Einstein's 1905 derivation of the mass-energy equivalence: is it valid? Is energy always equal to mass and *vice versa*?

Germano D'Abramo

Ministero dell'Istruzione, dell'Università e della Ricerca,
00041 Albano Laziale (RM), Italy.
germano.dabramo@gmail.com

November 24, 2022

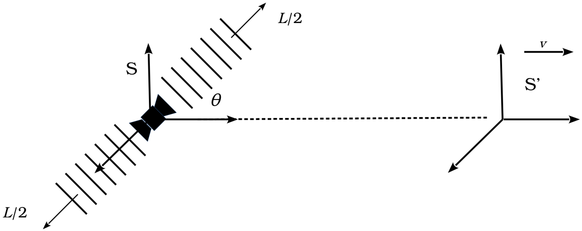
XXXIV International (ONLINE) Workshop on High Energy Physics
"From Quarks to Galaxies: Elucidating Dark Sides",
November 22-24, 2022.

Aim and limits

- In the existing literature, you can find many different proofs of mass-energy equivalence. Over the years, Einstein himself presented some 18 proofs of it, the last one in 1946 (Hetch, 2011).
- Some of the existing proofs do not require the machinery of SR (e.g., Einstein 1946 and Rohrlich 1990). Together with Einstein's first derivation, these proofs generally are considered to be valid as a special or limiting case.
- Other derivations dig deeply into the mathematics of SR, like those by Von Laue (1911) and Klein (1918). Today, they are considered the most rigorous and general proofs (definitive) of the equivalence.
- In the literature, you can also find criticism of Einstein's first derivation, starting from Planck (1907) onward.

Aim and limits

- I will not treat all these different proofs and criticism here, at least not in detail. Therefore, please do not expect a comprehensive and exhaustive presentation from a historical point of view.
- The focus of this presentation is almost exclusively on Einstein's 1905 derivation of the mass-energy equivalence (Einstein, 1905a).
- I will describe the results of my analysis of the logical structure of Einstein's 1905 derivation, its soundness, and the validity of the assumptions.
- I will also show why, in my view, the widely accepted interpretation of $E = mc^2$ is problematic.



- Energy balance in S: $\mathrm{E}_0 = \mathrm{E}_1 + \mathrm{L}$, where E_0 and E_1 are the **total energies** of the system before and after the emission. L is the total energy of the electromagnetic radiation.
- Energy balance in S': $\mathrm{H}_0 = \mathrm{H}_1 + \mathrm{L}'$, where H_0 and H_1 are the **total energies** of the system before and after the emission. $\mathrm{L}' = \frac{\mathrm{L}}{\sqrt{1-\frac{v^2}{c^2}}}$ is the total energy of the electromagnetic radiation (Einstein, 1905b) in S', where v is the velocity of S' relative to S, and c is the speed of light.

By subtracting the two energy balance equations, Einstein obtained the following relation:

$$(\mathrm{H}_0 - \mathrm{E}_0) - (\mathrm{H}_1 - \mathrm{E}_1) = \mathrm{L}\left\{\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1\right\}$$

What does the above equation mean? According to Einstein's reasoning:

- H and E are the **total energies** of the same body referred to two reference frames in uniform motion relative to each other.
- Thus, the difference $(\mathrm{H} - \mathrm{E})$ can only be equal to the **kinetic energy** K of the body relative to S' (it can differ only by an additive constant C that *wlog* can be taken equal to zero).

Einstein's 1905 derivation

- For $v \ll c$, Einstein's relation becomes

$$\mathrm{K}_0 - \mathrm{K}_1 = \frac{1}{2}\left[\frac{\mathrm{L}}{c^2}\right]v^2$$

- Since the emission is symmetric, the velocity v of the body does not change, and the only change can occur in its mass.

$$\frac{1}{2}m_0 v^2 - \frac{1}{2}m_1 v^2 = \frac{1}{2}\left[\frac{\mathrm{L}}{c^2}\right]v^2 \quad \rightarrow \quad m_0 - m_1 = \frac{\mathrm{L}}{c^2}$$

- Therefore, if a body gives off the energy L (in the form of radiation), its mass diminishes by L/c^2.
- In modern notation, a quantity of mass m can be completely converted into radiation energy E, and the relation between them is $E = mc^2$.

Received interpretation of mass-energy equivalence

1) A mass m can be transformed completely into energy E (mainly, radiation) with $E = mc^2$.

2) An amount of energy E (every kind of energy!) possesses an inertial/gravitational mass $m = E/c^2$. Therefore, if a body acquires an energy E, no matter what kind of energy it is, its mass increases by the amount $\Delta m = E/c^2$. (It is enough to apply Einstein's approach to absorption instead of emission!)

Incidentally, there is plenty of experimental proof of point 1) (e.g., Rainville *et al.*, 2005). However, as I will show you soon, I am not that sure about point 2).

Einstein's crucial assumption

- The crucial assumption in Einstein's 1905 derivation is:

$$\mathrm{H} - \mathrm{E} = \mathrm{K}$$

- In Einstein's 1905 paper, it was only briefly discussed and presented as "clear". However, it still appears innocent, obvious, and even necessary. But it is not!

- To see why, let us rewrite it as

$$\mathrm{H} = \mathrm{E} + \mathrm{K}$$

and remind that H and E are *the total energies from which the energies of the electromagnetic emission L' and L are drawn, respectively*.

Einstein's crucial assumption

- $\mathrm{H} = \mathrm{E} + \mathrm{K}$ means that Einstein was implicitly assuming that the kinetic energy K of the body, relative to S', does contribute to the increase in the 'internal reservoir' of energy H from which the electromagnetic emission originates in S'.

- Planck somehow made the point already in 1907. He said that Einstein's 1905 derivation was valid...

  > *"...under the assumption, permissible only as a first approximation, that the total energy of a body is composed additively of its kinetic energy and its energy referred to a system in which it is at rest."*

Criticism of Einstein's 1905 derivation

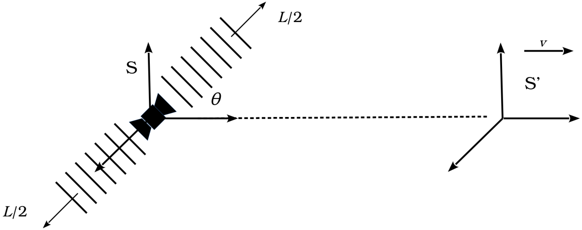
- Ives (1952) and Jammer (1961) asserted that Einstein's derivation was nothing but the result of a *petitio principii*. They generically asserted that $\mathrm{H} - \mathrm{E} = \mathrm{K}$ is unwarranted. Then, by using special relativity, they showed that, for $\mathrm{H} - \mathrm{E} = \mathrm{K}$ to be valid, the relation $\frac{\mathrm{L}}{(m_0 - m_1)c^2} = 1$ must be assumed beforehand.

- However, Stachel and Torretti (1982) analyzed Ives's, Jammer's, and Arzeliés's criticism and concluded that the logic behind Einstein's derivation is sound. Moreover, they also "proved" the equality $\mathrm{H} - \mathrm{E} = \mathrm{K}$ to be valid.

Intermezzo: the general mass-energy relationship (D'Abramo, 2020)

- If $\mathrm{H} - \mathrm{E} = \mathrm{K}$ is valid, a general mass-energy relationship can be heuristically derived without the need of special relativity or any other full-fledged physical theory (exception made for the principle of conservation of energy).
- This general mass-energy connection turns into a mass-energy equivalence when is applied to the case of a body emitting energy in the form of electromagnetic waves.
- Even within Maxwell's theory of light (and thus, no special relativity), one could have already come to mass-energy equivalence, albeit in the different form $E = \frac{1}{2}mc^2$.

Intermezzo: the general mass-energy relationship (D'Abramo, 2020)



- Energy balance in S: $\mathrm{E}_0 = \mathrm{E}_1 + \mathrm{L}$, where E_0 and E_1 are the **total energies** of the system before and after the emission. L is the total energy emitted by the body in any imaginable form.
- Energy balance in S': $\mathrm{H}_0 = \mathrm{H}_1 + \mathrm{L}'$, where H_0 and H_1 are the **total energies** of the system before and after the emission. L' is the total energy emitted by the body measured in S'.

Intermezzo: the general mass-energy relationship (D'Abramo, 2020)

- *Wlog*, we can write

$$\mathrm{L}' = \mathcal{F}(\mathrm{L}, v),$$

  where $\mathcal{F}$ is a suitable mathematical function, and v is the scalar relative velocity between S and S'.

- L' must be directly proportional to L, then

$$\mathrm{L}' = \mathrm{L}f(v),$$

  where $f(v)$ is a dimensionless function of the scalar velocity v.

- Consider the Maclaurin expansion of $f(v)$ up to $O(v^3)$

$$f(v) = \alpha + \beta v + \delta v^2 + O(v^3),$$

  where α, β, and δ are numerical coefficients.

Intermezzo: the general mass-energy relationship (D'Abramo, 2020)

- Since $f(0) = 1$, α must be equal to 1.
- For symmetry reasons, $f(-v) = f(v)$, therefore $\beta = 0$, along with all other terms with odd powers.
- Thus, we have,

$$f(v) = 1 + \delta v^2 + O(v^4),$$

with constant δ having the physical units of an inverse square velocity. This velocity is the 'characteristic velocity' of the peculiar emission process.

- Then,

$$\mathrm{L}' = \mathrm{L}(1 + \delta v^2 + O(v^4)).$$

Intermezzo: the general mass-energy relationship (D'Abramo, 2020)

- Back to the energy balance equations. They become

$$\begin{aligned} \mathrm{E}_0 &= \mathrm{E}_1 + \mathrm{L}, \\ \mathrm{H}_0 &= \mathrm{H}_1 + \mathrm{L}(1 + \delta v^2 + O(v^4)). \end{aligned}$$

- Like Einstein, we subtract the first equation from the second

$$(\mathrm{H}_0 - \mathrm{E}_0) - (\mathrm{H}_1 - \mathrm{E}_1) = \mathrm{L}(\delta v^2 + O(v^4)).$$

- Finally, with Einstein's assumption $\mathrm{H} - \mathrm{E} = \mathrm{K}$, we obtain

$$\mathrm{K}_0 - \mathrm{K}_1 = \mathrm{L}(\delta v^2 + O(v^4)).$$

Intermezzo: the general mass-energy relationship (D'Abramo, 2020)

- Let us define, like in Stachel and Torretti (1982), the inertial mass for a body in translational motion by

$$m = \lim_{v \to 0} \frac{K}{v^2/2}.$$

- Then, from $\mathrm{K}_0 - \mathrm{K}_1 = \mathrm{L}(\delta v^2 + O(v^4))$, it follows

$$-\Delta m = m_0 - m_1 = \lim_{v \to 0} \frac{(K_0 - K_1)}{v^2/2} = \lim_{v \to 0} \frac{L(\delta v^2 + O(v^4))}{v^2/2} = 2\delta \mathrm{L}.$$

Intermezzo: the general mass-energy relationship (D'Abramo, 2020)

- In short,

$$-\Delta m = 2\delta \mathrm{L},$$

namely, if a body gives off the energy L, its mass diminishes by $2\delta\mathrm{L}$.

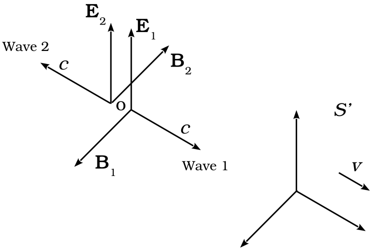
- Please, notice that this is not a mass–energy equivalence *per se*!
- If we apply it to a body releasing two projectiles of mass m in opposite directions with non-relativistic velocity v_0 (relative to the parent body), then it is possible to prove that $\delta = 1/v_0^2$. Since $\mathrm{L} = 2 \cdot \frac{1}{2} m v_0^2$ (the emitted energy, in this case, is only kinetic), then we have $-\Delta m = 2m$, quite unsurprisingly.

Intermezzo: the general mass-energy relationship (D'Abramo, 2020)

- Things get more interesting when we apply the general mass-energy relationship to the emission of energy in the form of electromagnetic waves: *radiation energy necessarily comes from mass reduction, and thus mass transforms into radiation energy*.
- Special relativity is not essential for the derivation of mass-energy equivalence: special relativity comes into play only in the numerical value of the constant δ.
- The constant δ has the physical units of an inverse square velocity, and in the case of electromagnetic phenomena, it must be heuristically proportional to $1/c^2$. In the case of Einstein's original derivation, we have that $\delta = 1/2c^2$.

Intermezzo: the general mass-energy relationship (D'Abramo, 2020)

- Even within 19th-century Maxwell's theory of light (and thus, no special relativity!), one could have already come to mass-energy equivalence, albeit in the different form $E = \frac{1}{2}mc^2$.
- Consider two plane waves of light, A, and B, emitted in opposite directions from the origin of the rest frame S along the x-axis, with total energy L. Within Maxwell's theory of light, S is the reference frame of the ether, while S' moves with $v \ll c$.



Intermezzo: the general mass-energy relationship (D'Abramo, 2020)

1) The total energy density associated with an electromagnetic wave is

$$u = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2}\frac{B^2}{\mu_0} = \epsilon_0 E^2,$$

where ϵ_0 and μ_0 are the permittivity and the permeability of free space, and $B = \sqrt{\epsilon_0 \mu_0}\, E$ for e.m. waves in a vacuum.

2) The electric field $\mathbf{E}'$ measured in the reference frame S' can be derived, via the Lorentz force $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ felt by a test charge q stationary in S', as

$$\mathbf{E}' = \frac{\mathbf{F}}{q} = \mathbf{E} + \mathbf{v} \times \mathbf{B}.$$

Intermezzo: the general mass-energy relationship (D'Abramo, 2020)

It can be shown that (D'Abramo, 2020):

- The total energy L', measured in S', is

$$\mathrm{L}' = \mathrm{L}\left(1 + \frac{v^2}{c^2}\right).$$

- And thus, $\delta = 1/c^2$, and $E = \frac{1}{2}mc^2$.

Intermezzo: the general mass-energy relationship (D'Abramo, 2020)

- With a more accurate but still non-relativistic transformation law for the electric field $\mathbf{E}$, we could derive the exact mass-energy formula ($E = mc^2$) even within Maxwell's theory of light.
- Consider an oscillating point charge q "stationary" in S and having its oscillation center moving at velocity $\mathbf{v}(t)(\ll c)$ relative to S'. By applying the Liénard-Wiechert potentials in S and S', we have that the electric field generated at a generic point $(\mathbf{x}, t)$ (the same point in space for S and S') is given by

$$\mathbf{E}(\mathbf{x}, t) = q\left[\frac{\mathbf{n}-\beta}{\gamma^2(1-\mathbf{n}\cdot\beta)^3R^2}\right]_{ret} + \frac{q}{c}\left[\frac{\boldsymbol{n}\times\{(\mathbf{n}-\beta)\times\dot{\beta}\}}{(1-\mathbf{n}\cdot\beta)^3R}\right]_{ret}$$

where $\gamma^2 = 1/(1-v^2/c^2)$ and $\beta = \mathbf{v}(t)/c$, $\dot{\beta} = [d\mathbf{v}(t)/dt]\cdot 1/c$. Quantities inside square brackets must be evaluated at the retarded time $t_r = t - \frac{R}{c}$

Back to Einstein's crucial assumption

$$\mathrm{H} = \mathrm{E} + \mathrm{K}$$

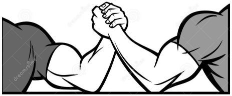
- Contrary to Stachel and Torretti, I believe that, from a physical viewpoint, the above assumption is not so obvious and unproblematic.
- As I said before, it means that Einstein was implicitly assuming that the kinetic energy K of the body relative to S' does contribute to the increase in the 'internal reservoir' of energy H from which the electromagnetic emission originates in S'.

Back to Einstein's crucial assumption

- The assumption is not problematic in the classical case of emission of energy in mechanical form (e.g., emission of non-relativistic mass projectiles), where both L and L' are just kinetic energies.
- On the contrary, with electromagnetic emissions or any non-mechanical process, it is not unproblematic.
- Consider the following pertinent analogies...

Back to Einstein's crucial assumption

- Suppose E to be the internal, metabolic energy of an arm wrestler seated before his contender and K his kinetic energy relative to a moving observer (the arm wrestler is at rest, and the observer is moving). In the observer rest frame, the arm wrestler cannot be stronger simply because now his total energy is $\mathrm{E}+\mathrm{K}$!



- The fact that both E and K have the same physical unit (joule) does not automatically imply that one kind of energy can always flow into the other!

Back to Einstein's crucial assumption

- Similarly, assuming the validity of $\mathrm{H} = \mathrm{E} + \mathrm{K}$ for electromagnetic emission is much like taking for granted that the kinetic energy of a car in motion relative to us can contribute for us to the increase in the energy content of the gasoline. Ultimately, K does contribute to the increase in the gasoline mass!

Back to Einstein's crucial assumption

- Assuming the validity of $\mathrm{H} = \mathrm{E} + \mathrm{K}$ is pretty much assuming the equivalence between mass and energy. Special relativity has little to do with mass-energy equivalence.
- Since it always (heuristically) holds that $\mathrm{L}' > \mathrm{L}$, and since $|\Delta\mathrm{H}| = \mathrm{L}' > \mathrm{L} = |\Delta\mathrm{E}|$, we have that $|\Delta\mathrm{H}| > |\Delta\mathrm{E}|$.
- By assuming $\mathrm{H} = \mathrm{E} + \mathrm{K}$, the variation of kinetic energy must necessarily be different from zero during the emission process, $\Delta\mathrm{K} \neq 0$.
- Since the velocity of the emitter does not change after the emission, $\Delta\mathrm{K}$ implies a variation in the mass of the emitter, $\Delta m \neq 0$.
- In this very peculiar sense, Einstein's original derivation is the result of a *petitio principii* (begging the question). This remark here is somewhat reminiscent of Planck's criticism (Planck, 1907).

Summing up

1) In Einstein's 1905 derivation, the essence of mass-energy equivalence is not in SR but in the assumption of the general validity of $\mathrm{H} = \mathrm{E} + \mathrm{K}$.

2) However, $\mathrm{H} = \mathrm{E} + \mathrm{K}$ is not true for all physical processes. It is not true *a priori*. Taking it as true for all physical processes is an *unjustified* and *arbitrary* step.

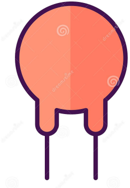
3) Since $\mathrm{H} = \mathrm{E} + \mathrm{K}$ mathematically leads to the mass-energy equivalence relation, Einstein's 1905 derivation is *circular*, although in a very peculiar and non-trivial way.

Summing up

4) Does this mean that the mass-energy equivalence and $E = mc^2$ do not hold? No, they have been experimentally verified to a high degree of accuracy (e.g., Rainville et al., 2005).

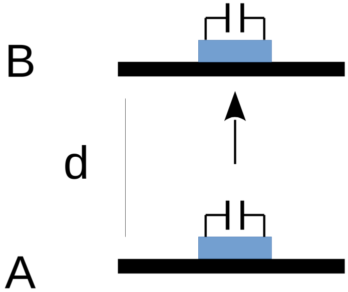
5) *If and when* mass transforms into energy (and vice versa), Einstein's 1905 derivation works well and gives the correct mathematical relationship $E = mc^2$.

6) Corroboration of a suspicion: there is no guarantee that every form of energy transforms into mass and *vice versa* (except for nuclear reactions). E.g., when we heat a teapot, does it increase its mass? Or, when we charge a capacitor, does it increase its mass?



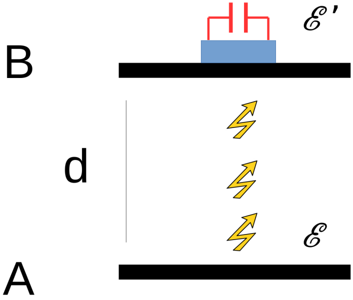
Indiscriminate energy-to-mass conversion is problematic

1) An apparatus of mass M is lifted for a distance d from place A to place B in a uniform gravitational field $\mathbf{g}$. The work performed on M is, therefore, equal to Mgd.



Indiscriminate energy-to-mass conversion is problematic

2) From place A, radiation of energy $\mathcal{E}$ is emitted towards the apparatus at place B. the energy absorbed by the apparatus is $\mathcal{E}' = \mathcal{E}\left(1 - \frac{gd}{c^2}\right)$ (Einstein, 1911). Then, energy $\mathcal{E}'$ is completely converted into electrical potential energy and stored in a capacitor inside the apparatus.



Indiscriminate energy-to-mass conversion is problematic

2 bis) According to the "indiscriminate energy-to-mass conversion" assumption, the total mass of the apparatus is now equal to

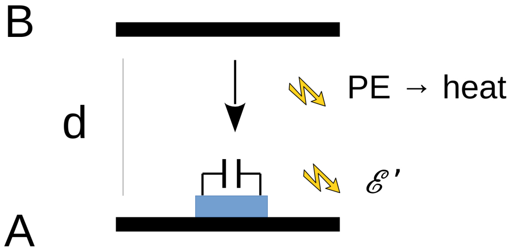
$$M + \frac{\mathcal{E}'}{c^2},$$

and its gravitational potential energy is equal to

$$Mgd + \frac{\mathcal{E}'}{c^2} gd.$$

Indiscriminate energy-to-mass conversion is problematic

3) The apparatus moves slowly back to place A, dissipating all its gravitational potential energy into heat by friction against a brake. The last step consists in the discharge of the capacitor inside the apparatus. The discharge necessarily releases the stored energy $\mathcal{E}'$.



Indiscriminate energy-to-mass conversion is problematic

Energy balance:

- Total energy put into the system:

$$E_{in} = Mgd + \mathcal{E}$$

- Total energy released at the end of the process:

$$E_{out} = \underbrace{\left(Mgd + \frac{\mathcal{E}'}{c^2} gd\right)}_{gravitational\ \ potential\ \ energy} + \underbrace{\mathcal{E}'}_{capacitor\ \ energy} = Mgd + \mathcal{E}'\left(1 + \frac{gd}{c^2}\right)$$

Indiscriminate energy-to-mass conversion is problematic

Energy balance:

- $\Delta E = E_{out} - E_{in}$ is not zero, and the total energy is not conserved!

$$\begin{aligned}\Delta E = E_{out} - E_{in} = \cancel{Mgd} + \mathcal{E}'\left(1 + \frac{gd}{c^2}\right) - \cancel{Mgd} - \mathcal{E} = \\ = \mathcal{E}\left(1 - \frac{gd}{c^2}\right)\left(1 + \frac{gd}{c^2}\right) - \mathcal{E} = \\ = -\mathcal{E}\frac{g^2 d^2}{c^4} \neq 0.\end{aligned}$$

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