XXXIV International (ONLINE) Workshop on High Energy Physics

From Quarks to Galaxies: Elucidating Dark Sides

RG EQUATIONS IN NON-RENORMALIZABLE THEORIES

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Motivation:

- The Standard Model is renormalizable
- Gravity is not renormalizable

Non-renormalizable theories are not accepted due to:

- UV divergences are not under control - infinite number of new types of divergences
- The amplitudes increase with energy (in PT) and violate unitarity

However:

- R-operation equally works for NR theories and leads to local counter terms
- Due to locality all higher order divergences are related to the lower ones

- These properties allow one to write down the RG equations for the scattering amplitudes, effective potential, etc which sum up the leading divergences (logarithms) and to find out the high energy/field behaviour

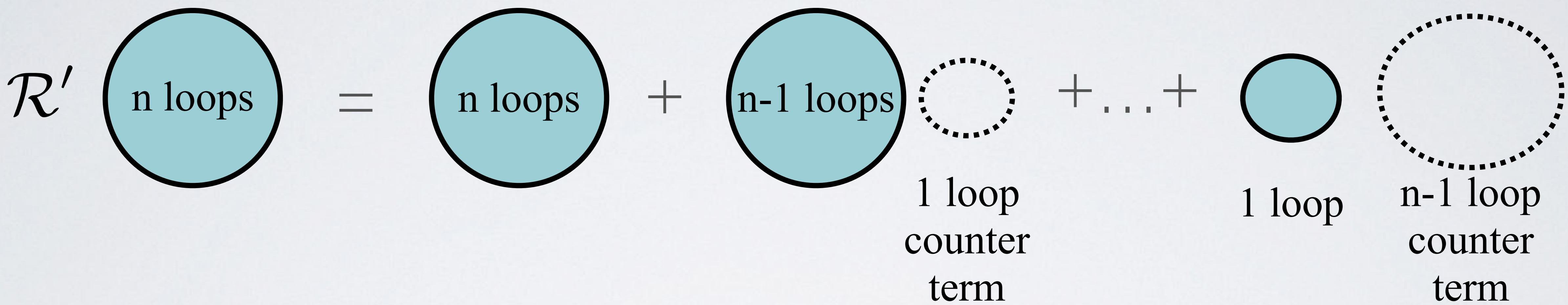
Based on: Phys. Lett. B734 (2014) 111, arXiv:1404.6998 [hep-th]
JHEP 11 (2015) 059, arXiv:1508.05570 [hep-th]
JHEP 12 (2016) 154, arXiv:1610.05549v2 [hep-th]
Phys.Rev. D95 (2017) no.4, 045006 arXiv:1603.05501 [hep-th]
Phys.Rev. D97 (2018) no.12, 125008, arXiv:1712.04348 [hep-th],
Phys.Lett. B786 (2018) 327-331, arXiv:1804.08387 [hep-th]
Symmetry 11 (2019) 1, 104, arXhiv:1812.11084 [hep-th]
Phys.Lett.B 797 (2019) 134801, arXiv:1904.08690 [hep-th]
Труды Мат. Инст. им. В.А. Стеклова, 2020, т. 308, с. 1–8
JHEP 06 (2022) 141, arXiv:2112.03091 [hep-th]

In collaboration with L.Bork, A.Borlakov, R. Iakhibbaev, D.Tolkachev and D.Vlasenko

BPHZ R-operation

$$\mathcal{R}'G_n = \frac{A_n^{(n)}(\mu^2)^{n\epsilon}}{\epsilon^n} + \frac{A_{n-1}^{(n)}(\mu^2)^{(n-1)\epsilon}}{\epsilon^n} + \ldots + \frac{A_1^{(n)}(\mu^2)^{\epsilon}}{\epsilon^n}$$

$+$ lower pole terms



$A_k^{(n)}(\mu^2)^{k\epsilon}$ terms appear after subtraction of (n-k) loop counter terms

<u>Statement:</u> $R'G_n$ is local, i.e. terms like $\log^k \mu^2/\epsilon^m$ should cancel for any k and m

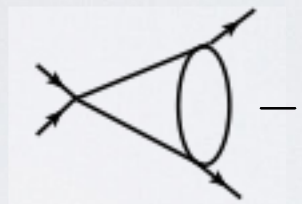
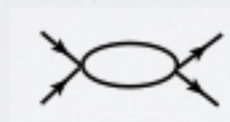
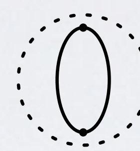
<u>Consequence:</u> $A_n^{(n)} = (-1)^{n+1}\frac{A_1^{(n)}}{n}$

The leading divergences are governed by 1 loop diagrams!

Two loop example

ϕ^4_4

 $= \left(\frac{A_2^{(2)}}{\epsilon^2} + \frac{A_1^{(2)}}{\epsilon}\right)(\frac{\mu^2}{s})^{2\epsilon}$

$\mathcal{R}'$  $=$  $-$   $= \left(\frac{A_2^{(2)}}{\epsilon^2} + \frac{A_1^{(2)}}{\epsilon}\right)(\frac{\mu^2}{s})^{2\epsilon} - \frac{A_1^{(1)}}{\epsilon}(\frac{\mu^2}{s})^{\epsilon}\frac{A_1^{(1)}}{\epsilon}$

$$= \frac{A_2^{(2)}}{\epsilon^2} - \frac{(A_1^{(1)})^2}{\epsilon^2} + 2\frac{A_2^{(2)}}{\epsilon}\log(\mu^2/s) - \frac{(A_1^{(1)})^2}{\epsilon}\log(\mu^2/s) = -\frac{1}{2}\frac{(A_1^{(1)})^2}{\epsilon^2} + \ldots$$

non-local terms to be cancelled

Leading divergence is given by the one-loop term $A_2^{(2)} = \frac{1}{2}(A_1^{(1)})^2$

- These statements are universal and are valid in non-renormalizable theories as well.
- The only difference is that the counter term $A_1^{(1)}$ depends on kinematics and has to be integrated through the remaining one-loop graph.
- As a result $A_2^{(2)}$ is not the square of $A_1^{(1)}$ anymore but is the integrated square .
- This last statement is the general feature of any QFT irrespective of renormalizability

Leading divergences

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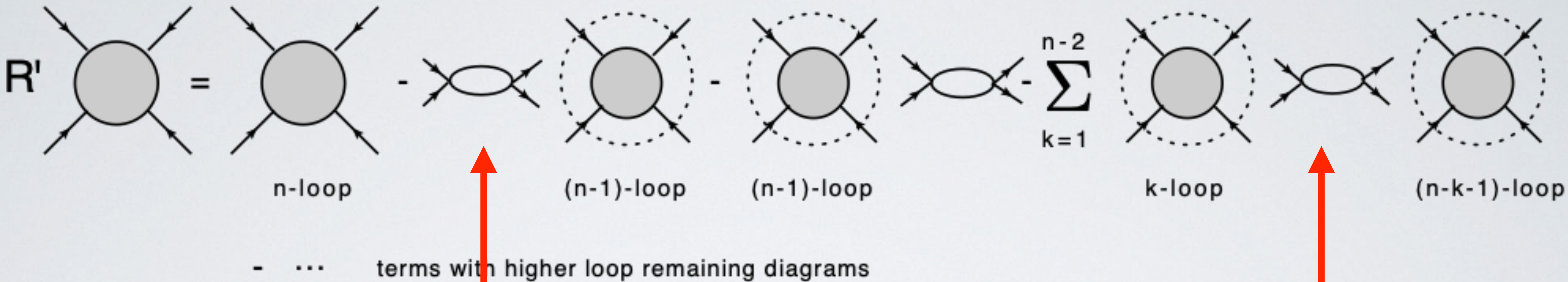
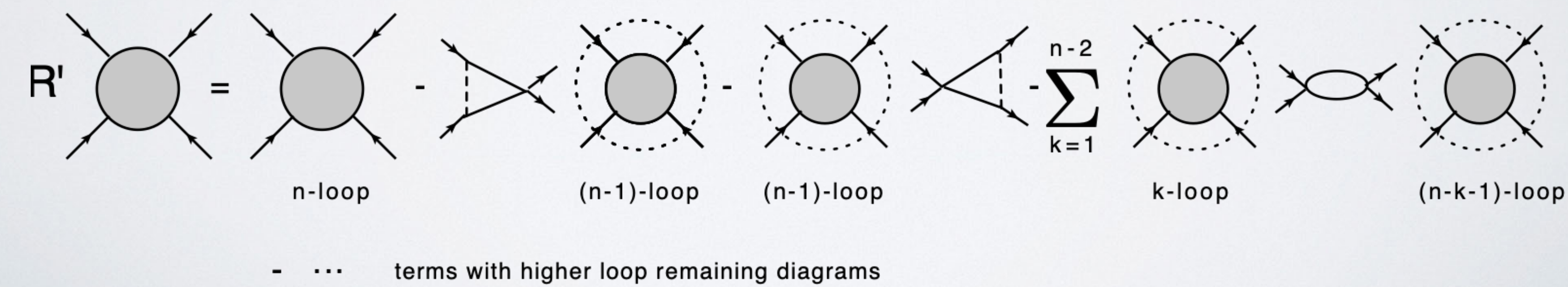
Quartic vertices



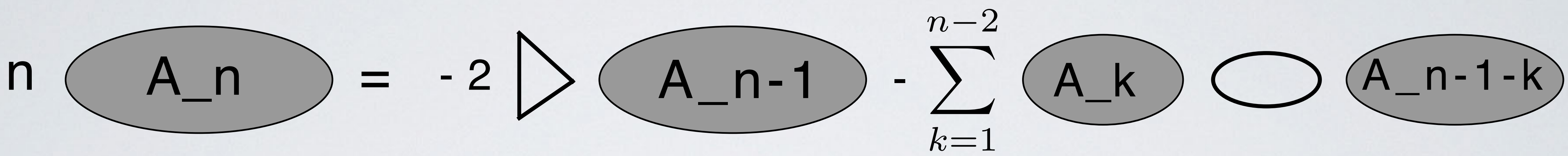
$A_n^{(n)}$ $A_1^{(n)}$ $A_1^{(n)}$

Cubic vertices



The Recurrence Relation

Kazakov,20



- This is the general recurrence relation that reflects the locality of the counter terms in any theory
- In renormalizable theories A_n is a constant and this relation is reduced to the algebraic one
- In non-renormalizable theories A_n depends on kinematics and one has to integrate through the one loop diagrams

Taking the sum $\sum_n A_n(-z)^n = A(z)$ one can transform the recurrence relation into integro-diff equation

$$\frac{d}{dz}A(z) = b_0\{-1 - 2\int_\triangle A(z) - \int_\bigcirc A^2(z)\}$$

$$\frac{d}{dz} = \frac{d}{d\log\mu^2}$$

This is the generalized RG equation valid in any (even non-renormalizable) theory!

RG Equation

SYM_D

D=6 N=2

$$\Sigma(s,t,z) = z^{-2}\sum_{n=3}^{\infty}(-z)^n S_n(s,t)$$

$$\frac{d}{dz}\Sigma(s,t,z) = s - \frac{2}{z}\Sigma(s,t,z) + 2s\int_0^1 dx \int_0^x dy\ (\Sigma(s,t',z)+\Sigma(t',s,z))|_{t'=xt+yu}$$

Linear equation

D=8 N=1

$$\Sigma(s,t,z) = \sum_{n=1}^{\infty}(-z)^n S_n(s,t)$$

$$\frac{d}{dz}\Sigma(s,t,z) = -\frac{1}{12} + 2s^2\int_0^1 dx \int_0^x dy\ y(1-x)\ (\Sigma(s,t',z)+\Sigma(t',s,z))|_{t'=tx+yu}$$
$$-s^4\int_0^1 dx\ x^2(1-x)^2\sum_{p=0}^{\infty}\frac{1}{p!(p+2)!}\left(\frac{d^p}{dt'^p}(\Sigma(s,t',z)+\Sigma(t',s,z))|_{t'=-sx}\right)^2\ (tsx(1-x))^p.$$

Non-linear equation

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Examples:

- Maximally supersymmetric gauge theory in D=6,8,10 dimensions SYM_D
- Scalar field theory in D=4,6,8,10 dimensions ϕ^4_D
- Gauge theory in D=4,6,8 dimensions YM
- Supersymmetric Wess-Zumino model with quartic superpotential in D=4 Φ^4_4

These are the toy models for (super) gravity - our aim

Perturbation Expansion for the 4-point Amplitudes for any D

SYM_D

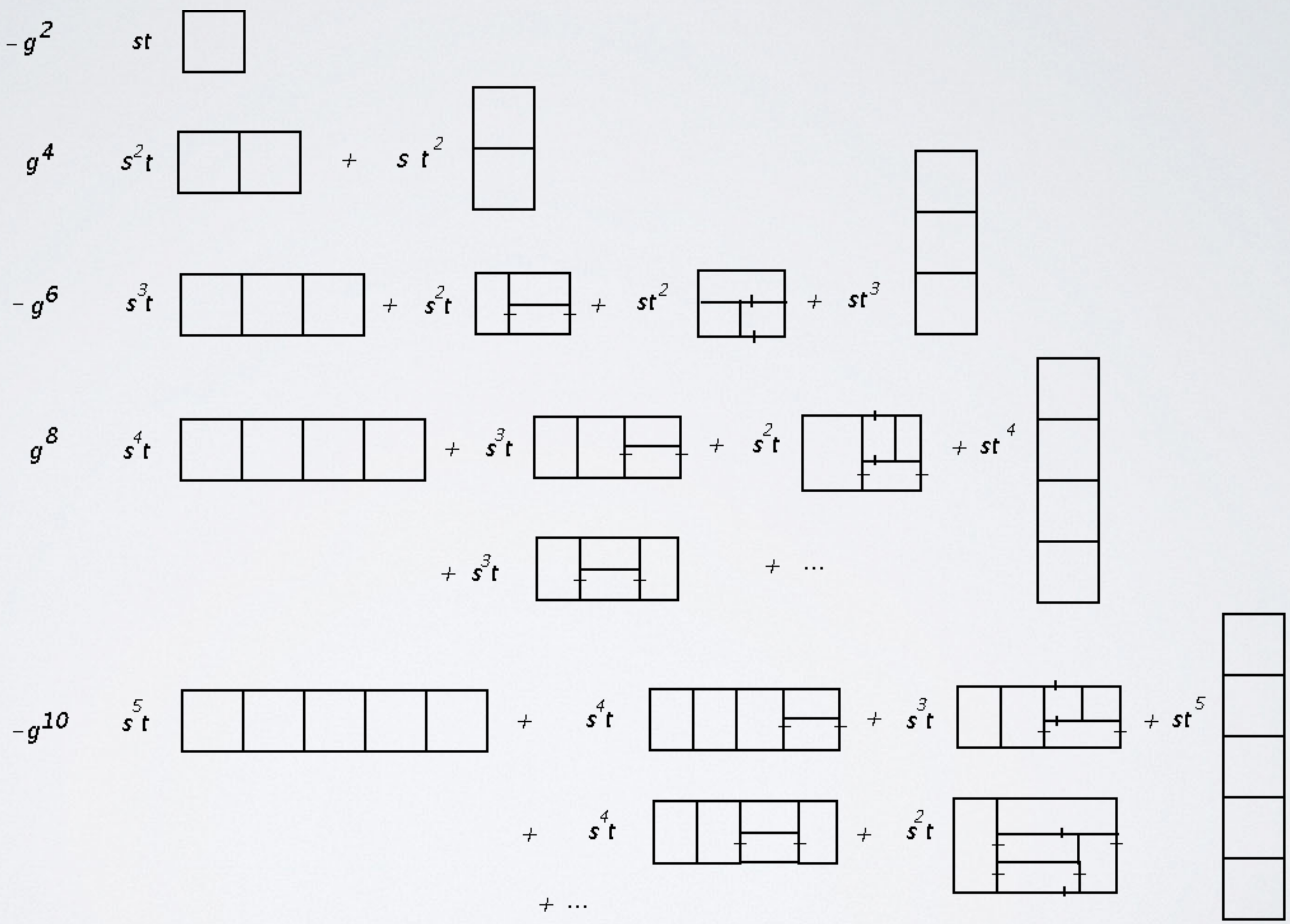
T. Dennen Yu-yin Huang 10 ,
S.Caron-Huot D.O'Connell 10

A_4/A_4^{tree}

No bubbles
No Triangles

First UV div at
L=[6/(D-4)] loops

IR finite



1

2

4

15

60

Universal expansion for any D in maximal SYM due to Dual conformal invariance

Bork,Kazakov,Kompaneets,Vlasenko, 13

Borlakov,,Kazakov,Tolkachev,Vlasenko, 15

SYM_D

D=6 N=2

S-channel $S_n(s,t)$ **T-channel** $T_n(s,t)$ $T_n(s,t) = S_n(t,s)$

Exact all-loop recurrence relation $S_3 = -s/3,\ T_3 = -t/3$

$$nS_n(s,t) = -2s \int_0^1 dx \int_0^x dy\ (S_{n-1}(s,t') + T_{n-1}(s,t'))$$

$n \geq 4$

$t' = t(x-y) - sy$

D=8 N=1

S-channel $S_n(s,t)$ **T-channel** $T_n(s,t)$ $T_n(s,t) = S_n(t,s)$

Exact all-loop recurrence relation $S_1 = \frac{1}{12},\ T_1 = \frac{1}{12}$

$$\begin{aligned} nS_n(s,t) &= -2s^2 \int_0^1 dx \int_0^x dy\ y(1-x)\ (S_{n-1}(s,t') + T_{n-1}(s,t'))|_{t'=tx+yu} \\ + \quad & s^4 \int_0^1 dx\ x^2(1-x)^2 \sum_{k=1}^{n-2} \sum_{p=0}^{2k-2} \frac{1}{p!(p+2)!}\ \frac{d^p}{dt'^p}(S_k(s,t') + T_k(s,t')) \times \\ & \times \frac{d^p}{dt'^p}(S_{n-1-k}(s,t') + T_{n-1-k}(s,t'))|_{t'=-sx}\ (tsx(1-x))^p \end{aligned}$$

The Scalar theory example

ϕ^4_D $\qquad D = 4, 6, 8, 10 \qquad [\lambda] = 2 - D/2$

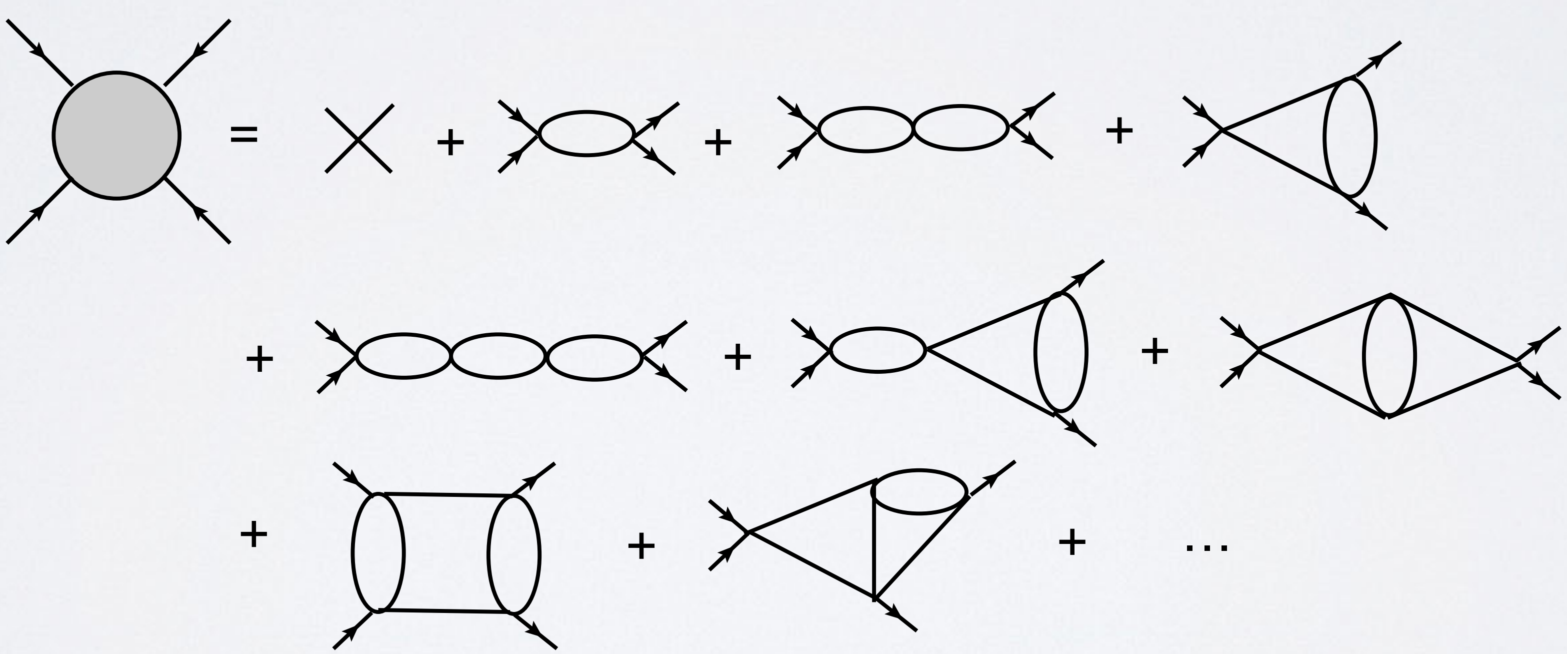
Kazakov,19

2->2 scattering amplitude on shell $\qquad m = 0 \qquad s + t + u = 0$

$$\Gamma_4(s,t,u) = \lambda(1 + \Gamma_s(s,t,u) + \Gamma_t(s,t,u) + \Gamma_u(s,t,u))$$

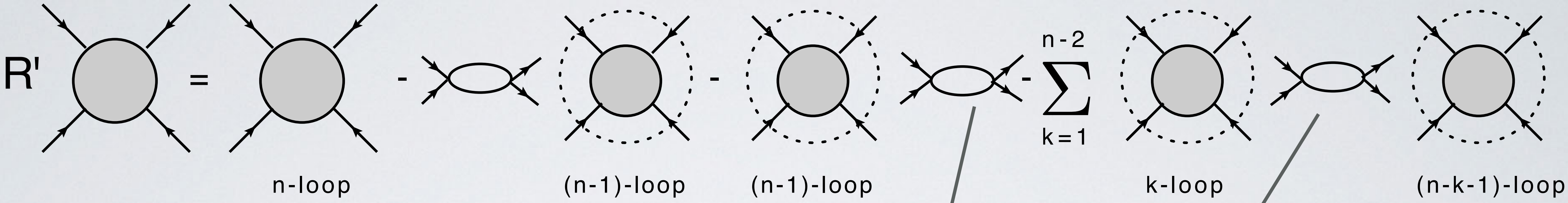
PT:

$$\Gamma_s = \sum_{n=1}^{\infty}(-z)^n S_n, \quad \Gamma_t = \sum_{n=1}^{\infty}(-z)^n T_n, \quad \Gamma_u = \sum_{n=1}^{\infty}(-z)^n U_n, \quad z \equiv \frac{\lambda}{\epsilon}$$

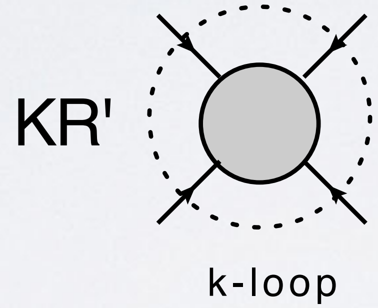


PT expansion (only s-channel is shown)

Recurrence Relations for the Leading Poles



- ⋯ terms with higher loop remaining diagrams



$$= \frac{A_k^{(k)'}}{\epsilon^k} = (-1)^{k+1}\frac{A_k^{(k)}}{\epsilon^k}$$

$$\begin{aligned}
nS_n(s,t,u) \;=\; & \frac{s^{D/2-2}}{\Gamma(D/2-1)}\int_0^1 dx[x(1-x)]^{D/2-2}\left(S_{n-1}(s,t',u')+T_{n-1}(s,t',u')+U_{n-1}(s,t',u')\right) \\
+ \; & \frac{1}{2}\frac{s^{D/2-2}}{\Gamma(D/2-1)}\int_0^1 dx[x(1-x)]^{D/2-2}\sum_{k=1}^{n-2}\sum_{p=0}^{(D/2-2)k}\sum_{l=0}^{p}\frac{1}{p!(p+D/2-2)!}\times \\
\times \; & \frac{d^p}{dt'^l du'^{p-l}}(S_k+T_k+U_k)\frac{d^p}{dt'^l du'^{p-l}}(S_{n-k-1}+T_{n-k-1}+U_{n-k-1})s^p[x(1-x)]^p t^l u^{p-l}
\end{aligned}$$

$t' = -xs, u' = -(1-x)s$

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Differential Equation

Summing up the recurrence relation $\sum_{n=2}^{\infty}(-z)^n$ one gets the diff equation

$$-\frac{d\Gamma_s(s,t,u)}{dz}=\frac{1}{2}\frac{\Gamma(D/2-1)}{\Gamma(D-2)}s^{D/2-2} \qquad \Gamma_s(z=0)=0$$

$$+\frac{s^{D/2-2}}{\Gamma(D/2-1)}\int_0^1 dx[x(1-x)]^{D/2-2}\left[\Gamma_s(s,t',u')+\Gamma_t(s,t',u')+\Gamma_u(s,t',u')\right]\Big|_{\substack{t'=-xs,\\ u'=-(1-x)s}}$$

$$+\quad\frac{1}{2}\frac{s^{D/2-2}}{\Gamma(D/2-1)}\int_0^1 dx[x(1-x)]^{D/2-2}\sum_{p=0}^{\infty}\sum_{l=0}^{p}\frac{1}{p!(p+D/2-2)!}\times$$

$$\times\quad\left(\frac{d^p}{dt'^l du'^{p-l}}(\Gamma_s+\Gamma_t+\Gamma_u)\Big|_{\substack{t'=-xs,\\ u'=-(1-x)s}}\right)^2 s^p[x(1-x)]^p t^l u^{p-l}$$

$$\frac{d\Gamma_s(s,t,u)}{d\log\mu^2}=-\frac{\lambda}{2}\frac{s^{D/2-2}}{\Gamma(D/2-1)}\int_0^1 dx[x(1-x)]^{D/2-2}\sum_{p=0}^{\infty}\sum_{l=0}^{p}\frac{1}{p!(p+D/2-2)!}\times$$

$$\times\left(\frac{d^p\bar{\Gamma}_4(s,t',u')}{dt'^l du'^{p-l}}\Big|_{\substack{t'=-xs,\\ u'=-(1-x)s}}\right)^2 s^p[x(1-x)]^p t^l u^{p-l}$$

$$\Gamma_s(\log\mu^2=0)=0$$

- YM_D   Both cubic and quartic vertices

Equation is more complicated but has the same main features

- Wess-Zumino modern in D=4

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta}\ \bar{\Phi}\Phi + \int d^2\bar{\theta}\ \frac{g}{4!}\bar{\Phi}^4 + \int d^2\theta\ \frac{g}{4!}\Phi^4,$$

$$C =< \Phi\Phi\Phi\Phi >, \quad \bar{C} =< \bar{\Phi}\bar{\Phi}\bar{\Phi}\bar{\Phi} >, \quad M =< \bar{\Phi}\bar{\Phi}\Phi\Phi > . \qquad C = CS + CT + CU, \quad M + MS + MT + MU$$

RG Equations

$$\frac{dCS}{dz} = sg^2 MS \otimes (CS + CT + CU)\,,$$

$$\frac{dMS}{dz} = \frac{1}{2}[sg^2(MS \otimes MS + MT \otimes MT + MU \otimes MU) + \bar{C}S \otimes CS + \bar{C}T \otimes CT + \bar{C}U \otimes CU],$$

$$A(s,t,u) \otimes B(s,t,u) = \int_0^1 dx \sum_{p=0}^{\infty} \sum_{l=0}^{p} \frac{1}{p!p!} \frac{d^p}{dt'^l du'^{p-l}} A(s,t',u') \frac{d^p}{dt'^l du'^{p-l}} B(s,t',u')\Big|_{\substack{t' = -xs, \\ u' = -(1-x)s}} s^p[x(1-x)]^p t$$

Effective Potential in Scalar Theory in D=4

Kazakov, Iakhibbaev, Tolkachev 22

Generating functional for Green functions

$$Z(J) = \int \mathcal{D}\phi \ \exp\left(i \int d^4x \ \mathcal{L}(\phi, d\phi) + J\phi \right)$$

$$W(J) = -i \log Z(J)$$

1PI generating functional

Effective action

$$\Gamma(\phi) = W(J) - \int d^4x J(x)\phi(x)$$

Legendre transformation

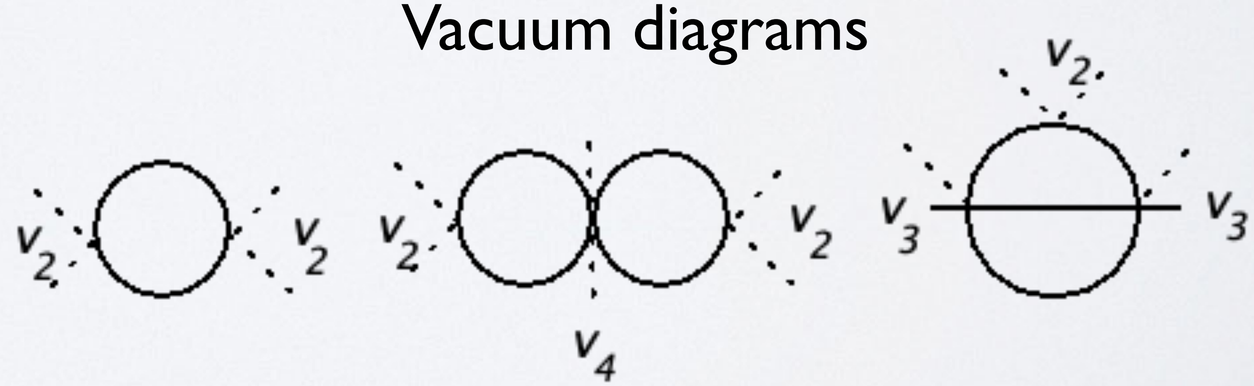
General scalar field theory in D=4

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - gV_0(\phi)$$

Shifted Classical action

$$S[\widehat{\phi} + \phi]$$

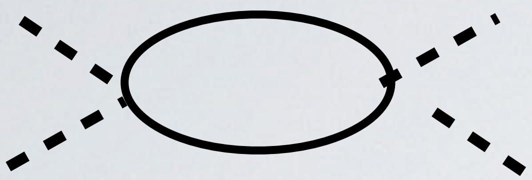
Vacuum diagrams



$$v_2(\phi) \equiv \frac{d^2V_0(\phi)}{d\phi^2}$$

$$v_n \equiv d^nV_0/d\phi^n$$

Divergences and Log ϕ behaviour



$$Diag \sim \frac{1}{\epsilon}(\frac{\mu^2}{m^2})^{\epsilon} \to \frac{1}{\epsilon} - \log\frac{m^2}{\mu^2}, \quad m^2 = g v_2(\phi)$$

The leading divergences ⟷ The leading logs

- In non-renormalizable theories divergences cannot be absorbed into the renormalization of the couplings and fields.
- If they are subtracted some way one is left with <u>infinite</u> arbitrariness.
- Coefficients of the leading divergences (logs) <u>do not depend</u> on this arbitrariness !

The aim is to calculate the leading divergences ~ $\frac{1}{\epsilon^n}$ in n-th order of PT

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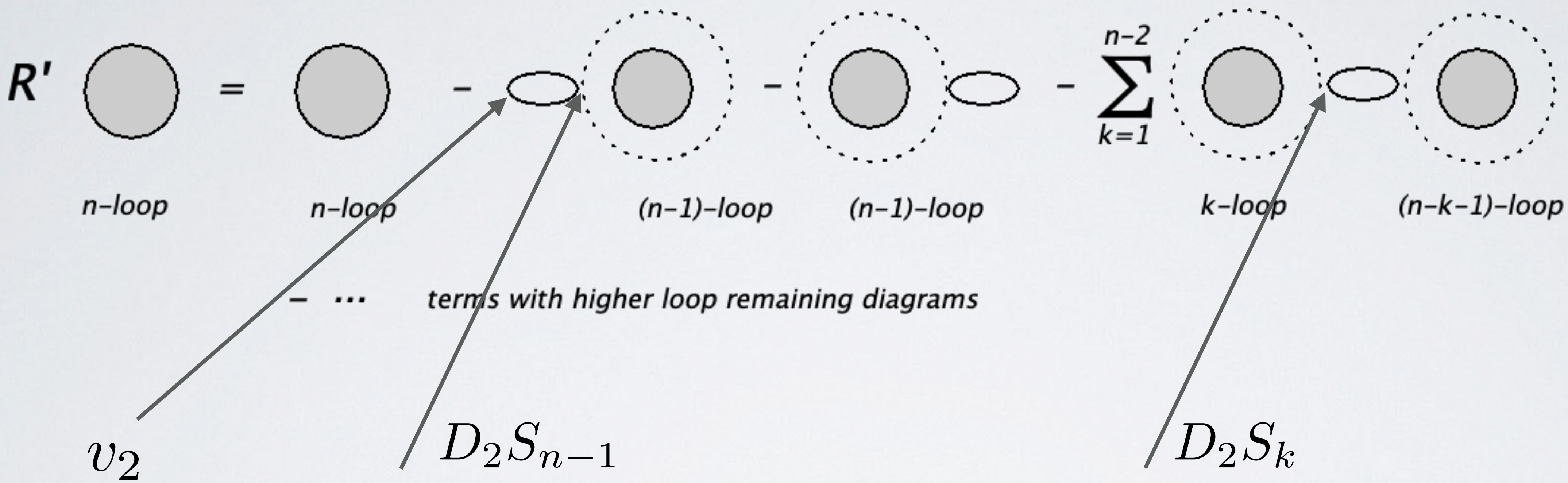
Recurrence relations for the leading poles

Action of R'-operation on divergent diagram



$$nS_n = \frac{1}{2} v_2 D_2 S_{n-1} + \frac{1}{4} \sum_{k=1}^{n-2} D_2 S_k D_2 S_{n-1-k}, \quad n \geq 2 \qquad S_1 = \frac{1}{4} v_2^2$$

$$nS_n = \frac{1}{4} \sum_{k=0}^{n-1} D_2 S_k D_2 S_{n-1-k}, \quad n \geq 1, \quad S_0 = V_0$$

RG pole equation for arbitrary potential

$$\Sigma(z,\phi) = \sum_{n=0}^{\infty} (-z)^n S_n(\phi) \qquad z = \frac{g}{\epsilon}$$

RG pole equation

$$\frac{d\Sigma}{dz} = -\frac{1}{4}(D_2\Sigma)^2 \qquad \Sigma(0,\phi) = V_0(\phi)$$

This a non-linear partial differential equation!

Effective potential

$$V_{eff}(g,\phi) = g\Sigma(z,\phi)|_{z\to -\frac{g}{16\pi^2}\log g v_2/\mu^2}\,. \qquad v_2(\phi) \equiv \frac{d^2 V_0(\phi)}{d\phi^2}$$

Example I: Power like Potential

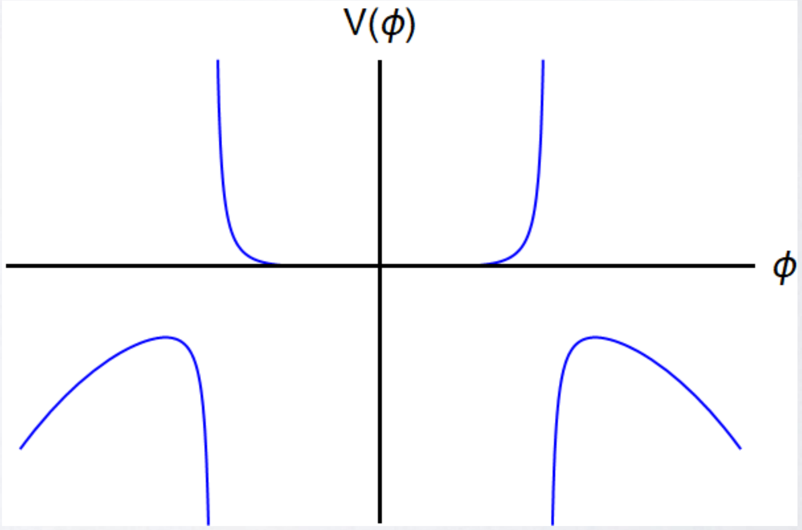
$$gV_0(\phi) = g\frac{\phi^p}{p!} \qquad y = g\phi^{p-4} \qquad \Sigma(z,\phi) = \frac{\phi^p}{p!} f(z\phi^{p-4})$$

$$f'(y) = -\frac{1}{4p!}\left[p(p-1)f(y) + (p-4)(3p-5)yf'(y) + (p-4)^2y^2f''(y)\right]^2$$

$$f(0) = 1, f'(0) = -\frac{1}{4}\frac{p(p-1)}{(p-2)!}$$

p=4

$$f'(y) = -\frac{3}{2}f(y)^2 \qquad f(y) = \frac{1}{1+\frac{3}{2}y}$$



$$V_{eff}(\phi) = \frac{g\phi^4/4!}{1 - \frac{3}{2}\frac{g}{16\pi^2}\log\left(\frac{g\phi^2}{2\mu^2}\right)}.$$

Example I: Power like Potential

p>4

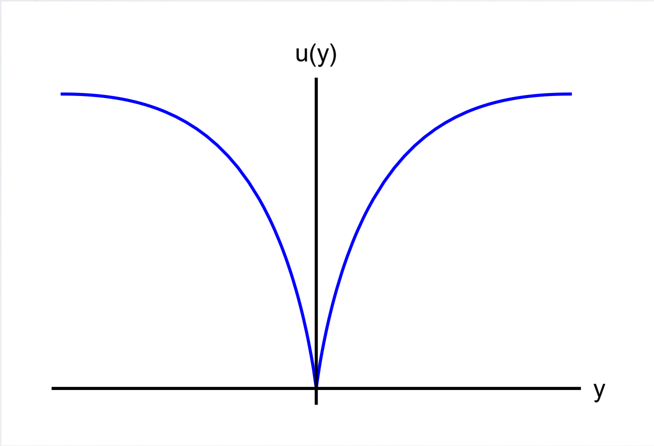
$$gV_0(\phi) = g\frac{\phi^p}{p!}$$

$$f(y) = u(y)/y$$

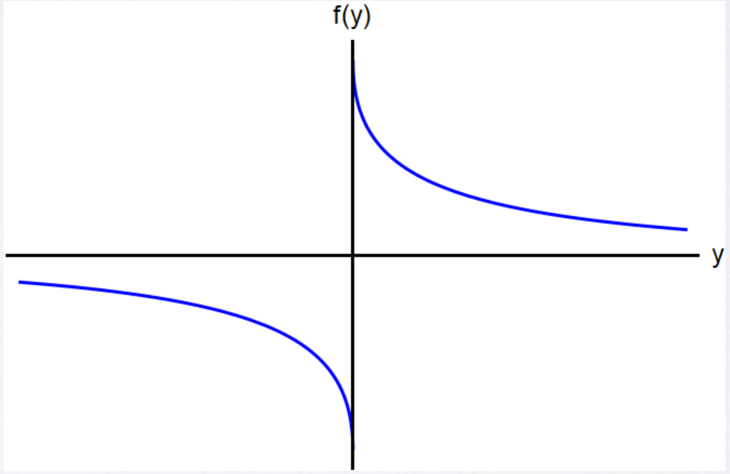
$$yu'(y) - u(y) = -\frac{1}{4p!}[12u(y) + (p-4)(p+3)yu'(y) + (p-4)^2y^2u''(y)]^2$$

$$u(\pm 0) = 0, u'(\pm 0) = \pm 1$$

Discontinuity at y=0



$y = 0$

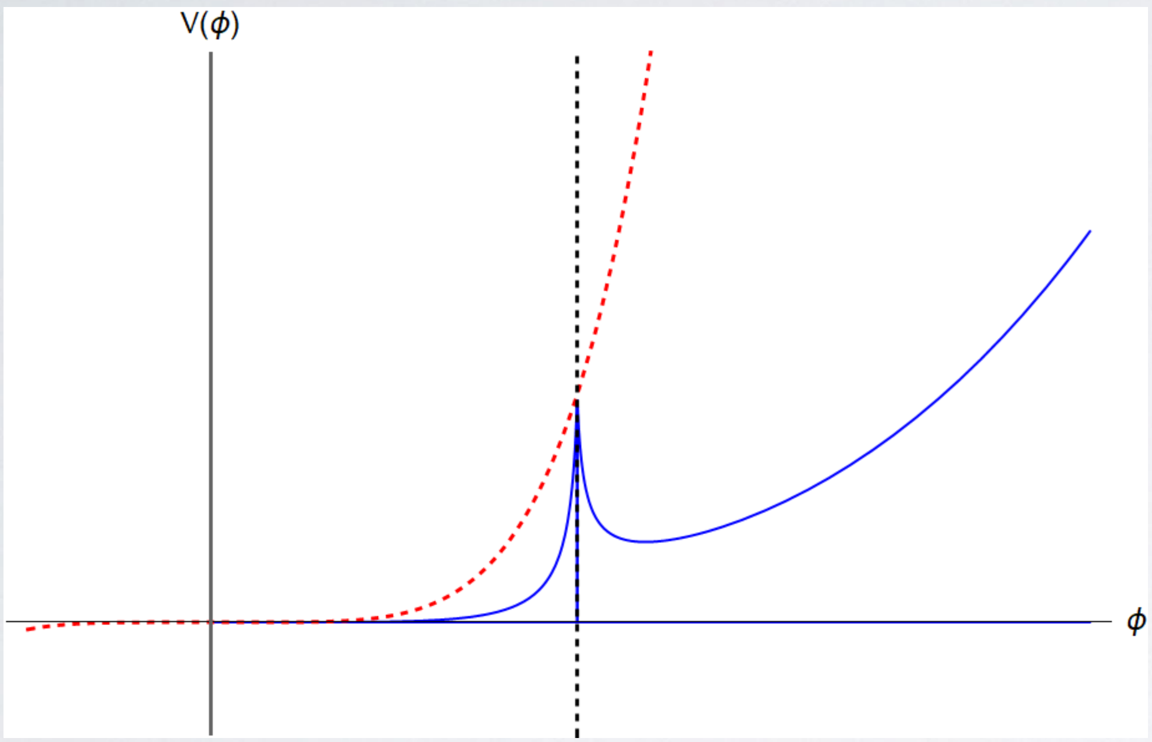


$y = 0$

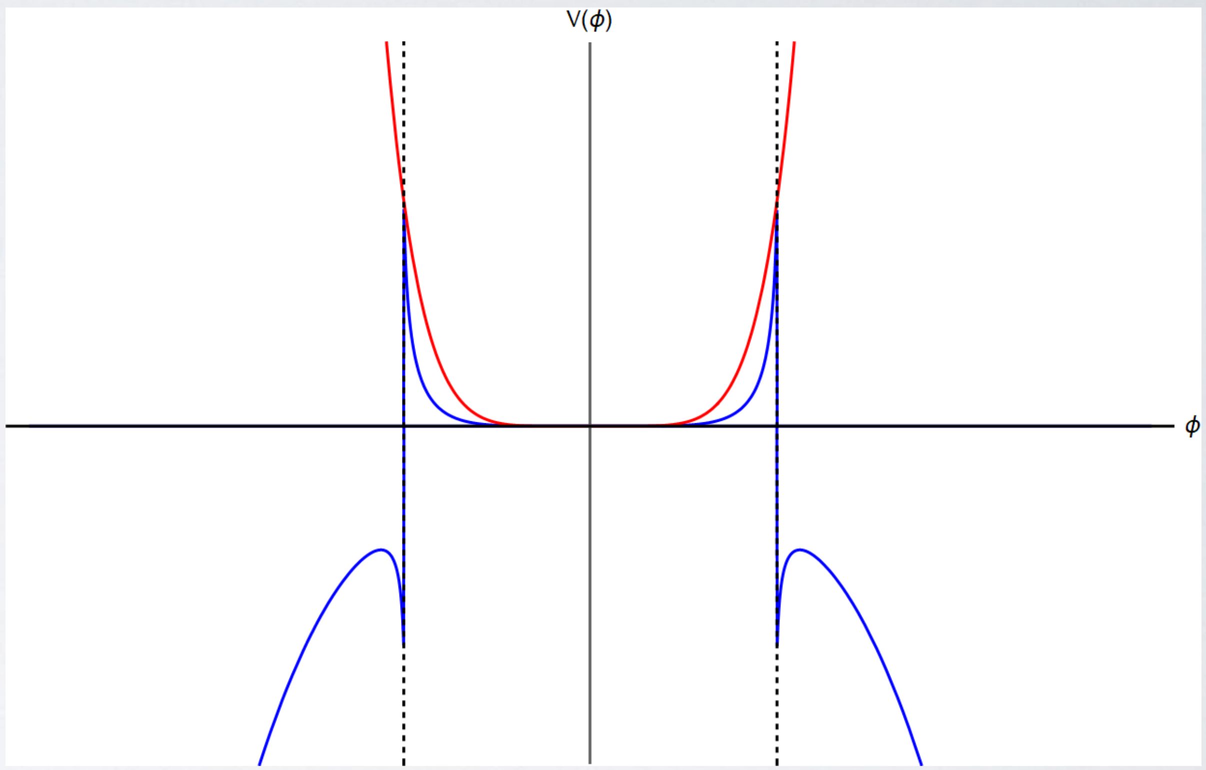
$$y \to -\frac{g}{16\pi^2}\phi^{p-4}\log\frac{g\phi^{p-2}}{\mu^2/(p-2)!}$$

Example I: Power like Potential

p=5



p=6



- Finite gap instead of an infinite barrier as for p=4
- Metastability of the quantum state
- No new minima

Applicability of approximation

Validity of PT

$$\frac{g}{16\pi^2}\phi^{p-4} < 1$$

Validity of LL approximation

$$\log \frac{g\phi^{p-2}}{(p-2)!\mu^2} > 1$$

Possible simultaneously for small coupling g and temporal field ϕ

Singular point $\frac{g\phi^{p-2}}{(p-2)!\mu^2} = 1$ is within validity region

p=6

$$\phi^2 < \frac{16\pi^2}{g} \Longrightarrow \log \frac{g\phi^4}{4!\mu^2} < \log \frac{(16\pi^2)^2}{24g\mu^2}$$ >1

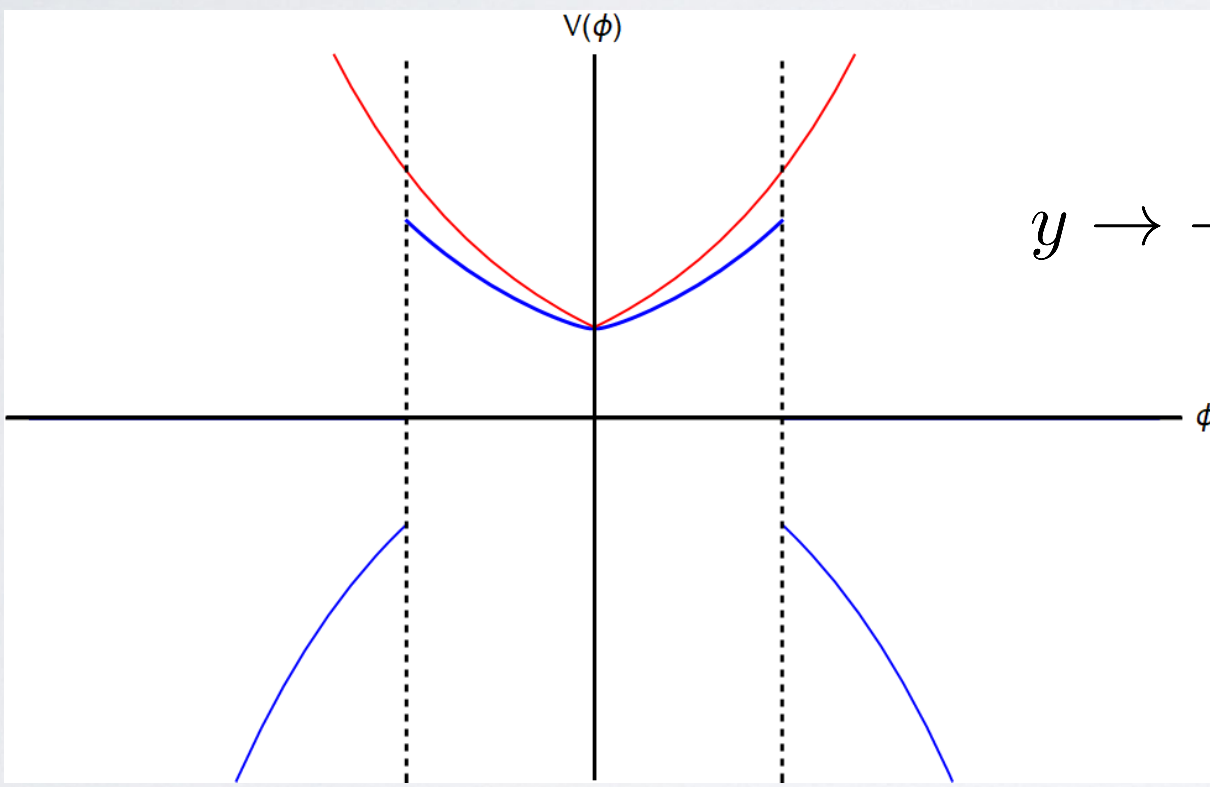
$$g\mu^2 < 1$$

$$y \to -\frac{g}{16\pi^2}\phi^{p-4}\log \frac{g\phi^{p-2}}{\mu^2/(p-2)!}$$

Example II: Exponential Potential

$$gV_0 = ge^{|\phi/m|} \qquad\qquad y = g/m^4 e^{\phi/m}$$

$$f'(y) = -\frac{1}{4}\left(y^2 f''(y) + 3yf'(y) + f(y)\right)^2$$



$$y \to -\frac{g}{16\pi^2 m^4} e^{|\phi/m|} \log[\frac{g}{16\pi^2 m^2 \mu^2} e^{|\phi/m|}]$$

- Finite gap
- Metastability of the quantum state
- No new minima

Applicability $\quad \frac{g}{16\pi^2 m^4} e^{|\phi/m|} < 1 \qquad \log[\frac{g}{m^2\mu^2} e^{|\phi/m|}] > 1$

Conclusion on Effective potential

- The effective potential in the LL approximation obeys the RG master equation which is a partial non-linear differential equation
- In some cases this equation is simplified to the ordinary differential one and can be solved at least numerically. I
- In all the cases that we studied the obtained ordinary differential equations obey the solution with a discontinuity.
- The effective potential has a metastable minima at the origin and no other minima exists.

- The main message is that under certain assumptions while studying the CW mechanism one may not be restricted by the renormalizable potentials but consider much wider possibilities. We provided the method of such analysis.

- This might be useful for cosmological applications where they are usually not limited by renormalizability since gravity makes it non-renormalizable anyway.

General Resume

- **The UV divergences in non-renormalizable theories are local and can be removed by local counter terms like in renormalizable ones**

- **The main difference is that the renormalization constant Z depends on kinematics and acts like an operator rather than simple multiplication**

- **Based on locality of the counter terms due to the Bogoliubov-Parasiuk theorem one can construct the recurrence relations that define all loop divergences starting from one loop**

- **The recurrence relations can be converted into the generalized RG equations just like in renormalizable theories**

- **The RG equations allow one to sum up the leading (subleading, etc) divergences in all loops and define the high-energy/field behaviour**