Covariant dynamics on momentum space

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November 24, 2022

Introduction and context

-QG $\rightarrow$ breakdown of the continuum picture of space-time at the scales near the Planck length
-e.g. string theory, loop qg, noncommutative geometry
-the same effect through modification of large momenta geometry
-first introduced by Snyder in 1947
-significant contribution of Soviet author in the 1960-80ies (Mir-Kasimov, kadyshevsky, et. al.)
-becomes fashionable in the west in the 1990-ies, with string theory and quantum groups

Introduction and context

-math properties, symmetries deformations well established
-formulation of dynamics still open, many different proposals
-not enough constarints: in principle, any Hamiltonian $H[\mathcal{P}]$ which

$$\lim_{\mathcal{P}\to\infty} H[\mathcal{P}] = H_{canonical}$$

will do
-in our approach covariance of the dynamics on momentum space postulated
-leads to the unique formulation of dynamics

Axioms

-list of definitions of and the demands on an energy-momentum background on which a physical field theory is to be defined:

1. Energy-momentum manifold $\mathcal{P}$ is a four-dimensional Riemmanian manifold described (locally) by a metric $g_{\mu\nu}(p)$ or (globally) by a distance function $d(p, q)$, which is a geodesic distance between the points p and q on the manifold.
2. Manifold $\mathcal{P}$ contains a distinguished unique point p_V called the origin, which is identified with the absolute vacuum. The coordinates of the vacuum are in any reference frame (any system of coordinates) $p_V = (0, 0, 0, 0)$. In addition to the origin, manifold $\mathcal{P}$ contains the set of points labeled the points at infinity, which is a set of points with a maximal (finite or infinite) distance from the origin. [1]

[1] We include points at infinity to the manifold from technical reasons (see axiom 5 below). This does not affect the dynamics of finite points in any way.

3. Elementary particles are the energy-momentum excitations with respect to the vacuum. Mass of an elementary particle on the point p on $\mathcal{P}$ is defined as its geodesic distance from the origin,
$$m(p) = d(0, p).$$
4. Manifold $\mathcal{P}$ must be able to accomodate a group of isometries that leave the origin invariant.
5. Manifold $\mathcal{P}$ must be able to accomodate a group of isometries that leave the points at infinity invariant.

-a minimal set of physically reasonable requirements for a momentum backgrounds
-relaxing any of them would lead to theories with much different physical concepts that the ones we are used to.[2]

[2] A less restrictive set of axioms was proposed in [**?**].

-the group in 4 is the usual Lorentz group, and its physical origin lies in the demand that all the observers agree on the value of the mass parameter (postion of the origin)
-the group in 5. is the group of translations or displacements, and it defines the energy-momentum conservation law (the momenta addition) in the interaction of elementary particles

- no two particles p and q are allowed to reduce or increase their mutual distance upon absorbing/emitting particles of the same energy-momentum k, i.e. $d(k \oplus p, k \oplus q) = d(p, q)$
-no finite number of finite displacements of some point that does not belong to the set of points at infinity could displace it to the point at infinity

Consequences:
- manifold $\mathcal{P}$ must necessarily be of a maximally symmetric type, with either positive, negative or vanishing curvature
-under the assumption that the radius of curvature is very large (Planck's mass), for the points p that lie close to the origin the manifold looks flat
-current theory of the Minkowskian $\mathcal{P}$ is to be be understood as a low energy approximation of a more fundamental theory
-an important consistency condition in the construction of a fundamental theory, namely, at any instance one should be able to reproduce the standard theory by setting the curvature to zero

Consequences:
-no explicit mention of any prefered system of coordinates implies that the dynamics must be defined in terms of geometrical invariants only
-otherwise, one would need to add to the set of axioms a rule which defines absolute physical quantities of energy-momentum in terms of some specific coordinates of $\mathcal{P}$
-axiom 3., which was given earlier in [**?**], represents geometrization of the usual dispersion relation $p^2 = m^2$
-it is only for the flat space that the points of the manifold are at the same time vectors (the radii-vectors), and $p^2 = \eta^{\mu\nu} p_\mu p_\nu$ consequently scalars. The latter is then naturally generalized for non-flat spaces to the square of the distance function from the origin.

Nonrelativistic QM: zero curvature, general coordinates
-we start with the canonical Heisenberg algebra,

$$[\hat{x}_i, \hat{\eta}^j] = i\hbar\delta_i^j, \qquad [\hat{x}_i, \hat{x}_j] = [\hat{\eta}^i, \hat{\eta}^j] = 0, \tag{1}$$

and from which the operators in the stationary Schrödinger equation are built,

$$(\hat{K}(\hat{\eta}_i) + \hat{V}(\hat{x}^i))\psi_n = E_n\psi_n, \tag{2}$$

-a spectral problem on an abstract Hilbert space
-one needs to choose the representation

-choosing momentum representation, assuming as usual Euclidean momentum space, with standard momentum coordinates, i.e. those in which the metric is Kroenecker's delta, then the momentum operator is just multiplication, and the position operator is gradient (a generator of infinitesimal translation on the Euclidean momentum space),

$$\hat{\eta}^i \to \eta^i, \quad \hat{x}_i \to -i\hbar \frac{\partial}{\partial \eta^i}. \tag{3}$$

Schrödinger's equation is then

$$\left(\frac{\eta^2}{2m} + V\left(\frac{\partial}{\partial \eta^i} \right) \right) \psi_n(\eta) = E_n \psi_n(\eta), \tag{4}$$

-but what if one chooses a different set of coordinates, i.e. identifies the momentum with p^i, related to the original via

$$\eta^i = h(\alpha^2 p^2) p^i, \tag{5}$$

where α is a constant of dimension inverse momentum and h an arbitrary function? In this case, the momentum metric is

$$g^{ij}(p) = \frac{\partial \eta^i}{\partial p^k}\frac{\partial \eta^j}{\partial p^l}\delta^{kl} = h^2\delta^{ij} + 4h'\alpha^2(h+\alpha^2p^2)p^ip^j \tag{6}$$

and the position operator becomes

$$\hat{x}_i \to -i\hbar \frac{\partial p^j}{\partial \eta^i}\frac{\partial}{\partial p^j} = -i\hbar\left(\frac{1}{h}\frac{\partial}{\partial p^i} - p^i\frac{2\alpha^2 h'}{h(h+2\alpha^2p^2h')}p^j\frac{\partial}{\partial p^j}\right), \tag{7}$$

where $h' = \partial h/\partial(\alpha^2p^2)$, which are just the rules for tensor transformation ((2,0) tensor and a covector) upon the change of coordinates.

-one obtains a deformed Heisenberg algebra

$$[\hat{x}_i, \hat{p}^j] = i\hbar \left(\frac{1}{h}\delta_i^j - \frac{2\alpha^2 h'}{h(h+2\alpha^2 p^2 h')}\delta_k^i p^k p^j \right), \quad [\hat{x}_i, \hat{x}_j] = [\hat{p}^i, \hat{p}^j] = 0 \tag{8}$$

-a special case of this transformation with $h = 1/(1+\alpha^2 p^2)$ was considered in [?], up to the leading order in α^2, where it was shown to lead to the minimal uncertainty in the position $\Delta x \geq \hbar\alpha$.
-how does the Schrödinger equation look like in this new setting? In [?, ?, ?], the kinetic energy operator was given as

$$\hat{K} = \frac{p^2}{2m} \tag{9}$$

- this choice not unique, not covariant

-n order to have a clear geometrical meaning of the kinetic energy operator, we define it as a geodesical distance from the origin divided with $2m$. In other words, the η^2 term in (4) is the square of the length of the shortest path from origin to the point η

$$\eta^2 = \left(\int_0^\eta \sqrt{\delta_{ij} d\eta^i d\eta^j} \right)^2 = d^2(0, \eta), \tag{10}$$

where $d(\eta, \eta')$ is the distance function (geodesic distance between points η and η'), so that for a general variable p defined through (5) the kinetic operator is

$$\hat{K} = \frac{1}{2m} \left(\int_0^p \sqrt{g_{ij} dp^i dp^j} \right)^2 = \frac{d^2(0, p)}{2m} = \frac{h^2 p^2}{2m} \tag{11}$$

-what concerns the potential energy operator, we discuss its geometric form on two specific examples.

a) Example A: harmonic oscillator potential

In the standard case, the isotropic harmonic oscillator (HO) potential is

$$\hat{V}_{HO} = \frac{m\omega^2}{2}\hat{x}^2 = -\frac{m\omega^2\hbar^2}{2}\frac{\partial^2}{\partial\eta^2} \tag{12}$$

-define it so as to give it a definite geometrical meaning

-this is achieved by defining it as the divergence of the gradient, which for the general choice of coordinates on the flat space is

$$\hat{V}_{HO} = -\frac{m\omega^2\hbar^2}{2}\Delta = -\frac{m\omega^2\hbar^2}{2\sqrt{\mathrm{det}g}}\frac{\partial}{\partial p^i}\left(\sqrt{\mathrm{det}g}g^{ij}\frac{\partial}{\partial p^j}\right) \tag{13}$$

-with the choices (11) and (13) for the kinetic and the potential energy operator the Schrödinger equation becomes completeley covariant, from whence it is clear that, by construction, the eigenvalues remain the same independent of the choice of transformation h in (5), so we may as well choose $h = 1$
-the only change between different coordinatizations is in the eigenvectors, which change according to

$$\psi_n(\eta) \rightarrow \psi_n(hp) \tag{14}$$

which is just the rule for the transformation of the scalar function upon the change of coordinates
-the same procedure described here applies to any potential which is given in terms of powers of $\hat{x}^2$.

a) Example B: Coulomb's potential

An example of the potential that can not be expanded in terms of $\hat{x}^2$ is the Coulomb potential. It is an integral operator, which in the standard case looks like[3]

$$\hat{V}_{Coul}\psi(\eta) = \frac{1}{\hbar}\int \frac{\psi(\eta')d^2\eta'}{|\eta - \eta'|}. \tag{15}$$

To make this coordinate invariant (covariant), one generalizes in a natural way

$$d^3\eta \to d\Omega_p = \sqrt{\det g}\, d^3p,\ |\eta - \eta'| \to d(p,p') = \sqrt{(h^2(\alpha^2 p'^2)p'^2 - h^2(\alpha^2 p^2} \tag{16}$$

which in combination with (11) gives a fully covariant Schrödinger equation. The spectrum remains that of the usual hydrogen atom, with only wave functions changing according to (14).

[3]We take $Ze^2/2\pi = 1$.

A note on the physical implications for the structure of configuration space

-it is argued that for some specific choice of transformation function, a "minimal length" emerges in the theory

- the position operator spectrum of a free particle remains continuous, as in the canonical case

-this does not dependend on the choice of the kinetic energy operator, but on the choice of the position operator (7)

$$\hat{x}_i e^{ix_j h p^j/\hbar} = x_i e^{ix_j h p^j/\hbar} \tag{17}$$

-no discretization of space emerges!

A note on the physical implications for the structure of configuration space
-the meaning of the uncertainty relation

$$\Delta x \geq \hbar \alpha, \tag{18}$$

which follows from the $[\hat{x}, \hat{p}]$ for certain choices of the transformation h [**?**], is, from a strictly instrumentalist point of view, that upon simultaneous measurement of position and momentum, the position can be measured only up to a certain precision, regardles of the precision of the momentum measurement. In the same way, if in the canonical case one were to design an experiment to measure simultaneously position and a function $g(\alpha^2 p^2)p^i$ of the momentum, where g is the inverse of the transformation function h, one would again arrive at the conclusion that there is a finite (non-vanishing) uncertainty in the measurment of the position in such experiment, irispective of the precision of the measurment of variable gp^i. This, however, would not imply the existence of a minimal length in the canonical setting.

Nonrelativistic QM, constant curvature (Snyder)

$$(\eta^1)^2 + (\eta^2)^2 + (\eta^3)^2 = \beta^{-2} \tag{19}$$

where β is a constant of the dimension of inverse momentum, and with the origin at the north pole
-expressing the physical momentum coordinates p^i in terms of the embedding space coordinates

$$\eta^i = h(\beta^2 p^2)p^i, \qquad \eta^3 = \sqrt{\beta^{-2} - h^2 p^2}, \tag{20}$$

the momentum metric is given as

$$g^{ij}(p) = h^2\delta^{ij} + \beta^2 \frac{4h'(h - \beta^2 p^2 h') - h^4}{1 - \beta^2 p^2 h^2} p^i p^j \tag{21}$$

-the position operator is

$$\hat{x}_i = \beta \hat{J}_{i3} = -i\hbar\beta \frac{\sqrt{1-\beta^2 p^2 h^2}}{h} \left[\frac{\partial}{\partial p^i} + \frac{2\beta^2 h'}{h - 2\beta^2 p^2 h'} \delta_{ik} p^k p^j \frac{\partial}{\partial p^j} \right], \tag{22}$$

which leads to a deformed Heisenberg algebra

$$[\hat{x}_i, \hat{p^j}] = i\hbar \frac{\sqrt{1-\beta^2 p^2 h^2}}{h} \left(\delta_i^j + \frac{2\beta^2 h'}{h - 2\beta^2 p^2 h'} \delta_{ik} p^k p^j \right), \quad [\hat{x}_i, \hat{x}_j] = \beta^2 \hat{J}_{ij} \tag{23}$$

where $\hat{J}_{12}$ is the angular momentum operator
-all the geometrical considerations from the flat space case apply equally here
-kinetic energy operator is again given by

$$\hat{K} = \frac{d^2(0,p)}{2m} = \frac{1}{2m} \beta^{-2} \arccos^2 \sqrt{1-\beta^2 p^2 h^2} \tag{24}$$

-what concerns the potential, the HO as before -Coloumb case now

$$d^2\eta \to d\Omega_p = \sqrt{\mathsf{det} g}\, d^2 p \tag{25}$$

$$|\eta - \eta'| \to d(p,p') = \beta^{-1} \arccos \left(\beta^2 h_p h_{p'} p^i p'^i - \sqrt{1-\beta^2 h_p^2 p^2}\sqrt{1-\beta^2 h \right. \tag{26}$$

where $h_p = h(\beta^2 p^2)$ and $h_{p'} = h(\beta^2 p'^2)$, final note on the consequence of the generalization of the momentum space geometry on the configurational space. In this case, unlike in the previous one, the spectrum of the position operator is discrete. In one dimension, for instance, we have

$$\hat{x}e^{in\theta} = \hbar\beta n e^{in\theta}, \tag{27}$$

where n must be integer to enable the vanishing of the wave function at $\theta = \pm\pi/2$, which represent the points at infinity. This implies the emergence of the minimal length $\hbar\beta$ in the theory. For a detailed exposition of the spatial lattice in three dimensions, we refer to [**?**]

....if there is any time left....

Scalar field theory

-the stage is set - we consider the geometry

$$\frac{\eta_0^2}{c^2} - \eta_1^2 - \eta_2^2 - \eta_3^2 - \eta_4^2 = \mathcal{P}^2 \tag{28}$$

-the law of the momenta addition, or the energy-momentum conservation law is given by

$$p_\mu \oplus k_\mu = h^{-1}\left(\mathcal{P}^{-2}\mathcal{K}^2\right)\mathcal{K}_\mu, \tag{29}$$

where

$$\mathcal{K}_\mu = h_p p_\mu + h_k k_\mu \left(\sqrt{1 + \mathcal{P}^{-2} h_p^2 p^2} + \frac{\mathcal{P}^{-2} h_p h_k (pk)}{1 + \sqrt{1 + \mathcal{P}^{-2} h_k^2 k^2}} \right). \tag{30}$$

-the enrgy-momentum integral is changed accordingly,

$$\int d^4p \to \int d\Omega_p = \int \sqrt{detg} d^4p \tag{31}$$

-finally, the delta function generalizes to

$$\delta(p - k) \to \delta(p \ominus k), \tag{32}$$

wchich is defined in the distributional sense as

$$\int d\Omega_p f(p)\delta(p \ominus k) = f(k). \tag{33}$$

Feynman rules

$$\mathcal{L} = \phi(p^2 - m^2)\phi - \frac{g}{n!}\phi^n \tag{34}$$

is generalized generalized to

$$\mathcal{L} = \phi(d^2(p,0) - m^2)\phi - \frac{g}{n!}\phi^n \tag{35}$$

in order to accomodate for a generalization in the dispersion relation given in axiom 4 above

-the free Feynman propagator

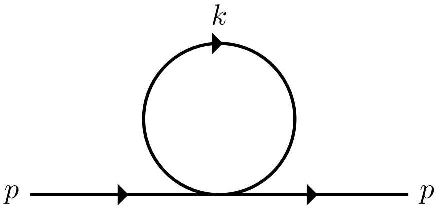
$$D_F(p) = \frac{1}{d^2(p,0) - m^2 \pm i\epsilon}, \tag{36}$$

with the $+i\epsilon$ a choice consistent with causality, as in the canonical case
vertex factors remain the same, with the momentum conservation imposed via generalized delta function,

$$\delta(p_{out} \ominus p_{in}) \tag{37}$$

and the integration over undefined momenta proceeds according to (31).

For the sake of definitness, let us evaluate the leading order correction in the simplest case of a free propagator in ϕ^4 theory



$$i\mathcal{M} = ig \int \frac{d\Omega_k}{(2\pi)^4} \frac{i}{d^2(k,0) - m^2 + i\epsilon}. \tag{38}$$

For this purpose angular coordinates are used, with angles $(\omega, \rho, \theta, \varphi)$. We preform Wick's rotation on the embedding space, as is done in the standard case, to obtain Euclidean integrals. Rotating η_0 from (**??**) for $\pi/2$ in the complex plane, the surface becomes a sphere with imaginary radius, and the distance function depends only on the polar angle ω, $d = i\mathcal{P}\omega$. This gives

$$i\mathcal{M} = -\frac{ig\mathcal{P}^4}{8}\int_0^{\pi/2} d\omega \sin^3\omega \frac{i}{\mathcal{P}^2\omega^2 + m^2} \approx 0.086 g\mathcal{P}^2 \tag{39}$$