein condensate of massive bosons. These can be gauge bosons Z and W, the scalar Higgs boson H, as well as boson quasiparticles of paired fermions (Cooper effect).

The wave function of a Bose-Einstein condensate is a scalar field. Lagrangian of a complex scalar field ψ

$$L = g^{ik}\psi^*_{,i}\psi_{,k} - U\left(|\psi|^2\right), \quad U(0) = 0.$$

In expansion of the potential

$$U\left(|\psi^2|\right) = \left(\frac{mc}{\hbar}\right)^2 |\psi|^2 + \frac{1}{2}\lambda|\psi|^4 + \dots$$

m is the rest mass of a boson. Scalar functions ψ and ψ^* satisfy the Klein-Gordon equation:

$$\frac{1}{\sqrt{\det g_{ik}}}\left(\sqrt{\det g_{ik}}\, g^{lm}\psi_{,l}\right)_{,m} = -\frac{\partial U}{\partial|\psi^2|}\psi.$$

This equation is invariant against the sign change of $\det g_{ik}$: $\sqrt{-1}$ in the numerator and denominator are cancelled. Static spherically symmetric scalar field in the state of definite energy E per particle

$$\psi_E\left(x^i\right) = e^{-iEx^0/\hbar c}\psi(r)$$

depends formally on two coordinates x^0 и r. But in statics in the Klein-Gordon

$$g^{rr}\psi'' + \left(\left(g^{rr}\right)' + \frac{1}{2}\left(\ln\left(\det g_{ik}\right)\right)' g^{rr} \right)\psi' = \left(\frac{1}{\hbar^2 c^2}\left(g^{00}E^2 - m^2c^4\right) - \lambda|\psi|^2 \right)\psi$$

and Einstein

$$\left(g^{rr}\right)' + \frac{1+g^{rr}}{r} = \kappa r T_0^0,$$

$$g^{rr}\left(\frac{1}{r} - \left(\ln g^{00}\right)'\right) + \frac{1}{r} = \kappa r T_r^r$$

equations time x^0 is a cyclic variable. The x^0 coordinate is not explicitly included in the components T_0^0 and T_r^r of the energy-momentum tensor:

$$T_0^0 = \frac{1}{\hbar^2 c^2}\left(g^{00}E^2 + m^2c^4\right)|\psi|^2 + \frac{1}{2}\lambda|\psi|^4 - g^{rr}|\psi'|^2,$$

$$T_r^r = \frac{1}{\hbar^2 c^2}\left(-g^{00}E^2 + m^2c^4\right)|\psi|^2 + \frac{1}{2}\lambda|\psi|^4 + g^{rr}|\psi'|^2.$$

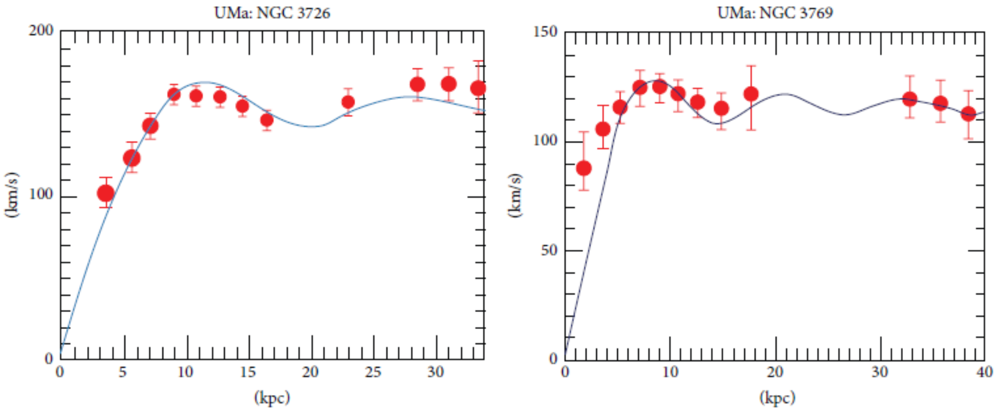
Three equations (one Klein-Gordon and two Einstein) for three functions $\psi(r)$, $g^{00}(r)$ and $g^{rr}(r)$ determine the static state of the gravitating Bose-Einstein condensate.

The metric component $g^{rr}(r)$ is the coefficient of the highest derivative of the Klein-Gordon equation. From the point of view of the existence and uniqueness theorem at gravitational radii $r = r_g$ and $r = r_h > r_g$ (at which in the Schwarzschild metric $g^{rr}(r) = 0$) the solution $\psi(r)$ exists, but is not unique. For an arbitrarily large black hole mass, the presence of an internal gravitational radius r_g ensures the existence of a static solution, regular at the center. The sphere with gravitational radius r_h is the boundary of a black hole with dark matter. The non-uniqueness of solutions with boundary conditions at gravitational radii r_g and r_h confirms the possibility of existence of a massive black hole in a regular static state surrounded by dark matter.

It follows from Einstein equations that at the boundary $r = r_h$ the components of the energy-momentum tensor $T_0^0(r_h) = 0$ and $T_r^r(r_h) = 1/\kappa r_h^2$. The covariant divergence of a vector field is a scalar that satisfies the Klein-Gordon equation. The scalar wave function of the Bose condensate also satisfies the Klein-Gordon equation, but only with a different quantum mass. One may joke that the divergence of the longitudinal field of dark matter in the region $r > r_h$ is the wave function of the Bose condensate inside the black hole $r < r_h$, "turned inside out". The condition of continuity of pressure at the interface made it possible to determine the dependence of the plateau velocity on the black hole mass:

$$V_{plat} = c\frac{M_{Пл}^2}{4\sqrt{\mu m M}}.$$

Here $M_{Пл} = \sqrt{\hbar c/k} = 2{,}177 \times 10^{-5}\,г$ – Planck mass, M – mass of a black hole, μ and m are the rest masses of the quanta of dark matter and black hole, respectively. $k = 6.67 \times 10^{-8}\, см^3/г \times сек^2$ is the gravitation constant.



Figures 4.

The rotation curves of two spiral galaxies (No. 3726 and No. 3769) in the Ursa Major cluster are shown in Figures 4. The acronym NGC stands for New General Catalog of Nebulae and Star Clusters. The vertical axis is the velocity V in km/sec, and the horizontal axis is the distance r from the galactic center in kiloparsecs. Points with error bars are observations. Solid curves are approximations according to the formula

$$V(r)=c\frac{M_{Pl}^2}{4\sqrt{\mu m M}}\sqrt{1-\frac{\lambda\!\!\!^{-}}{2r}\sin\left(\frac{2}{\lambda\!\!\!^{-}}r\right)}.$$

Galaxy #3726 (left picture) has the plateau velocity V=150 km/sec. The De Broglie wavelength $\lambda\!\!\!^{-}\approx 16\,\text{kpc}$. The rest mass of the dark matter quantum is $\mu=\hbar/c\lambda\!\!\!^{-}\approx 0.76\times10^{-60}$ г.

Galaxy #3769 on the (right side of Figure 4) has a plateau velocity of V= 120 km/sec and a wavelength of $\lambda\!\!\!^{-}\approx 13\,\text{kpc}$. The rest energy of massive bosons of the Standard Model of Elementary Particles is about 100 GeV. For quantitative estimates, we will assume that the rest mass of the black hole bosons is $m\approx 1.78\times10^{-22}$ г. It turns out that the masses of black holes in the centers of these galaxies $M_{3726}\approx 2\times10^{34}$ г and $M_{3769}\approx 2.3\times10^{34}$ г. The accuracy of estimating masses of these two black holes is low, as long as it is not clear which bosons make up the Bose-Einstein condensate.

Gravitational field of a black hole in the dark matter halo

Outside a black hole $r>r_h$ Einstein equation

$$\lambda'+\frac{\lambda}{r}=\kappa r\left[\left(\frac{1}{\lambda\!\!\!^{-}}\varphi^r\right)^2-\left(\varphi^m_{;m}\right)^2\right]$$

.

is a linear inhomogeneous ordinary differential equation. Its complete solution consists of the sum of the general solution of the homogeneous equation and the special solution of the inhomogeneous equation. Special solution to the inhomogeneous equation

$$\lambda(r)=2\frac{V_{plat}^2}{c^2}\left[\frac{\lambda\!\!\!^{-}}{2r}\sin\left(\frac{2}{\lambda\!\!\!^{-}}r\right)-\left(\frac{\lambda\!\!\!^{-}}{r}\right)^2\sin^2\left(\frac{r}{\lambda\!\!\!^{-}}\right)\right],\quad r>r_h$$

is the contribution of dark matter to gravity. The general solution to the homogeneous equation $\lambda'+\dfrac{\lambda}{r}=0$ in our case is the Schwarzschild solution $\lambda(r)=r_h/r$. This is the contribution of a black hole to the gravitational field in the region $r>r_h$, occupied by dark matter.

Dependence of star speed

$$V(r)=\sqrt{V_{plat}^2\left(1-\frac{\lambda\!\!\!^{-}}{2r}\sin\left(\frac{2}{\lambda\!\!\!^{-}}r\right)\right)+c^2\frac{r_h}{2r}}$$

on the mass M of the black hole located in the center of the galaxy manifests itself in two ways. Due to dark matter, plateaus velocity $V_{plat} = c\dfrac{M_{\text{Пл}}^2}{4\sqrt{\mu m M}}$ decreases with increasing of black hole mass $\sim M^{-1}$. And simply due to the attraction to the black hole (in view of $r_h = \dfrac{2kM}{c^2}$) the speed of the star $V(r)$ increases with increasing mass $\sim\sqrt{M}$. At distances from the center $r \sim \lambda\!\!\!^{-}$ contributions to gravity directly from the black hole and through dark matter are of the same order for the black hole mass

$$M \sim \widetilde{M} \equiv \frac{M_{\text{Пл}}^2}{\left(16m\mu^2\right)^{1/3}}.$$

When $M >> \widetilde{M}$ the speed of a star $V(r)$ decreases proportionally $1/\sqrt{r}$, according to Newton's theory. And vice versa, with a black hole mass $M << \widetilde{M}$ the rotation curve of the stars in the galaxy $V(r)$ transfers to a plateau. With the rest energy of bosons ~100 GeV of the Bose-Einstein condensate (with mass $m \approx 1.78\times 10^{-22}$ г) and with the mass of a quant of the longitudinal vector field $\mu = \hbar / c\lambda\!\!\!^{-} \approx 0.76\times 10^{-60}$ г, we have $\widetilde{M} \approx 4\times 10^{37}$ г.

A comparison of the observed (vertical axis) centripetal acceleration with the Newtonian one (horizontal axis) for 240 different galaxies is presented in Figure 5.

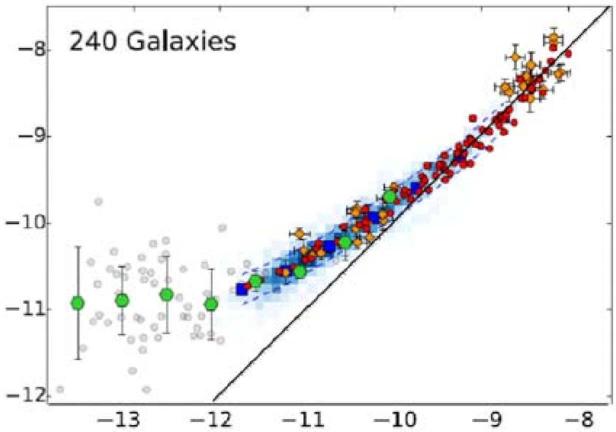


Figure 5.

Without dark matter, all points would lie on a straight line at the angle 45 degrees from the axes. For stars of different galaxies moving in a circle of the same radius, the accelerations are proportional to the masses of black holes at the centers of their galaxies. Therefore, the logarithms of black hole masses are actually plotted along the axes.

Masses $M_{3726} \approx 2\times 10^{34}$ г and $M_{3769} \approx 2.3\times 10^{34}$ г of the spiral galaxies NGC3726 and NGC3769 of the Ursa Major cluster are much smaller than $\widetilde{M} \approx 4\times 10^{37}$ г. Their place is at the left bottom side in Figure 5. The mass $M_{\text{МП}} = 8.6\times 10^{39}$ г of the black hole in the center of our фтMilky Way galaxy is two orders of magnitude bigger than $\widetilde{M}$. Dark matter does not play a significant role in our Milky Way galaxy. Among the 240 galaxies in Figure 5, our Milky Way's location is at the right top.

You can find details in my articles:

1. Vector fields in multidimensional cosmology. *Phys. Rev.* **D84**, 064037 (2011) arXiv:1105.4420

2. Vector fields in multidimensional cosmology. Proceedings of PIRT-2011 Moscow, p.211, (2012)
3. Towards the theory of evolution of the Universe. *Phys. Rev.* **D85**, 123544 (2012) arXiv:1201.2562
4. Galaxy rotation curves driven by massive vector fields: Key to the theory of dark sector. *Phys. Rev.* **D87**, 103510 (2013) arXiv:1303.7062
5. Macroscopic theory of dark sector. Physical Interpretation of Relativity Theory Proceedings of International Scientific Meeting PIRT-2013. Moscow: 1-4 July, 2013. p440. Bauman Moscow State Technical University (2013)
6. Macroscopic theory of dark sector. *Journ. of Gravity* **2014**, 586958 (2014)
7. Phenomenological description of dark energy and dark matter by vector fields. Physical Interpretation of Relativity Theory. Proceedings of International Scientific Meeting PIRT-2015. Moscow: 29 June - 02 July, 2015. p 384. Bauman Moscow State Technical University (2015)
8. Description of dark energy and dark matter by vector fields. Proceedings of the Tenth Asia-Pacific International Conference on Gravitation, Astrophysics, and Cosmology. Dedicated to the Centenary of Einstein's General Relativity. p 135 (2016) World Scientific Publishing Co.
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10. О равновесном состоянии гравитирующего конденсата Бозе-Эйнштейна *ЖЭТФ* **154**(5), 1000 (2018)
11. Static State of a Black Hole Supported by Dark Matter. *Universe* **5**(9), 198 (2019)
12. Black hole in balance with dark matter. *International Journal of Modern Physics* **A35**, (2&3), 2040050 (2020)
13. Black Hole and Dark Matter. Phase Equilibrium. *J. Phys.* CS **1557**, 012030 (2020)
14. Guessing the Riddle of a Black Hole. *Universe* **6**(8), 113 (2020)
15. Gravitational Radius in view of Existence and Uniqueness Theorem. *J. Phys.* CS **2081**, 012026 (2021)
16. Bose-Einstein Condensate in Synchronous Coordinates. *Phys. Sci. Forum* **7**, 47 (2023)
17. Черная дыра и темная материя в синхронной системе координат. *ЖЭТФ* **163**(5), 660 (2023)
18. О гравитационном поле чёрной дыры в синхронной системе координат В книге: Физические интерпретации теории относительности (PIRT-2023) Сборник тезисов XXIII Международной научной конференции. Москва, С. 115 (2023)