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based on 2203.09307 [hep-ph] (space-like region)

work in progress (time-like region)

The XXXIV International (ONLINE) Workshop on High Energy Physics From Quarks to Galaxies: Elucidating Dark Sides, November 22-24, 2022, Protvino.

Fractional Analytic QCD for space-like and time-like processes

OUTLINE

1. Introduction
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Abstract

A review of the main elements of (fractional) analytical QCD is presented. The main part of the review is focused on the introduction of the Shirkov-Solovtsov and Bakulev-Mikhailov-Stefanis approaches and their recent extension beyond the leading order of perturbation theory. We present various representations in Euclidean and Minkowski spaces, details of their construction and show their applicability.

0. History. QED.

Consider so-called polarization operator $D(k^2)$ in QED. Leading logarithmic terms of $D(k^2)$ in the n order of perturbation theory with $|k^2| >> m^2$ (m is the electron mass) have the following form:

$$(e^2 F(K^2, m^2))^n / K^2, \; K^2 = -k^2 \geq 0, \; F(K^2, m^2) = \frac{1}{3\pi} \ln\left(\frac{K^2}{4m^2}\right) .$$

Resummation of the large logarithms leads to (Landau,Abrikosov,Khalatnikov:1954):

$$D_{\rm per}(k^2) = \frac{1}{K^2} \frac{1}{1 - \frac{e^2}{3\pi} \ln\left(\frac{K^2}{4m^2}\right)} .$$

Then, there is the pole *(so-called Landau pole)* at K_p^2:

$$K_p^2 = 4m^2 e^{3\pi/e^2}$$

and QED is not applicable at $K^2 \geq K_p^2$ (Landau,Pomeranchuk:1955).

With another side, there is so-called Kallen-Lehmann representation:

$$D(k^2) = \frac{1}{K^2} + \int_{4m^2}^{\infty} dz \, \frac{I(z)}{z + K^2}, \quad I(z) = ImD(i\varepsilon - K^2)$$

and $D_{\rm per}(k^2)$ is not in agreement with the Kallen-Lehmann representation.

Combination of the Kallen-Lehmann representation and perturbation theory *(or same, perturbation theory for $I(z)$)* has been considered in (Redmond:1958), (Redmond,Uretsky:1958), (Bogolyubov,Logunov,Shirkov:1959).

We follow (Bogolyubov,Logunov,Shirkov:1959).

From calculation (Landau,Abrikosov,Khalatnikov:1954) they obtained that $I_{\rm per}(z) = 0$ for $z < 4m^2$ and for $z \geq 4m^2$:

$$I_{\rm per}(z) = \frac{e^2}{3\pi z} \frac{1}{\left(\left(1 - \frac{e^2}{3\pi} \ln\left(\frac{z-4m^2}{4m^2}\right)\right)^2 + \frac{e^2}{9}\right)} .$$

Using $I_{\rm per}(z)$ in the Kallen-Lehmann representation they obtained at $|k^2| >> m^2$

$$D(k^2) = \frac{1}{K^2} \frac{1}{1 - \frac{e^2}{3\pi} \ln\left(\frac{K^2}{4m^2}\right)} + \frac{(3\pi)/e^2}{K^2 - K_p^2} .$$

The additional term cancels exacly Landau pole at $K^2 = K_p^2$. Moreover, it cannot be obtained in the framework of perturbation theory, since it cannot be expanded in e^2-series.

Thus, the combination of perturation theory and Kallen-Lehmann representation (i.e. perturbation theory for spectral function) does not lead to the Landau problem in QED.

In the general case the QCD couplant is defined as a product of propagators and a vertex function. Therefore, one might pose a question concerning the analytic properties of this quantity. This matter has been examined (Ginzburg,Shirkov:1965).

It was shown that in this case the integral representation of the Kallen-Lehmann type holds for the running coupling, too. Proceeding from these motivations, the analytic approach was lately extended to Quantum Chromodynamics by D.V. Shirkov and I.L. Solovtsov.

1. Introduction

According to the general principles of (local) quantum field theory (QFT) (Bogolyubov,Shirkov:1959); (Oehme:1994) observables in the spacelike domain can have singularities only with negative values of their argument Q^2.
On the other hand, for large values of Q^2, these observables are usually represented as power series expansion by the running coupling constant (couplant) $\alpha_s(Q^2)$, which, in turn, has a ghost singularity, the so-called Landau pole, for $Q^2 = \Lambda^2$.
To restore analyticity, this pole must be removed.

Strong couplant $\alpha_s(Q^2)$ obeys the renormalized group equation

$$L \equiv \ln \frac{Q^2}{\Lambda^2} = \int^{\overline{a}_s(Q^2)} \frac{da}{\beta(a)}, \quad \overline{a}_s(Q^2) = \frac{\alpha_s(Q^2)}{4\pi}, \quad a_s(Q^2) = \beta_0\,\overline{a}_s(Q^2)$$

with some boundary condition and the QCD β-function:

$$\beta(a_s) \ = \ - \sum_{i=0} \beta_i \overline{a}_s^{i+2} = -\beta_0 \overline{a}_s^2 \,(1 + \sum_{i=1} b_i a_s^i), \quad b_i = \frac{\beta_i}{\beta_0^{i+1}}\,,$$

where the first fifth coefficients, i.e. β_i with $i \leq 4$, are exactly known (Baikov,Chetyrkin,Kuhn: 2017).

So, already at leading order (LO), when $a_s(Q^2) = a_s^{(1)}(Q^2)$, we have

$$a_s^{(1)}(Q^2) = \frac{1}{L}\,,$$

i.e. $a_s^{(1)}(Q^2)$ does contain a pole at $Q^2 = \Lambda^2$.

In a series of papers (Shirkov,Solovtsov: 1996,1997); (Milton,Solovtsov,Solovtsova: 1997); (Shirkov: 2001) authors have developed an effective approach to eliminate the Landau singularity without introducing extraneous IR regulators.

The idea: the dispersion relation, which connects the new analytic couplants: $A_{\rm MA}(Q^2)$ in Euclidean space and $U_{\rm MA}(s)$ in Minkowski space, with the spectral function $r_{\rm pt}(s)$, obtained in the framework of perturbative theory. In LO

$$A^{(1)}_{\rm MA}(Q^2) = \frac{1}{\pi}\int_0^{+\infty} \frac{ds}{(s+t)}\, r^{(1)}_{\rm pt}(s)\,, \quad r^{(1)}_{\rm pt}(s) = {\rm Im}\, a^{(1)}_s(-s-i\epsilon)\,,$$

$$U^{(1)}_{\rm MA}(s) = \frac{1}{\pi}\int_s^{+\infty} \frac{d\sigma}{\sigma}\, r^{(1)}_{\rm pt}(\sigma)$$

So, let's repeat once again: the spectral function is taken directly from perturbation theory, but the analytical couplants $A_{\rm MA}(Q^2)$ and $U_{\rm MA}(s)$ are restored using dispersion relations.
This approach is called *Minimal Approach* (MA) (Cvetic, Valenzuela: 2008) or *Analytic Perturbation Theory* (APT) (Shirkov, Solovtsov:1996,1997); (Milton,Solovtsov,Solovtsova:1997); (Shirkov:2001)

Thus, MA QCD is a very convenient approach that combines the general (analytical) properties of quantum field quantities and the results obtained within the framework of perturbative QCD, leading to the appearance of the MA couplants $A_{\rm MA}(Q^2)$ and $U_{\rm MA}(s)$, which are close to the usual strong couplant $a_s(Q^2)$ in the limit of large values of its argument and completely different at $Q^2 \leq \Lambda^2$.

A further development of APT is the so-called fractional APT (FAPT), which extends the principles of constructing to non-integer powers of couplant, which arise for many quantities having non-zero anomalous dimensions (Bakulev,Mikhailov,Stefanis: 2005,2008,2010), with some privious study (Karanikas,Stefanis: 2001)
and reviews (Bakulev: 2008), (Stefanis: 2013).

The results in FATP have a very simple form in LO perturbation theory, but they are quite complicated in higher orders.

In (Kotikov,Zemlyakov: 2022), in Euclidean space FART was extended to higher orders of perturbation theory using the so-called $1/L$-expansion of the usual couplant.

For an ordinary coupling constant, this expansion is applicable only for large values of its argument Q^2, i.e. for $Q^2 >> \Lambda^2$.

In the case of an analytic coupling constant, the situation changes greatly and this expansion is applicable for all values of the argument. This is due to the fact that the non-leading expansion corrections disappear not only at $Q^2 \to \infty$, but also at $Q^2 \to 0$, which leads to non-zero (small) corrections only in the region $Q^2 \sim \Lambda^2$.

1.1. Other models of analytic couplant

MA coupling (Shirkov,Solovtsov: 1996,1997); (Milton,Solovtsov,Solovtsova: 1997); (Shirkov: 2001):

$$A_{SS}^{(1)}(Q^2) = \frac{1}{L} + \frac{1}{1-t}, \quad t = \frac{Q^2}{\Lambda^2}, \quad L = ln(t)\,,$$

with the infrared finite value:

$$A_{SS}^{(1)}(0) = 1\,.$$

There are several other models of analytic couplant. We will show a few of them.

1. Let us start with the model developed by (Alekseev,Arbuzov:1998), (Alekseev:1998). By making use of a special solution to the Schwinger-Dyson equations for the gluon propagator, these authors proposed the following expression for the QCD running coupling

$$A_{AA}^{(1)}(Q^2) = \frac{1}{L} + \frac{1}{1-t} + \frac{c}{t} + \frac{1-c}{t + m_g^2/\Lambda^2},$$

where m_g is the gluon mass and c denotes a dimensionless parameter fixed by the phenomenological value of the gluon condensate. The running coupling has enhancement in the infrared domain.

2. Another similar model for the QCD analytic coupland:

$$A^{(1)}_{Latt}(Q^2) = \frac{1}{L} + \frac{1}{1-t} + \frac{\nu}{t},$$

cames from an analysis of the lattice simulation data on the lowenergy behavior of the QCD coupland (Boucaud et al.:2000), (Burgio et al.:2002). The model also possesses the infrared enhancement.

3. By making use of a certain phenomenological reasoning, Webber suggested the coupland of the following form (Webber:1998)

$$A^{(1)}_{W}(Q^2) = \frac{1}{L} + \frac{1}{1-t}\frac{t+b}{1+b}\left(\frac{1+c}{t+c}\right)^p,$$

with a specific choice of the parameters: $b = 1/4$, $c = 4$, $p = 4$. This model has infrared finite value:

$$A^{(1)}_{W}(0) = \frac{1}{2}.$$

4. Nesterenko model (Nesterenko:2000,2001) at LO:

$$\frac{d\ln[A_N^{(1)}(Q^2)]}{d\ln Q^2} = -A_{MA}^{(1)}(Q^2) = -\frac{1}{\pi}\int_0^{+\infty} \frac{ds}{(s+t)}\, r_{\rm pt}^{(1)}(s)\,,$$

that leads to

$$A_N^{(1)}(Q^2) = \frac{t-1}{tL}$$

The model also possesses the infrared enhancement.

5. A generalization of the Nesterenko model (Srivastava et al.:2001) $(0 < p \leq 1)$:

$$A_{SPPW}^{(1)}(Q^2) = \left[\frac{1}{A_{SPPW}^{(1)}(\Lambda^2)} + \int_0^{\infty} d\sigma\, \frac{(t-1)t^p}{(\sigma+t)((\sigma+1)(1+t^p)}\right]^{-1},$$

which equal to $A_N^{(1)}(Q^2)$, when $p=1$. The model also possesses the infrared enhancement.

5. 2δ and 2δ models of analytic QCD (Ayala,Contreras,Cvetic:2012), (Ayala,Cvetic:2015), (Ayala,Cvetic,Kogerler:2017):

$$r^{(1)}_{\rm pt}(\sigma) \to r^{(1)}_{n}(\sigma) = \pi \sum_{j=1}^{n} F_j \, \delta(\sigma - M_j^2) + \theta(\sigma - M_0^2) \, r^{(1)}_{\rm pt}(\sigma) \,,$$

where $M_1^2 < M_2^2 < ... < M_n^2 < M_0^2$ and F_j are some constants. Here $(n = 2)$ and $(n = 3)$ for 2δ and 2δ models.

Several phenomenological models for the QCD coupland have been proposed in (Krasnikov,Pivovarov:2001). It should be mentioned that the ideas, similar to that of the MA perturbation theory, were also used in analysis of the electronpositron annihilation into hadrons (Geshkenbein,Ioffe:1999), (Howe,Maxwell:2002,2004), investigation of the inclusive τ lepton decay (Geshkenbein,Ioffe, Zablyuk:2001), (Geshkenbein:2003), the study of Bjorken sum rule (Ayala et al.: 2017,2018,2020), ...

The study of the power corrections to the strong coupland was performed in (Grunberg:1995,2001), (Fischer:1997), (Caprini,Fischer:2002). There is also a number of methods of the Renormalization group improvement of perturbative series for the QCD observables (see, e.g., (Grunberg:1992), (Pineda,Soto:2000), (Maxwell,Mirjalili:2000), (Kiselev:2002), (Ahmady:2003), (Elias:2003),...).

This talk is organized as follows.

In Section 2 we firstly review the basic properties of the usual strong couplant and its $1/L$-expansion.

Section 3 contains fractional derivatives (i.e. ν-derivatives) of the usual strong couplant, which $1/L$-expansions can be represented as some operators acting on the ν-derivatives of the LO strong couplant. This is the key idea of this paper, which makes it possible to construct $1/L$-expansions of ν-derivatives of MA couplants for high-order perturbation theory, which are presented in Section 4 and 5.

Sections 6 contains the application of this approach to the Bjorken sum rule.

In conclusion, some final discussions are given.

2. Strong coupling constant

As shown in Introduction, the strong couplant $a_s(Q^2)$ obeys the renormalized group equation. When $Q^2 >> \Lambda^2$, it can be solved by iterations in the form of $1/L$-expansion (for simplicity we present here the first 3 terms of the expansion). [In (Kotikov,Zemlyakov: 2022) the 5 terms of the expansion have been considered in an agreement with the number of known coefficients β_i]:

$$a^{(1)}_{s,0}(Q^2) = \frac{1}{L_0}, \quad a^{(i+1)}_{s,i}(Q^2) = a^{(1)}_{s,i}(Q^2) + \sum_{m=2}^{i} \delta^{(m)}_{s,i}(Q^2), \quad L_i = \ln\frac{Q^2}{\Lambda^2_k},$$

where the corrections $\delta^{(m)}_{s,k}(Q^2)$ can be represented as follows

$$\delta^{(2)}_{s,k}(Q^2) = -\frac{b_1 \ln L_k}{L_k^2}, \quad \delta^{(3)}_{s,k}(Q^2) = \frac{1}{L_k^3}\left[b_1^2(\ln^2 L_k - \ln L_k - 1) + b_2\right].$$

We show exactly that at any order of perturbation theory, the couplant $a_s(Q^2)$ contains its own parameter Λ of dimensional transmutation, which is fitted from experimental data.
It relates with the normalization $\alpha_s(M_Z^2)$ as

$$\Lambda_i = M_Z \, \exp\{-\frac{1}{2}[\frac{1}{a_s(M_Z^2)} + b_1 \, \ln a_s(M_Z^2)$$
$$+ \int_0^{\overline{a}_s(M_Z^2)} \, da \, \left(\frac{1}{\beta(a)} + \frac{1}{a^2(\beta_0 + \beta_1 a)}\right)]\} \, ,$$

where $\alpha_s(M_Z) = 0.1176$ in PDG20.

The coefficients β_i depend on the number f of active quarks, which changes at thresholds $Q_f^2 \sim m_f^2$. Here we will not consider the f-dependence of Λ_i^f and $a_s(f, M_Z^2)$. Since we will mainly consider the region of low Q^2, we will use the results for $\Lambda_i^{f=3}$.

2.2 Discussions

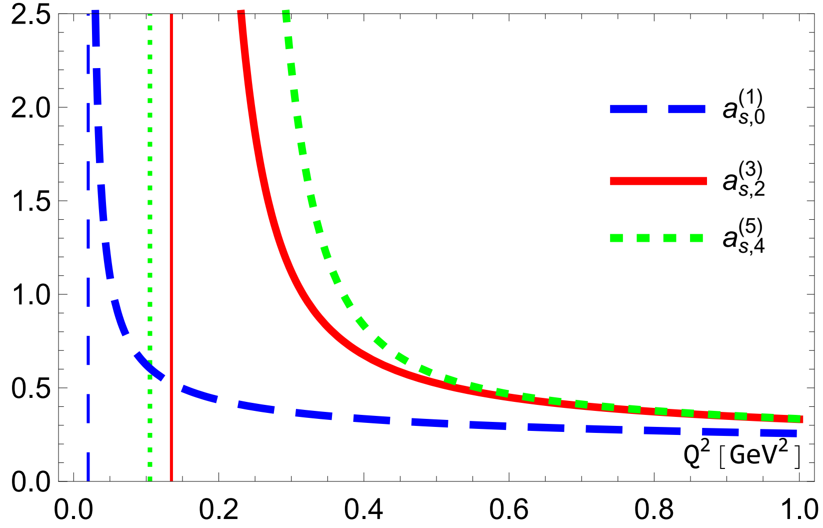

Figure 1: The results for $a_{s,i}^{(i+1)}(Q^2)$ and $(\Lambda_i^{f=3})^2$ (vertical lines) with $i = 0, 2, 4$.

In Fig. 1 one can see that the strong couplants $a_{s,i}^{(i+1)}(Q^2)$ become to be singular at $Q^2 = \Lambda_i^2$. The Λ_0 and Λ_i $(i \geq 1)$ values are rather different (Chen,Liu,Wang,Waqas,Peng: 2021):

$$\Lambda_0^{f=3} = 142 \ \text{MeV}, \quad \Lambda_1^{f=3} = 367 \ \text{MeV}, \quad \Lambda_2^{f=3} = 324 \ \text{MeV},$$
$$\Lambda_3^{f=3} = 328 \ \text{MeV}\,.$$

3. Fractional derivatives

Following (Cvetic,Valenzuela: 2006) we introduce the derivatives (in the $(i+1)$-order of perturbation theory)

$$\tilde{a}_{n+1}^{(i+1)}(Q^2) = \frac{(-1)^n}{n!} \frac{d^n a_s^{(i+1)}(Q^2)}{(dL)^n},$$

which will be very convenient in the case of the analytic QCD.

The series of derivatives $\tilde{a}_n(Q^2)$ can successfully replace the corresponding series of the a_s-powers. Indeed, every derivative decrease the power of a_s but it comes together with the additional β-function $\sim a_s^2$, appeared during the derivative. So, every application of derivative produces the additional a_s, and, thus, indeed the series of derivatives can be used instead of the series of the a_s-powers.

At LO, the series of derivatives $\tilde{a}_n(Q^2)$ exactly coincide with a_s^n. Beyond LO, the relation between $\tilde{a}_n(Q^2)$ and a_s^n was established in (Cvetic,Valenzuela: 2006), (Cvetic,Kogerler,Valenzuela: 20110) and extended to the fractional case, where $n \to$ a non-integer ν, in (Cvetic,Kotikov: 2012).

Now we consider the $1/L$ expansion of $\tilde{a}^{(k)}_{\nu}(Q^2)$. After some calculatins, we have

$$\tilde{a}^{(1)}_{\nu,0}(Q^2) = \left(a^{(1)}_{s,0}(Q^2)\right)^{\nu} = \frac{1}{L_0^{\nu}},$$
$$\tilde{a}^{(i+1)}_{\nu,i}(Q^2) = \tilde{a}^{(1)}_{\nu,i}(Q^2) + \sum_{m=1}^{i} C^{\nu+m}_{m}\, \tilde{\delta}^{(m+1)}_{\nu,i}(Q^2),$$
$$\tilde{\delta}^{(m+1)}_{\nu,i}(Q^2) = \hat{R}_m \frac{1}{L_i^{\nu+m}}, \quad C^{\nu+m}_{m} = \frac{\Gamma(\nu+m)}{m!\Gamma(\nu)},$$

where

$$\hat{R}_1 = b_1[\hat{Z}_1(\nu) + \frac{d}{d\nu}], \quad \hat{R}_2 = b_2 + b_1^2[\frac{d^2}{(d\nu)^2} + 2\hat{Z}_1(\nu+1)\frac{d}{d\nu} + \hat{Z}_2(\nu+1)].$$

The representation of the $\tilde{\delta}^{(m+1)}_{\nu,i}(Q^2)$ corrections as $\hat{R}_m$-operators is very important to use. This will make it possible to present high-order results for the analytic couplant in a similar way.

Here

$$Z_2(\nu) = S_1^2(\nu) - S_2(\nu),$$
$$Z_1(\nu) \equiv S_1(\nu) = \Psi(1+\nu) + \gamma_{\rm E}, \quad S_2(\nu) = \zeta_2 - \Psi'(1+\nu),$$

and

$$\hat{Z}_1(\nu) = Z_1(\nu) - 1, \quad \hat{Z}_2(\nu) = Z_2(\nu) - 2Z_1(\nu) + 1.$$

Note that operators like $\hat{R}_m$ were used earlier in (Bakulev,Mikhailov,Stefanis: 2005,2008,2010).

4. MA coupling (Euclidean space)

There are several ways to obtain analytical versions of the strong couplant a_s (see, e.g. (Bakulev: 2008)).

Here we will follow MA approach (Shirkov, Solovtsov: 1996), (Milton,Solovtsov,Solovtsova: 1997), (Shirkov: 2001)

as discussed in Introduction.

To the fractional case, the MA approach was generalized by Bakulev, Mikhailov and Stefanis (hereinafter referred to as the BMS approach) (Bakulev,Mikhailov,Stefanis: 2005,2008,2010).

We first show the LO BMS results, and later we will go beyond LO, following our results for the usual strong couplant obtained in the previous section.

4.1 LO

The LO minimal analytic coupling $A^{(1)}_{\mathrm{MA},\nu}$ have the form (Bakulev,Mikhailov,Stefanis: 2005)

$$A^{(1)}_{\mathrm{MA},\nu,0}(Q^2) = \left(a^{(1)}_{\nu,0}(Q^2)\right)^{\nu} - \frac{\mathrm{Li}_{1-\nu}(z_0)}{\Gamma(\nu)} \equiv \frac{1}{L_0^{\nu}} - \Delta^{(1)}_{\nu,0}\,,$$

where

$$\mathrm{Li}_{\nu}(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^{\nu}} = \frac{z}{\Gamma(\nu)} \int_0^{\infty} \frac{dt\; t^{\nu-1}}{(e^t - z)}, \quad z_i = \frac{\Lambda_i^2}{Q^2}$$

is the Polylogarithmic function.

For $\nu = 1$ we recover the famous Shirkov-Solovtsov result (Shirkov, Solovtsov: 1996)

$$A^{(1)}_{\mathrm{MA},0}(Q^2) \equiv A^{(1)}_{\mathrm{MA},\nu=1,0}(Q^2) = a^{(1)}_{s,0}(Q^2) - \frac{z_0}{1-z_0} = \frac{1}{L_0} - \frac{z_0}{1-z_0}\,.$$

4.2 Beyond LO

Following to the LO analytic couplant, we consider the difference between the derivatives of usual and MA couplants:

$$\tilde{A}_{\mathrm{MA},n+1}(Q^2) = \frac{(-1)^n}{n!} \frac{d^n A_{\mathrm{MA}}(Q^2)}{(dL)^n} \,.$$

For the differences of fracted derivatives of usual and MA couplants

$$\tilde{\Delta}^{(i+1)}_{\nu,i} \equiv \tilde{a}^{(i+1)}_{\nu,i} - \tilde{A}^{(i+1)}_{\mathrm{MA},\nu,i}$$

we have the following results

$$\tilde{\Delta}^{(i+1)}_{\nu,i} = \tilde{\Delta}^{(1)}_{\nu,i} + \sum_{m=1}^{i} C^{\nu+m}_{m} \, \hat{R}_m \left(\frac{\mathrm{Li}_{-\nu-m+1}(z_i)}{\Gamma(\nu+m)} \right) ,$$

where the operators $\hat{R}_i$ $(i = 1, 2, 3, 4)$ are shown above.

After some evaluations, we obtain

$$\tilde{\Delta}^{(i+1)}_{\nu,i} = \tilde{\Delta}^{(1)}_{\nu,i} + \sum_{m=1}^{i} C^{\nu+m}_{m} \, \overline{R}_m(z_i) \left(\frac{\mathrm{Li}_{-\nu-m+1}(z_i)}{\Gamma(\nu+m)} \right) ,$$

where

$$\overline{R}_1(z) = b_1[\gamma_{\rm E} - 1 + {\rm M}_{-\nu,1}(z)],$$
$$\overline{R}_2(z) = b_2 + b_1^2[{\rm M}_{-\nu-1,2}(z) + 2(\gamma_{\rm E} - 1){\rm M}_{-\nu-1,1}(z) + (\gamma_{\rm E} - 1)^2 - \zeta_2],$$

and

$$\mathrm{Li}_{\nu,k}(z) = (-1)^k \frac{d^k}{(d\nu)^k}\,\mathrm{Li}_\nu(z) = \sum_{m=1}^{\infty} \frac{z^m \ln^k m}{m^\nu}, \quad {\rm M}_{\nu,k}(z) = \frac{\mathrm{Li}_{\nu,k}(z)}{\mathrm{Li}_\nu(z)}\,.$$

So, we have for MA analytic couplants $\tilde{A}^{(i+1)}_{\mathrm{MA},\nu}$ the following expressions:

$$\tilde{A}^{(i+1)}_{\mathrm{MA},\nu,i}(Q^2) = \tilde{A}^{(1)}_{\mathrm{MA},\nu,i}(Q^2) + \sum_{m=1}^{i} C^{\nu+m}_{m} \tilde{\delta}^{(m+1)}_{\mathrm{MA},\nu,i}(Q^2)$$

where

$$\tilde{A}^{(1)}_{\mathrm{MA},\nu,i}(Q^2) = \tilde{a}^{(1)}_{\nu,i}(Q^2) - \frac{\mathrm{Li}_{1-\nu}(z_i)}{\Gamma(\nu)},$$

$$\tilde{\delta}^{(m+1)}_{\mathrm{MA},\nu,i}(Q^2) = \tilde{\delta}^{(m+1)}_{\nu,i}(Q^2) - \overline{R}_m(z_i)\,\frac{\mathrm{Li}_{-\nu+1-m}(z_i)}{\Gamma(\nu+m)}$$

and $\tilde{\delta}^{(k+1)}_{\nu,m}(Q^2)$ are given above.

There are three more representations for $\tilde{A}^{(1)}_{\mathrm{MA},\nu,i}(Q^2)$ (see (Kotikov, Zemlyakov: 2005)) that give exactly the same numerical results. Each of the representations is useful in its own kinematic range.

4.3. The case $\nu = 1$

For the case $\nu = 1$,

$$A^{(i+1)}_{\mathrm{MA},i}(Q^2) \equiv \tilde{A}^{(i+1)}_{\mathrm{MA},\nu=1,i}(Q^2) = A^{(1)}_{\mathrm{MA},i}(Q^2) + \sum_{m=1}^{i} \tilde{\delta}^{(m+1)}_{\mathrm{MA},1,i}(Q^2)$$

where

$$A^{(1)}_{\mathrm{MA},i}(Q^2) = \tilde{a}^{(1)}_{\nu=1,i}(Q^2) - \mathrm{Li}_0(z_i) = a^{(1)}_{s,i}(Q^2) - \mathrm{Li}_0(z_i),$$

$$\tilde{\delta}^{(m+1)}_{\mathrm{MA},1,i}(Q^2) = \tilde{\delta}^{(m+1)}_{1,i}(Q^2) - \overline{R}_m(z_i)\,\frac{\mathrm{Li}_{-m}(z_i)}{m!}$$

and

$$\mathrm{Li}_0(z) = \frac{z}{1-z}, \quad \mathrm{Li}_{-1}(z) = \frac{z}{(1-z)^2}, \quad \mathrm{Li}_{-2}(z) = \frac{z(1+z)}{(1-z)^3}.$$

The results can be used for phenomenological studies beyond LO in the framework of the minimal analytic QCD.

4.4 Discussions

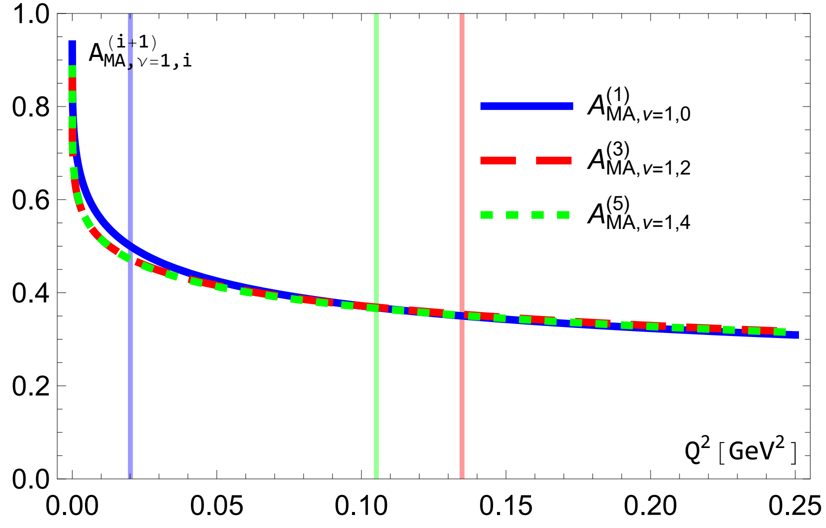

Figure 2: The results for $A^{(i+1)}_{\mathrm{MA},\nu=1,\mathrm{i}}(Q^2)$ with $i = 0, 2, 4$.

From Fig. 2 we can see differences between $A^{(i+1)}_{\mathrm{MA},\nu=1,\mathrm{i}}(Q^2)$ with $i = 0, 2, 4$, which are rather small and have nonzero values around the position $Q^2 = \Lambda_i^2$.

Thus, we can conclude that contrary to the case of the usual couplant, considered in Fig. 1, the $1/L$-expansion of the MA couplant is very good approximation at any Q^2 values. Moreover, the differences between $A^{(i+1)}_{\mathrm{MA},\nu=1,\mathrm{i}}(Q^2)$ and $A^{(1)}_{\mathrm{MA},\nu=1,0}(Q^2)$ are small. So, the expansions of $A^{(i+1)}_{\mathrm{MA},\nu=1,\mathrm{i}}(Q^2)$ $i \geq 1$ through the one $A^{(1)}_{\mathrm{MA},\nu=1,0}(Q^2)$ done in (Bakulev,Mikhailov,Stefanis: 2005,2008,2010) very good approximations.

Note that above representation of $\delta^{(i+1)}_{\mathrm{MA},\nu=1,\mathrm{i}}(Q^2)$ looks very similar to its expansion in terms of $A^{(i+1)}_{\mathrm{MA},\nu=1,\mathrm{i}}(Q^2)$ done in (Bakulev,Mikhailov, Stefanis: 2005,2008,2010).

Also the approximation

$$A^{(i+1)}_{\mathrm{MA},\nu=1,\mathrm{i}}(Q^2) = A^{(1)}_{\mathrm{MA},\nu=1,0}(k_i Q^2), \quad (i = 1, 2)\,,$$

introduced in (Pasechnik,Shirkov,Teryaev,Solovtsova,Khandramai: 2010,2012) and used in (Kotikov,Krivokhizhin,Shaikhatdenov: 2012), (Sidorov,Solovtsova: 2014) is very convenient, too.

Indeed, since the corrections $\delta^{(i+1)}_{\mathrm{MA},\nu=1,\mathrm{i}}(Q^2)$ are very small, then one can see that the MA couplants $A^{(i+1)}_{\mathrm{MA},\nu=1,\mathrm{i}}(Q^2)$ are very similar to the LO ones taken with the corresponding Λ_i.

5. MA couplant in timelike region

Transition to the Minkowskian space is defined through the contour integration (Shirkov:2005) [with earlier studies in (Schrempp, Schrempp:1980), (Pennington,Ross:1981), (Krasnikov,Pivovarov:1982), (Radyushkin:1982)]

$$\tilde{U}_{\nu}^{(l)}(\mathrm{s}) = \frac{1}{2\pi i} \int_{-s-i\epsilon}^{-s+i\epsilon} \frac{\tilde{A}_{\mathrm{MA},\nu}^{(l)}(\sigma)}{\sigma} d\sigma$$

The one-loop result has the form

(Bakulev,Mikhailov,Stefanis: 2005)

$$U_{\nu}^{(1)}(\mathrm{s}) = \tilde{U}_{\nu}^{(1)}(\mathrm{s}) = \frac{\sin[(\nu-1)\, g(s)]}{\pi(\nu-1)(\pi^2+L_s^2)^{(\nu-1)/2}} \stackrel{\nu\to 1}{\Longrightarrow} \frac{g(s)}{\pi}\,,$$

where

$$L_s = \ln\frac{s}{\Lambda^2}, \quad g(s) = \arccos\left(\frac{L_s}{\sqrt{\pi^2+L_s^2}}\right) = \frac{\pi}{2} - \arctan\left(\frac{L_s}{\pi}\right)\,.$$

It is convenient to introduce the results for

$$D_\nu^{(i)}(\mathrm{s}) = \left(\frac{d}{d\nu}\right)^i U_\nu^{(1)}(\mathrm{s}),$$

which can be represented in the form

$$D_\nu^{(i)}(\mathrm{s}) = \frac{h_{si}^{(i)}(s)\sin\left[(\nu-1)g(s)\right] + h_{co}^{(i)}(s)\cos\left[(\nu-1)g(s)\right]}{\pi(\nu-1)(\pi^2+L_s^2)^{(\nu-1)/2}}\,.$$

After some calculations, we have

$$h_{si}^{(1)}(s) = -\left(\frac{1}{\nu-1} + G(s)\right), \quad h_{co}^{(1)}(s) = g(s)\,,$$

$$h_{si}^{(2)}(s) = \frac{2}{(\nu-1)^2} + \frac{2G}{\nu-1} + G^2 - g^2, \quad h_{co}^{(2)}(s) = -2g\left(\frac{1}{\nu-1} + G\right),$$

where

$$G(\mathrm{s}) = \frac{1}{2}\ln\left(\pi^2 + L_s^2\right)\,.$$

So, it's possible to write the expressions for $\tilde{U}_\nu(\mathrm{s})$ beyond the LO as

$$\tilde{U}_\nu^{(i+1)}(\mathrm{s}) = \tilde{U}_\nu^{(1)}(\mathrm{s}) + \sum_{m=1}^{i} C_m^{\nu+m}\, \tilde{\delta}_\nu^{(m+1)}(\mathrm{s}),$$

$$\tilde{\delta}_\nu^{(m+1)}(\mathrm{s}) = \hat{R}_m\, \tilde{U}_{\nu+m}^{(1)}(\mathrm{s}), \quad C_m^{\nu+m} = \frac{\Gamma(\nu+m)}{m!\Gamma(\nu)},$$

In explicit form:

$$\tilde{\delta}_\nu^{(2)}(\mathrm{s}) = \frac{b_1}{\nu\pi(\pi^2+L_s^2)^{\nu/2}}\{g\cos(\nu g) + [\hat{Z}_1(\nu-1) - G]\sin(\nu g)\},$$

$$\tilde{\delta}_\nu^{(3)}(\mathrm{s}) = \frac{1}{(\nu+1)\pi(\pi^2+L_s^2)^{(\nu+1)/2}}(b_2\sin((\nu+1)g)$$
$$+b_1^2\{[\hat{Z}_2(\nu) - 2G\hat{Z}_1(\nu) + G^2 - g^2]\sin((\nu+1)g)$$
$$+2g[\hat{Z}_1(\nu) - G]\cos((\nu+1)g)\})$$

There is another representation for $\tilde{U}_\nu^{(i+1)}(\mathrm{s})$ that gives exactly the same numerical results. Each of the representations is useful in its own kinematic range.

5.1. ν=1

For the case $\nu = 1$ we get

$$\tilde{U}^{(i+1)}_{\nu=1}(\mathrm{s}) = \tilde{U}^{(1)}_{\nu=1}(\mathrm{s}) + \sum_{m=1}^{i} \tilde{\delta}^{(m+1)}_{\nu=1}(\mathrm{s}),$$

where

$$U_1^{(1)}(\mathrm{s}) = \frac{g(s)}{\pi} = \frac{1}{\pi}\arccos\left(\frac{L_s}{\sqrt{L_s^2+\pi^2}}\right) = \frac{1}{\pi}\left(\frac{\pi}{2} - \arctan\left(\frac{L_s}{\pi}\right)\right)$$

and

$$\tilde{\delta}^{(2)}_{\nu=1}(\mathrm{s}) = \frac{b_1}{\pi(\pi^2+L_s^2)^{1/2}}\{g\cos(g) - [1+G]\sin(g)\},$$

$$\tilde{\delta}^{(3)}_{\nu=1}(\mathrm{s}) = \frac{1}{2\pi(\pi^2+L_s^2)}(b_2\sin(2g) + b_1^2[G^2-g^2-1]\sin(2g)) \,.$$

We would like to note that

$$\cos(g) = \frac{L_s}{\sqrt{L_s^2 + \pi^2}}, \quad \sin(g) = \frac{\pi}{\sqrt{L_s^2 + \pi^2}}$$

and

$$\sin(2g) = 2\sin(g)\cos(g) = \frac{2\pi L_s}{L_s^2 + \pi^2},$$

$$\cos(2g) = \cos^2(g) - \sin^2(g) = \frac{L_s^2 - \pi^2}{L_s^2 + \pi^2}.$$

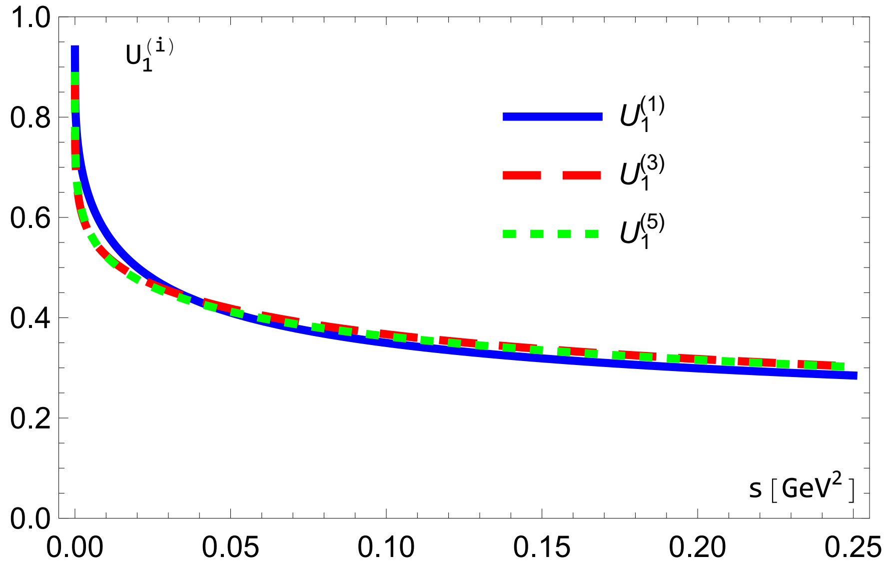

Figure 3: 1,3 and 5 orders of $U_1^{(i)}$.

From Fig. 3 we can see differences between $U^{(i+1)}_{\rm MA,\nu=1,i}(s)$ with $i = 0, 2, 4$, which are rather small and have nonzero values around the position $s = \Lambda_i^2$ (similar to above results in Euclidean space).

6. Bjorken sum rule

The polarized Bjorken sum rule is defined as the difference between proton and neutron polarized structure function (SFs) g_1 integrated over the whole x interval

$$\Gamma_1^{p-n}(Q^2) = \int_0^1 dx\, [g_1^p(x,Q^2) - g_1^n(x,Q^2)].$$

Based on the various SF measurements, $\Gamma_1^{p-n}(Q^2)$ has been extracted at various values of squared momenta Q_j^2 (0.054 GeV$^2 \leq Q_j^2 < 5$ GeV^2).

Theoretically, it has the Operator Produxt Expansion form

$$\Gamma_1^{p-n}(Q^2) = \left|\frac{g_A}{g_V}\right| \frac{1}{6}\left(1 - D_{\rm BS}(Q^2)\right) + \sum_{i=2}^{\infty} \frac{\mu_{2i}(Q^2)}{Q^{2i-2}},$$

where $|g_A/g_V|$=1.2723 $\pm$ 0.0023 is the ratio of the nucleon axial charge, $(1 - D_{BS}(Q^2))$ is the leading-twist contribution, and μ_{2i}/Q^{2i-2} is the higher-twist contributions.

Since we will consider very low Q^2 values, the above representation of the higher-twist contributions are not so convenient and it is better to use so-called its "massive" conter-part (Teryaev: 2013), (Khandramai,Teryaev,Gabdrakhmanov: 2016,2017):

$$\Gamma_1^{p-n}(Q^2) = \left|\frac{g_A}{g_V}\right| \frac{1}{6}\left(1 - D_{\rm BS}(Q^2)\right) + \frac{\tilde{\mu}_4}{Q^2 + M^2}\,,$$

where the values of $\tilde{\mu}$ and M^2 has been fitted in (Ayala,Cvetič,Kotikov,Shaikhatdenov: 2018) in the different types of models for analytic QCD.

The perturbative part has the following form

$$D_{\rm BS}(Q^2) = \frac{4}{\beta_0}\, a_s \left(1 + d_1 a_s + d_2 a_s^2 + d_3 a_s^3\right)$$
$$= \frac{4}{\beta_0} \left(\tilde{a}_1 + \tilde{d}_1 \tilde{a}_2 + \tilde{d}_2 \tilde{a}_3 + \tilde{d}_3 \tilde{a}_4\right),$$

where

$$\tilde{d}_1 = d_1, \quad \tilde{d}_2 = d_2 - b_1 d_1, \quad \tilde{d}_3 = d_3 - \frac{5}{2} b_1 d_2 + (\frac{5}{2} b_1^2 - b_2)\, d_1,$$

For $f = 3$ case, we have

$$\tilde{d}_1 = 1.59, \quad \tilde{d}_2 = 2.51, \quad \tilde{d}_3 = 10.58\,.$$

In the MA model, the perturbative part has the form:

$$D_{\rm MA,BS}(Q^2) = \frac{4}{\beta_0}\, (A^{(k)}_{\rm MA,k-1} + \sum_{m=2}^{k} \tilde{d}_{m-1}\, \tilde{A}^{(k)}_{\rm MA,\nu=m,k-1})\,.$$

Moreover, from (Ayala,Cvetič,Kotikov,Shaikhatdenov: 2018)

it is possible tom see that

$$M^2 = 0.439, \quad \tilde{\mu}_4 = -0.082\,.$$

6.1 Discussions

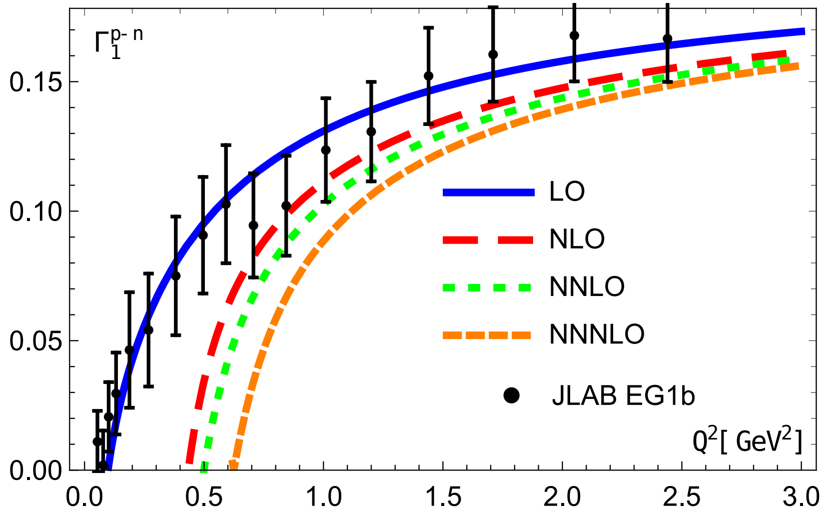

Figure 4: The results for $\Gamma_1^{p-n}(Q^2)$ in the first four orders of perturbation theory with the "massive" twist-four term.

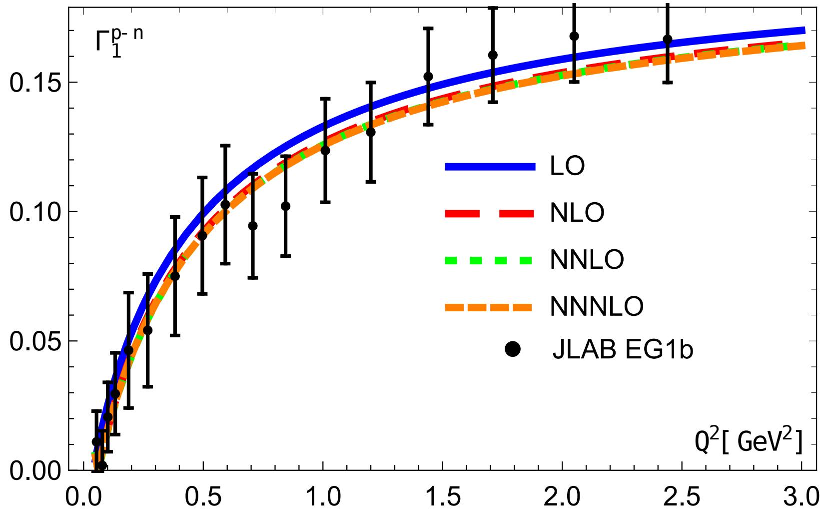

Figure 5: Same as in Fig. (4) but in analytic theory.

The results of calculations are shown in Figs. 4 and 5. Here we use the Q^2-independent M and $\tilde{\mu}_4$ values and the twist-two parts shown above for the cases of usual PT and APT, respectively.

As can be seen in Fig. 4, results obtained using usual couplants are good only at LO and deteriorate as the PT order increases. The good agreement at LO is due to the use of $\Lambda_{\rm LO}$, which is small, and therefore the investigated range of Q^2 is higher than $\Lambda^2_{\rm LO}$. Visually, the results are close to those obtained in (Khandramai,Pasechnik,Shirkov,Solovtsova,Teryaev: 2012), where the usual twist-four form has been used.

Thus, the usage of the "massive" twist-four form does not improve the results, since at $Q^2 \to \Lambda_i^2$ usual couplants become to be singular, that leads to large and negative results for the twist-two part. With increasing the PT order usual couplants become to be singular at larger Q^2 values (see Fig. 1) and the Bjorken sum rule tends to negative values with increasing values of Q^2. So, the discrepancy between theory and experiment increases with the increase in the PT order.

In the case of using MA couplants, our results are close to those obtained in (Ayala,Cvetič,Kotikov,Shaikhatdenov: 2018), which is not surprising, since we used the parameters, obtained in this paper. Moreover, we see that the results based on different orders of perturbation theory are close to each other, in contrast to the case of using the usual couplants.

7. Conclusions

In this talk, we have showen results of several models of QCD analytic coupland.
We have focused on the introduction of the Shirkov-Solovtsov and Bakulev-Mikhailov-Stefanis approaches and their recent extension beyond the leading order of perturbation theory.
We have considered $1/L$-expansions of the ν-derivatives of the strong couplant a_s expressed as combinations of operators $\hat{R}_m$ applied to the LO couplant $a_s^{(1)}$.
Applying these operators to the ν-derivatives of the LO MA couplants $A_{\rm MA}^{(1)}$ and $U_{\rm MA}^{(1)}$, we have got different representations for the ν-derivatives of the MA couplants: $\tilde{A}_{\rm MA,\nu}^{(i)}$ in Euclidean space and $\tilde{U}_{\rm MA,\nu}^{(i)}$ in Minkowski space, i.e. , in each i-order of PT.

The high-order corrections are negligible in the $Q^2 \to 0$ and $Q^2 \to \infty$ asymptotics and are nonzero in a neighborhood of the point $Q^2 = \Lambda^2$. Thus, in fact, they are really only small corrections to the LO MA couplant $A^{(1)}_{\mathrm{MA},\nu}(Q^2)$.

As can be clearly seen, all our results have a compact form and do not contain complicated special functions, such as the Lambert W-function (Magradze: 1999), which already appears in two-loop order as an exact solution to the usual couplant and which was used to estimate the MA couplants in (Bakulev,Mikhailov,Stefanis: 2010).

As a example, we considered the Bjorken sum rule and obtained results similar to previous studies in (Pasechnik,Shirkov,Teryaev,Solovtsova,Khandramai: 2008,2009,2011), (Ayala,Cvetic,Kotikov,Shaikhatdenov: 2018) because the high order corrections are small. The results based on usual perturbation theory do not not agree with the experimental data at $Q^2 \leq 1.5$ GeV^2. MA APT leads to good agreement with the data when we used the "massive" version for high-twist contributions.

In the future, we plan to finish the timelike case.