a primer on unimodular gravity

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einstein's 1915 \neq einstein's 1919

Traceless einstein equations. Related to Mie's 1910 theory trying to obtain gravitation from electromagnetism

$$L \equiv \sum_{i=1}^{4} C_i \mathcal{O}^{(i)}$$

$$\mathcal{O}^{(1)} \equiv \frac{1}{4}\partial_\mu h_{\rho\sigma}\partial^\mu h^{\rho\sigma}$$

$$\mathcal{O}^{(2)} \equiv -\frac{1}{2}\partial^\rho h_{\rho\sigma}\partial_\mu h^{\mu\sigma}$$

$$\mathcal{O}^{(3)} \equiv \frac{1}{2}\partial_\mu h \partial_\lambda h^{\mu\lambda}$$

$$\mathcal{O}^{(4)} \equiv -\frac{1}{4}\partial_\mu h \partial^\mu h$$

C_1=1 (Normalization)

TDiff implies C_2=1

Fierz-Pauli implies C_3=C_4=1

$$C_3 = \frac{2}{n}$$

$$C_4 = \frac{n+2}{n^2}$$

WTDiff

This can be obtained from Fierz-Pauli through

$$h_{\mu\nu} \to h_{\mu\nu} - \frac{1}{n} h \eta_{\mu\nu}$$

This is NOT a field redefinition (not invertible)

only three gauge invariances are really needed in prder to met from massive spin 2 to massless spin 2

2=5-3

TDiff=volume preserving diffs (connected to the identitv)

$$x \to x'; \quad \det \frac{\partial x'}{\partial x} = 1$$

$$\partial_\mu \xi^\mu = 0$$

this is the linear limit of

THE NONLINEAR THEORY

only unimodular metrics are allowed in the path integral

no problem with conformal factor (GHP)

How to integrate over unimodular geometries only?

Lagrange multiplier enforcing the unimodular constraint?

weyl invariant formulation

$$\hat{g}_{\mu\nu} \equiv (T_U g)_{\mu\nu} \equiv |g|^{-\frac{1}{n}} \; g_{\mu\nu}$$

$$S_{UG} = -M_P^{n-2} \int d^n x \, |g|^{\frac{1}{n}} \left(R + \frac{(n-1)(n-2)}{4n^2} \frac{\nabla_\mu g \nabla^\mu g}{g^2} \right) \quad (5)$$

Symmetry group is TDiff X Weyl (4 generators)

g is inert under TDiff

traceless EM

$$R_{\mu\nu} - \frac{1}{n} g_{\mu\nu} = \Theta_{\mu\nu}$$

$$\Theta_{\mu\nu} \equiv \frac{(n-2)(2n-1)}{4n^2}\left(\frac{\nabla_\mu g \nabla_\nu g}{g^2} - \frac{1}{n}\frac{(\nabla g)^2}{g^2} g_{\mu\nu}\right) -$$

$$-\frac{n-2}{2n}\left(\frac{\nabla_\mu \nabla_\nu g}{g} - \frac{1}{n}\frac{\nabla^2 g}{g} g_{\mu\nu}\right) \qquad (6)$$

weyl gauge g=1

$$R_{\mu\nu} - \frac{1}{n} R g_{\mu\nu} = \kappa^2 \left(T_{\mu\nu} - \frac{1}{n} T g_{\mu\nu} \right)$$

$$\nabla^\nu R_{\mu\nu} = \frac{1}{2} \nabla_\mu R.$$

$$\frac{n-2}{2} \nabla_\mu R = -\frac{\kappa^2}{n} \nabla_\mu T$$

which integrates to

$$\frac{n-2}{2d} R + \frac{2\kappa^2}{d} T = constant \equiv -\lambda$$

UG is equivalent to GR with “some” Lambda

The dynamics itself determines this Lambda

In the Weyl symmetric formulation Noether charges change

Example: exponential expansion without lambda

$$R_{\mu\nu} = \frac{1}{4} R g_{\mu\nu}$$

$$ds^2 = b(t)^{-3/2}\, dt^2 - b(t)^{1/2}\, \delta_{ij} dx^i dx^j$$

$$b(t) = H_0^{\frac{4}{3}} \left(3t - t_0\right)^{\frac{4}{3}}$$

de sitter space in the unimodular gauge

quantization: BRST

$$s_D g_{\mu\nu} = s_W g_{\mu\nu} = 0$$

$$s_D h_{\mu\nu} = \nabla_\mu c^T_\nu + \nabla_\nu c^T_\mu + c^{T\rho} \nabla_\rho h_{\mu\nu} + \nabla_\mu c^{T\rho} h_{\rho\nu} + \nabla_\nu c^{T\rho} h_{\rho\mu}$$

$$s_W h_{\mu\nu} = 2c\,(g_{\mu\nu} + h_{\mu\nu})$$

$$c^T_\mu = \Theta_{\mu\nu} c^\nu = (g_{\mu\nu}\Box - \nabla_\mu \nabla_\nu - R_{\mu\nu})\, c^\nu = (Q_{\mu\nu} - \nabla_\mu \nabla_\nu)\, c^\nu$$

$$\begin{array}{lllll} h^{(0,0)}_{\mu\nu}, & c^{(1,1)}_\mu, & b^{(1,-1)}_\mu, & f^{(0,0)}_\mu, & \phi^{(0,2)}, \\ \pi^{(1,-1)}, & \pi'^{(1,1)}, & \bar{c}^{(0,-2)}, & c'^{(0,0)}, & \\ c^{(1,1)}, & b^{(1,-1)}, & f^{(0,0)} & & \end{array} \tag{2.15}$$

where $c^{(1,1)}_\mu$ denotes c_μ, $h^{(0,0)}_{\mu\nu}$ stands for $h_{\mu\nu}$ and the superscript (n, m) carries the Grassmann number, n, (defined modulo two) and ghost number, m. In this language, the BRST operators s_D and s_W enjoy Grassmann number 1 and ghost number 1, each.

field	s_D	s_W
$g_{\mu\nu}$	0	0
$h_{\mu\nu}$	$\nabla_\mu c^T_\nu + \nabla_\nu c^T_\mu + c^{\rho T}\nabla_\rho h_{\mu\nu} + \nabla_\mu c^{\rho T} h_{\rho\nu} + \nabla_\nu c^{\rho T} h_{\rho\mu}$	$2c^{(1,1)}\left(g_{\mu\nu} + h_{\mu\nu}\right)$
$c^{(1,1)\mu}$	$\left(Q^{-1}\right)^\mu_\nu \left(c^{\rho T}\nabla_\rho c^{T\nu}\right) + \nabla^\mu \phi^{(0,2)}$	0
$\phi^{(0,2)}$	0	0
$b^{(1,-1)}_\mu$	$f^{(0,0)}_\mu$	0
$f^{(0,0)}_\mu$	0	0
$\bar{c}^{(0,-2)}$	$\pi^{(1,-1)}$	0
$\pi^{(1,-1)}$	0	0
$c'^{\,(0,0)}$	$\pi'^{\,(1,1)}$	0
$\pi'^{\,(1,1)}$	0	0
$c^{(1,1)}$	$c^{T\rho}\nabla_\rho c^{(1,1)}$	0
$b^{(1,-1)}$	$c^{T\rho}\nabla_\rho b^{(1,-1)}$	$f^{(0,0)}$
$f^{(0,0)}$	$c^{T\rho}\nabla_\rho f^{(0,0)}$	0

Table 1. BRST transformations of the fields involved in the path integral.

for the $|g| = 1$ fixed background are the traceless Einstein equations

$$R_{\mu\nu} - \frac{1}{4} R g_{\mu\nu} = 0 \qquad (3.41)$$

which, altogether with Bianchi identities, imply the following for the operators appearing in the effective action

$$R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} = E_4 \qquad (3.42)$$

$$R_{\mu\nu} R^{\mu\nu} = \frac{1}{4} R^2 \qquad (3.43)$$

$$R = \text{constant} \qquad (3.44)$$

on shell effective action

$$W_{\infty}^{\text{on-shell}} = \frac{1}{16\pi^2} \frac{1}{n-4} \int d^n x \left(\frac{119}{90} E_4 - \frac{83}{120} R^2 \right)$$

(lambda forbidden by weyl invariance)

Not clear whether UG and GR are fully equivalent at the quantum level

Thank you