“New Wine in an Old Bottle? ”

„Novel Outlook on the Eigenvalue Problem for the Orbital Angular Momentum Operator“

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Analysis of differential equations for eigenfunctions in spherical coordinates:

$$(\theta;\varphi|L;m) = L(L+1)\,(\theta;\varphi|L;m); \qquad (1)$$

$$= (\theta\ ;\ \partial/\partial\theta\ ;\ (\partial/\partial\varphi)^2)\ ;$$

$$_z\,(\varphi) = -\,i\,(\partial/\partial\varphi)\,(\varphi) = m\,(\varphi); \qquad (2)$$

=0≤θ≤π; and 0≤φ<2π - spherical coordinates; ^{2} - operator of the momentum squared; L(L+1) – the eigenvalue of the ^{2}; (θ;φ|L;m) – the eigenfunction of ^{2}; $_z$ – the third component of the momentum operator; (φ) - the eigenfunction of the $_z$.

$$(\theta;\varphi|L;m) = (\theta|L;m)\ (\varphi);$$

$$(\partial/\partial\theta\ ;\ \theta\ ;\ (i\partial/\partial\varphi)^2)\,(\theta|L;m)\,(\varphi) = (\varphi)\,(\partial/\partial\theta\ ;\ \theta\ ;\ m^2)\ (\theta|L;m);$$

$$(\partial/\partial\theta\ ;\ \theta\ ;\ m^2)\ (\theta|L;m) = L(L+1)\ (\theta|L;m)\ ;$$

$(\theta|L;m) = a\,(\theta|L;m) + b\,(\theta|L;m)$;

Even Odd

$(\theta|L;m) = (\theta|L;-m)$; $(\theta|L;m) = -\,(\theta|L;-m)$;

$\varphi) = (\varphi) + i\,(\varphi)$;

$d(\varphi)/d\varphi = -\,m\,(\varphi)$; $d(\varphi)/d\varphi = m\,(\varphi)$;

$(\varphi)\,/d = -\,(\varphi)$

$(\varphi|m) = c\,(\varphi|m) + d\,(\varphi|m)$;

$(\varphi|m) = (\varphi|-m)$; $(\varphi|m) = -\,(\varphi|-m)$;

Both equations reduce to the equation for the Gauss hypergeometric function:

1) $(\theta|L;m)$ ---- $z = \cos^2\theta$; $4\beta^2 = m^2$; $(\theta|L;m) = (1-z)^{\beta}\ F(z|L;\beta)$

$$[z(1-z)\,(d^2/dz^2) + [1/2 - z(3/2+2\beta)\,(d/dz) + (\lambda-2\beta-4\beta^2)/4]F(z|L;\beta) = 0; \qquad (3)$$

$\lambda = L(L+1)$;

$F(z|L;\beta) = {}_2F_1\,(a, b; c; z)$;

$a = [1/2+2\beta+(1/4+\lambda)^{1/2}]/2$; $b=[1/2+2\beta-(1/4+\lambda)^{1/2}]/2$; $c=1/2$;

2) (z) ------ $= z$; $4\beta^2 = m^2$;

$$[z(1-z)d^2/dz^2 + [c-(a+b+1)z]d/dz - ab]\,(z) = 0; \qquad (4)$$

$c=1/2$; $a = \beta$; $b = -\,\beta$;

Let's start from the equation (3). From the set of Kummer pairs, we choose two linearly independent solutions corresponding to specific boundary conditions

$$(\theta|L;m) = (1-\cos^2\theta)^{\beta}\ {}_2F_1(1/2+\beta+L/2,\ \beta-L/2;1/2;\ \cos^2\theta)\ ;$$

$$(\theta|L;m) = (1-\cos^2\theta)^{\beta}\ (\cos^2\theta)^{1/2}\ {}_2F_1(1+\beta+L/2,1/2+\beta-L/2;\ 3/2;\ \cos^2\theta);$$

$$\beta = \{|m|/2;\ \pm m/2\};$$

$$\beta = \pm m/2\ ;$$

$$(\theta|L;\ \pm m) = (1-\cos^2\theta)^{\pm m/2}\ {}_2F_1(1/2 \pm m/2 + L/2,\ \pm m/2-L/2;1/2;\ \cos^2\theta)\ ;$$

$$(\theta|L;m) = (1-\cos^2\theta)^{\pm m/2}\ (\cos^2\theta)^{1/2}\ {}_2F_1(1\pm m/2+L/2,1/2\pm m/2-L/2;\ 3/2;\ \cos^2\theta);$$

$$(\theta|L;m) \pm (\theta|L;\ -m); \qquad (\theta|L;m) \pm (\theta|L;\ -m);$$

$$\beta = |m|/2\ ;$$

$$(\theta|L;\ |m|) = (1-\cos^2\theta)^{|m|/2}\ {}_2F_1(1/2-|m|/2+L/2,\ |m|/2-L/2;1/2;\ \cos^2\theta) =$$

$$= \theta(m)\ (\theta|L;m) + \theta(-m)\ (\theta|L;\ -m) = \ (\theta|L;m)\ ;$$

$$(\theta|L;|m|) = (1-\cos^2\theta)^{-|m|/2}(\cos^2\theta)^{1/2}\ {}_2F_1(1-|m|/2+L/2,1/2+|m|/2-L/2;\ 3/2;\ \cos^2\theta) =$$

$$= \theta(m)\ (\theta|L;m) + \theta(-m)\ (\theta|L;\ -m) = \ (\theta|L;m)\ .$$

The Physical requirement of normalization Mathematical condition of regularity (polynomization!?) of hypergeometric functions (k – integer):

for $(\theta\|L;\ m)$	$m/2-L/2 = -k;$	$m = L - 2k \geq 0\ ;$
$(\theta\|L;\ m)$	$1/2 + m/2-L/2 = -k;$	$m = (L-1) - 2k \geq 0\ ;$
$(\theta\|L;\ -m)$	$-m/2-L/2 = -k;$	$m = -L + 2k \leq 0\ ;$
$(\theta\|L;\ -m)$	$1/2 - m/2-L/2 = -k;$	$m = -(L-1) + 2k \leq 0\ ;$

(θ|L; |m|) |m|/2-L/2 = -k; |m| = L – 2k ;

(θ|L; |m|) 1/2 + |m|/2-L/2 = -k; |m| = (L-1) – 2k ;

- (L-2k) ≤ m ≤ (L-2k) ; - (L-1-2k) ≤ m ≤ (L-1-2k) .

Integer "m" integer "L"

Is "m" i n t e g e r or n o t ?

Lore (from textbooks) - "m" can be only integer – two different arguments.

Argument of the first type:

I. Physical requirement - wave function must be single-valued ;(?)

II. if "m" is non-integer - φ) = ; - not single-valued;(?)

"m" - must be i n t e g e r .

Comments:

φ) = ;

;

φ) = ; - single-valued ;

The first type of argument is not correct.

Arguments of the second type:

III. Physical requirement –

The wave function must be periodic with regard ; (?)

IV. if "m" is noninteger - φ) = ; - not periodic

"m" - must be i n t e g e r ;

Comment: Physical requirement –

The observables must be periodic with respect to rotations at 2π ;

Wave functions are not observables ;

The second type of argument is not correct.

φ) satisfies the orthonormalization condition -

If |m| = (L – 2k), or |m| = (L-1 – 2k), than - (m′-m) - integer,

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φ|m)* φ|m′) = = [– 1]/i(m′-m) =

=

On non uniqueness of

In cartesian coordinates

x;y|m) = f(z;;) --- not single-valued for non integer m;

Using the Euler-Moivre formula:

= ;

= = = [;

= = - becomes single-valued

When for the eigenfunctions the specific pair of Kummer functions is used, and the non-uniqueness of the exponential function is determined by the Euler-Moivre prescription, the physical requirement of normalizability

is realized by the condition

|m| = L – 2k ; |m| = (L-1) – 2k ; so |m| = L – k .

Legendre functions as eigenfunctions of

The sum of the two functions of the Kummer pair is again the solution of the same equation:

a (θ|L;m) + b (θ|L;m) = (θ|L;m);

For the Legendre functions:

(θ|L;m) = (θ|L;m) + (θ|L;m) ;

(θ|L;m) = (θ|L;m) + (θ|L;m) ; are fixed coefficients.

(θ|L;m) is regular only if

and (θ|L;m) - is regular, when (L;m) are only integer ;

and (θ|L;m) - never becomes regular ;

The spectrum of numerical values of (L;m) depends on the choice of Kummer pairs

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Algebraic method

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The method is based on the following mathematical principles:

1*:

2*: (θ;φ|L; m) = (L;m) (θ;φ|L; m ±1) ;

3*: (θ;φ|L; ± L) = 0 ;

These relations hold for all values of m

When “m” is integer ---- 2* and 3* ;

-L ≤ m ≤ L and all the (θ;φ|L; m) are regular

If we introduce a mathematical (not-physical!) condition as the 4th* mathematical principles (as is done in textbooks):

4*: = -L +k ; k = {0; 1; 2; •••} and = {-L; -L+1; ••• ; +L; •••}

or

= L - k ; k = {0; 1; 2; •••} and = {L ; L-1; •••; -L; •••}

(; L) =

When (; L) integer, all (θ;φ|L; m) functions are regular;

When (; L) is not integer (including half-integer values):

3.1: (θ;φ|L; + L) = 0 ;

3.2: ((θ;φ|L; + L) – is ;

3.3: $(\theta;\varphi|L; -L) = 0$;

3.4: $((\theta;\varphi|L; -L)$ – is ;

3.5: $(\theta|L; m) = (\theta|L; -m); -L \le m \le L$; - set is finite;

3.4: $(\theta|L; m) = (\theta|L; -m); m\text{-}L; m L$; - set is infinite.

Pauli rejected the requirement of periodicity and proposed instead

$((\theta;\varphi|L; +L)$ and $((\theta;\varphi|L; -L)$ – must be regular for any k ;

(L ; m) –must be only integer

Question: is the above a physical requirement?

No, it is not

Arguments, that this principle is only mathematical and not physical.

1: Operators - do not correspond to physical quantities ;

2: Operators - correspond operators of physically observable quantities, but they don’t commute with and with each other - so, their corresponding observable quantities are not observable at the same time.

Principle of quantum mechanics:

Hilbert space vectors must be constructed using only the complete set of commutative operators and not by considering operators outside of this set.

Counter example for Pauli's assumption: define

= [$^{+}$ + exp(i2φ)$^{-}$]/2 = ξ(1- ξ^{2})$^{-1/2}$(i/φ); ξ= cos ;

Acting with :

=

= $_2F_1$(1/2 -|m|/2 +L/2, |m|/2 - L/2;1/2;) -

is singular, when (|m|-p) for all „m", integer as well as non integer .

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Some remarks on the Euler-Moivre theorem

1: Invariance to translations at 2πk - = ;

2: exponentiation operator - X = ;

3: = ; = ;

and – functions are not uniquely defined ;

4: = ; = ;

5: Euler-Moivre theorem - = ; = ;

and - functions are uniquely defined ;

6: The relations derived from the Euler-Moivre theorem violates translational invariance -

;

A similar example

7: X = ; (-X) = = ; - Invariance to inversion of sign;

8: (X) = - not uniquely defined function;

9: Due to = ±1, there is an infinite number of possible rules by which this function can be redefined as unique ;

The basic redefinition is reduced to three cases of prescription:

= {±X ; |X|} ;

Prescriptions: = ±X - violates invariance to inversion of sign;

Prescription: = |X| - saves invariance to sign inversion;

Question:

whether there is a prescription that will preserve the starting symmetry = for equality = ;

Answer - yes, there is

1: is considered as the solution of equation:

$_z$ (x;y) = m (x;y); (x;y) (x;y) = ;

Let's define the real and imaginary parts of this exponential function as the solutions of the corresponding equations in spherical coordinates:

(φ) /d = - (φ) ;

which can be easily reduced to the equation of hypergeometric functions:

[z(1-z)d²/dz² + (1/2- z)d/dz -/4] (z) = 0 ; z = ;

Define (z) :

() = $_2F_1$(m/2, -m/2 ;1/2;);

() = $_2F_1$(m/2, -m/2 ;1/2;);

Due to the invariance of the equation to the transformation z → (1-z), the second pair of solutions can also be indicated:

() = $_2F_1$(m/2, -m/2 ;1/2;);

() = $_2F_1$(m/2, -m/2 ;1/2;);

These functions are invariant to the **φ → (φ+2kπ)** transformation, no matter how we determine the exponential functions and

() = () ;

= [() + i()];

C o n c l u s i o n

It would be better if the spectrum of the orbital moment were observed according to the experimental results, which would be quite natural.