SCREW CALCULUS: FROM MACHINERY TO TWISTORS

R. N. Rogalyov

NRC "Kurchatov institute" - IHEP

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Outline

1. Force as sliding vector
2. Forces as screws
3. Operations on screws
4. Angular velocities, applications in machinery etc
5. Historical comment
6. Plücker coordinates
7. Penrose's description of light rays

History of physics: Genesis of some ideas between engineering mechanics and relativistic equations

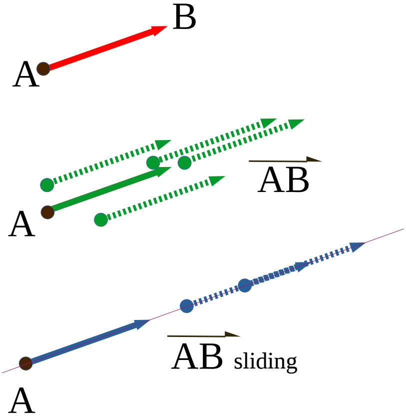
The third Newton's Law:

Forces, emerging from an interaction of two pointlike bodies, are

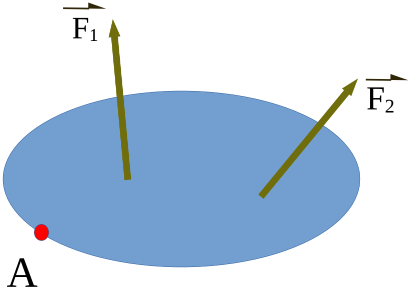
1. equal in magnitude;
2. opposite in direction;
3. act along a common straight line;
4. applied to different bodies;
5. of the same nature (friction, deformation, gravitational electrostatic etc.)

Force is characterized by line of action $\Longrightarrow$
$\vec{F}$ is sliding vector rather than free vector

FREE, BOUND AND SLIDING VECTORS



- A bound vector is a directed line segment.
- The equivalence class of line segments is called free vector if all segments with the same direction and magnitude are equivalent.
- The equivalence class of line segments is called sliding vector if segments of a given line with the same direction and magnitude are equivalent.



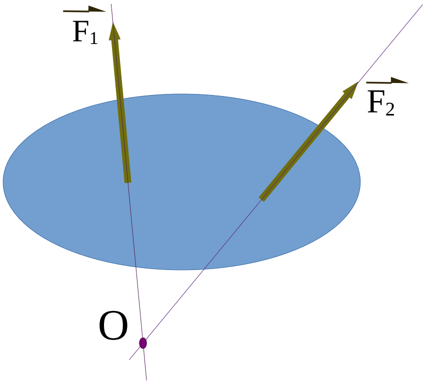
$$\vec{F} = \vec{F}_1 + \vec{F}_2$$

$$\Longrightarrow \quad \mathbf{M} = \mathbf{M}_1 + \mathbf{M}_2$$

Problem:

To find $\vec{F} = \vec{F}_1 + \vec{F}_2$:

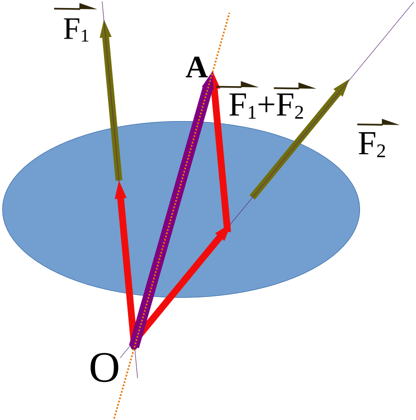
The moment **M** of the force $\vec{F}$ with respect to any point A equals sum of the momenta of the forces $\vec{F}_1$ and $\vec{F}_2$ with respect to this point



If forces are added as sliding vectors, their momenta with respect to any point are added too.

In the planar case, addition of sliding vectors is simple:

1. Find the intersection point of the vector lines, let it be designated by O

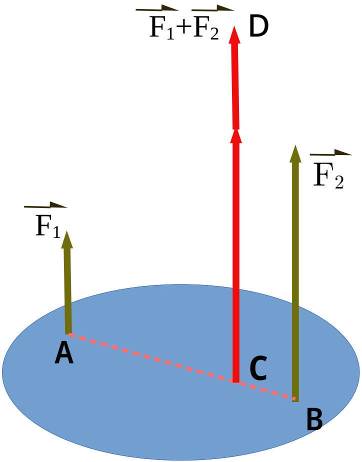


2. Place the initial points of $\vec{F}_1$ and $\vec{F}_2$ to the point O and

3. add them according to the parallelogram rule:

$$\vec{F} = \vec{F}_1 + \vec{F}_2$$

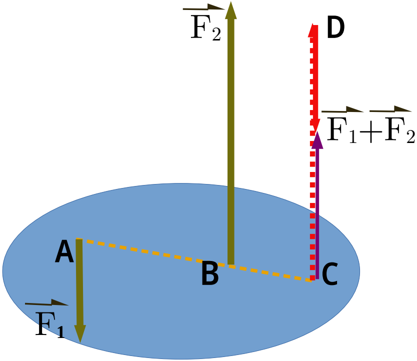
4. the resultant vector $\vec{F}_1 + \vec{F}_2$ is the sliding vector represented by $\vec{OA}$



If sliding vectors $\vec{F}_1$ and $\vec{F}_2$ are collinear, then their sum is represented by $\vec{CD}$ determined as follows:
Let A be the initial point of F_1,
B is the initial point of F_2 and

$$C \in [AB] : \quad \frac{|\vec{F}_1|}{|\vec{F}_2|} = \frac{|CB|}{|AC|}$$

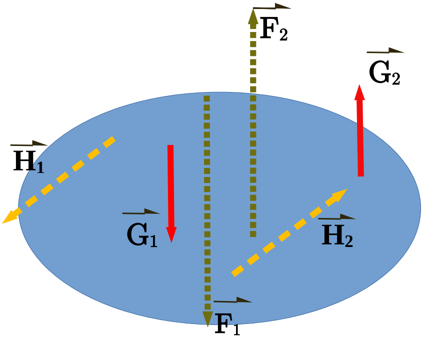
then $\vec{CD} = \vec{F}_1 + \vec{F}_2$



If sliding vectors $\vec{F}_1$ and $\vec{F}_2$ are anticollinear, then their sum is determined as follows:
Let A be the initial point of $\vec{F}_1$,
B be the initial point of $\vec{F}_2$ and

$$C \in [AB] : \quad \frac{|\vec{F}_1|}{|\vec{F}_2|} = \frac{|CB|}{|AC|}$$

then $\vec{F}_1 + \vec{F}_2$ can slide along the line CD



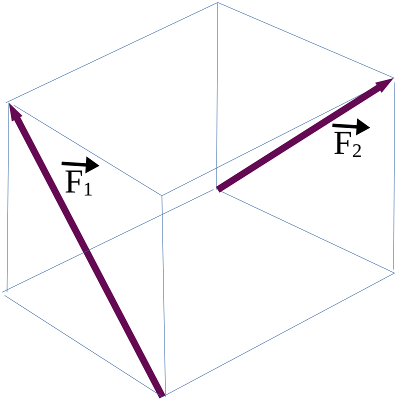
All vectors in this figure are in the same plane

If sliding vectors $\vec{F}_1$ and $\vec{F}_2$ are anticollinear and $|\vec{F}_1| = |\vec{F}_2|$, then their sum is pure torque.

Pairs of forces $(\vec{F}_1, \vec{F}_2)$, $(\vec{H}_1, \vec{H}_2)$, $(\vec{G}_1, \vec{G}_2)$ represent the same force couple (or torque)

$$\vec{\mathbf{M}} \perp \vec{F}_1, \vec{H}_1, \vec{G}_1 \ .$$

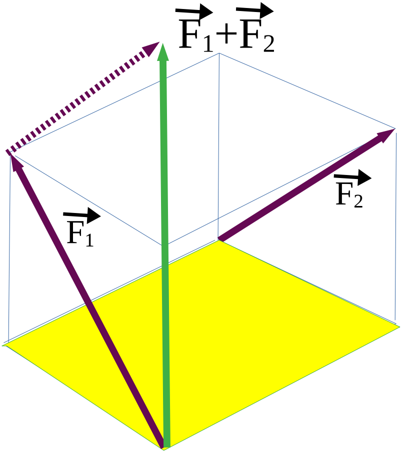
$\vec{\mathbf{M}}$ is free vector orthogonal to the plane of force couple.



How to add forces
acting along skew lines?

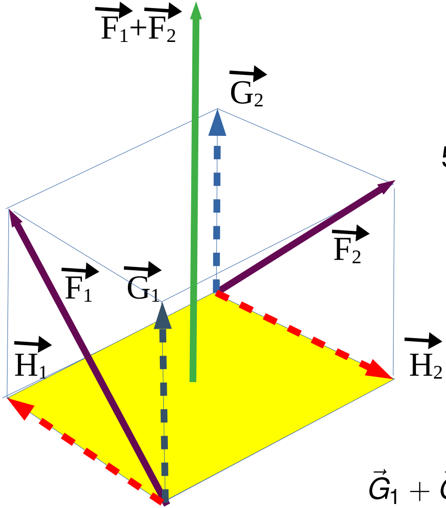
Moment of their sum with respect to an arbitrary point should be equal to the sum of their momenta

Solution:



1. Add $\vec{F}_1$ and $\vec{F}_2$
as free vectors
2. Choose a plane $\perp$ $(\vec{F}_1 + \vec{F}_2)$
3. Move initial points of $\vec{F}_1$ and $\vec{F}_2$ to this plane

4. Decompose both $\vec{F}_1$ and $\vec{F}_2$ into $\|$ and $\perp$ components with respect to the direction of $\vec{F}_1 + \vec{F}_2$
considering $\vec{F}_1$ and $\vec{F}_2$ as sliding vectors



$$\vec{F}_1 = \vec{G}_1 + \vec{H}_1$$
$$\vec{F}_2 = \vec{G}_2 + \vec{H}_2$$

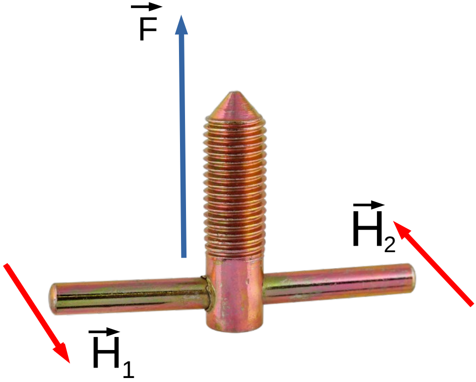
5. Then

$$\vec{F}_1 + \vec{F}_2 =$$
$$= (\vec{G}_1 + \vec{G}_2) + (\vec{H}_1 + \vec{H}_2)$$

$\vec{H}_1 + \vec{H}_2$ is pure torque;

$\vec{G}_1 + \vec{G}_2$ is the force $(\vec{F}_1 + \vec{F}_2)|_{\text{free}}$

Sum of forces is not a force!



Sum of forces is a screw!

Sum of forces can be described by the vector pair $(\vec{F},\vec{M})$, $\vec{F}\|\vec{M}$.

$\vec{F}$ is a sliding vector
$\vec{M}$ is a free (pseudo)vector

Such pair is called a SCREW

$$\vec{F}^{(sliding)}+\vec{G}^{(sliding)}\neq\vec{H}$$

$$(\vec{F},\vec{M})+(\vec{G},\vec{L})=(\vec{H},\vec{J})$$

SCREW_1+SCREW_2 = SCREW

$+$

$=?$

How to add screws?

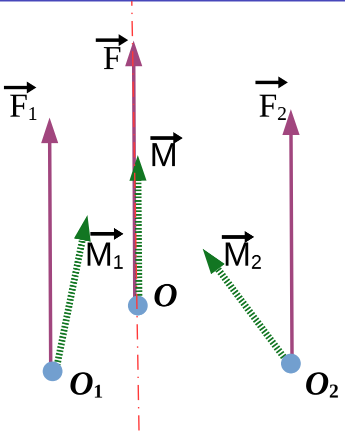
$$(\vec{F}, \vec{M}) + (\vec{G}, \vec{L}) = ?$$

Screws should be placed
on intersecting lines $\Longrightarrow$

Let O be the intersection point.
How to translate a screw to O ?

If $\vec{F} = \vec{OA}$ and $\vec{G} = \vec{OB}$
then

$$\Longrightarrow \; (\vec{F}, \vec{M}) + (\vec{G}, \vec{L}) = (\vec{F} + \vec{G}, \vec{M} + \vec{L})$$



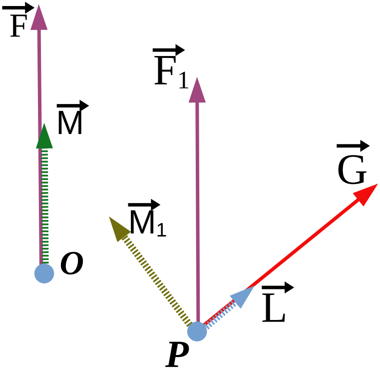
Representation of a sum of forces acting on a rigid body by the pair of vectors $(\vec{F}, \vec{M})$ is not unique:

The pair $(\vec{F}_1, \vec{M}_1)$, where $\vec{M}_1 = \vec{M} + \vec{OO_1} \times \vec{F}|_{free}$, represents the same physics.

Any vector pair $(\vec{F}_2, \vec{M}_2)$ such that $\vec{F}_2 \not\parallel \vec{M}_2$ determines a screw $(\vec{F}, \vec{M})$

such that $\vec{M} \| \vec{F}$. Choose O_2 on the line of F_2, find O: $\vec{O_2O} = \dfrac{\vec{M}_2 \times \vec{F}_2}{|\vec{F}_2|^2}$;

$\vec{M} = \vec{M}_2 + \vec{O_2O} \times \vec{F}_2^{(free)}$; $\vec{F}$ is sliding beginning in O, $\vec{F}^{(free)} = \vec{F}_2^{(free)}$



$(\vec{F}, \vec{M}) + (\vec{G}, \vec{L})$ is determined as follows:

1. fix initial points O and P of these vector pairs
2. transport $(\vec{F}, \vec{M})$ from O to P: one obtains $(\vec{F}_1, \vec{M}_1)$
3. find $(\vec{F}_1, \vec{M}_1) + (\vec{G}, \vec{L})$
4. find the main axis of the obtained vector pair
5. transport the obtained vector pair to its main axis

Each vector pair $(\vec{P}', \vec{J}')$: $\vec{P}' \nparallel \vec{J}'$
can be transported to its main axis $(\vec{P}, \vec{J})$: $\vec{P} \| \vec{J}$

KOMPLEX NUMBERS FOR SCREWS

$z = a + b\varepsilon : \ \varepsilon^2 = 0;$ $\quad$ $\boxed{\imath \to \varepsilon \quad \Longrightarrow \ \text{complex} \to \text{komplex}}$

$$(a + \varepsilon b)(c + \varepsilon d) = ac + \varepsilon(ad + bc)$$

Trigonometric functions:

$$\sin(\phi + \varepsilon d) = \sin\phi + \varepsilon d \cos\phi,$$

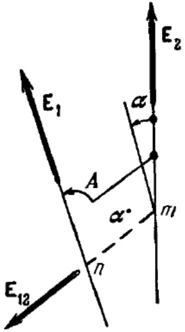
$$\cos(\phi + \varepsilon d) = \cos\phi - \varepsilon d \cos\phi$$

One can use $\vec{F} + \varepsilon\vec{M}$ instead of $(\vec{F}, \vec{M})$

unit screw: $(\vec{e}, \vec{0}) : \vec{e} \cdot \vec{e} = 1$

If $\vec{F} \| \vec{M} \| \vec{e}$ then $\vec{F} + \varepsilon\vec{M} = (|\vec{F}| + \varepsilon|\vec{M}|)(\vec{e}, \vec{0})$;

$$|\vec{F} + \varepsilon\vec{M}| = \sqrt{(\vec{F} + \varepsilon\vec{M}) \cdot (\vec{F} + \varepsilon\vec{M})} = |\vec{F}| + \varepsilon\frac{\vec{F} \cdot \vec{M}}{|\vec{F}|}$$



Complex angle between unit screws $\mathbf{E}_1$ and $\mathbf{E}_2$:

$$A = \alpha + \varepsilon \alpha^0 \; ;$$

$$\mathbf{E}_1 \cdot \mathbf{E}_2 = \cos A$$

Scalar product:

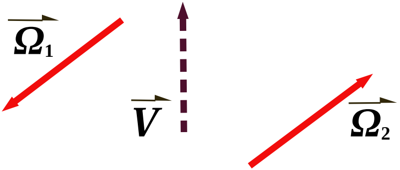
$$\mathbf{F} \cdot \mathbf{G} = (\vec{F} + \varepsilon \vec{M}) \cdot (\vec{G} + \varepsilon \vec{L}) =$$
$$= |\mathbf{F}||\mathbf{G}| \cos(\Phi_{\mathbf{F},\mathbf{G}}) =$$
$$= \vec{F} \cdot \vec{G} + \varepsilon(\vec{M} \cdot \vec{G} + \vec{F} \cdot \vec{L})$$

Cross product:

$$\mathbf{F} \times \mathbf{G} = (\vec{F} + \varepsilon \vec{M}) \times (\vec{G} + \varepsilon \vec{L}) =$$
$$= |\mathbf{F}||\mathbf{G}| \sin(\Phi_{\mathbf{F},\mathbf{G}}) =$$
$$= \vec{F} \times \vec{G} + \varepsilon(\vec{M} \times \vec{G} + \vec{F} \times \vec{L})$$

Projections on axes, mixed product, linear transformations, functions of screw argument, ...

Angular velocity Ω is also sliding vector

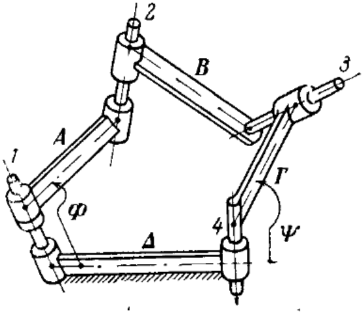


Addition of angular velocities is similar to addition of forces, in particular:

- Couple of forces gives pure torque,
  $\vec{F}_1 + \vec{F}_2 \equiv \vec{M}$.
- Couple of angular velocities is velocity,
  $\vec{\Omega}_1 + \vec{\Omega}_2 \equiv \vec{V}$.

Dynamical screw $(\vec{F}, \vec{M})$ — wrench
Kinematical screw $(\vec{\Omega}, \vec{V})$ — twist

SCREW ALGEBRA IS HELPFUL IN MACHINERY



Revolute-prismatic joints

Komplex angle of rotation of nth joint,

$$\Phi_n = \phi_n + \varepsilon d_n \ ,$$

where ϕ_n is the angle between links attached to the nth axis,

d_n is the distance between joints along the nth axis,

provides the screw of rotation

$$\mathbf{R}_n = \mathbf{E}_n \tan \frac{\Phi_n}{2}$$

$\mathbf{E}_n$ is the unit screw along nth axis.

Each link is characterized by komplex number, for example,

$$A = \alpha_{12} + \varepsilon a_{12} \ ,$$

α_{12} is the angle and a_{12} is the distance between axes 1 and 2

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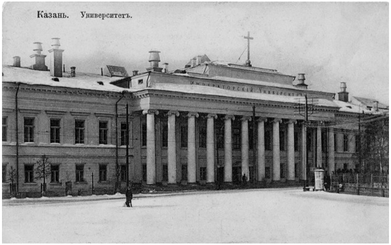
Kharkov University



Kotelnikov Petr Ivanovich
(1809-1879)
worked in mechanics and geometry
Kazan university

Kotelnikov Alexandr Petrovich (1865-1944)

one of the authors of screw theory

Kotelnikov Vladimir Alexandrovich (1908-2005)

- Louis Poinsot (1777-1859) was the inventor of geometrical mechanics, showing how a system of forces acting on a rigid body could be resolved into a single force and a couple.
- William Kingdon Clifford (1845–1879) suggested komplex numbers
- Robert Stawell Ball (1840–1913), Royal Astronomer of Ireland, developed a mathematical framework for kinematics and statics of mechanisms.
- Michel Floréal Chasles (1793-1880) proved that that the most general rigid body displacement is produced by a translation along a line (called its screw axis) followed by a rotation about an axis parallel to that line

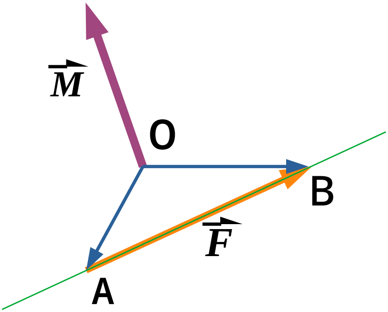
Also contributed: Julius Plücker (1801–1868),
Hermann Grassmann (1809–1877) etc.

Something to be noticed

- An elementary particle is characterized by helicity and momentum.
- Velocity of rigid body is characterized by helicity and momentum or by a kinematical screw.
- Sum of forces acting in a rigid body is also a screw, however
- Force in relativistic case is a fixed vector.
- Uniform motion of a relativistic rigid body can be described by kinematical screw, however
- Angular velocity of a relativistic body cannot be changed if the Born rigidity condition is met and
- Helicity of an elementary particle cannot be changed without interaction.

Generalization of screws to the relativistic case is of interest

Yet another approach to sliding vectors



Let force $\overrightarrow{F}$ be the sliding vector $\overrightarrow{AB}$.

Taking the free vector $\vec{\mathcal{F}}$ associated with $\overrightarrow{F}$ and the free vector

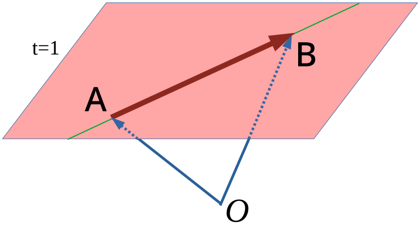
$$\overrightarrow{M} = \overrightarrow{OA} \times \overrightarrow{F} ,$$

we can unambiguously fix the sliding vector $\overrightarrow{F}$. This being so, $\overrightarrow{M} \perp \vec{\mathcal{F}}$

$$(\vec{\mathcal{F}}, \overrightarrow{M}) \equiv \overrightarrow{F}$$

$\overrightarrow{M}$ is the moment of $\overrightarrow{F}$ with respect to the origin **O**

Force as bivector



We embed 3D affine space associated with forces into 4D Euclidean space so that the 3D space is a hyperplane $t = 1$ (shown in rose).

Sliding vector $\overrightarrow{AB}$ (a force) maps into bivector $\overrightarrow{OA} \wedge \overrightarrow{OB}$, where O is the origin in 4D space. Let

$$\begin{pmatrix} \overrightarrow{OA} \\ \overrightarrow{OB} \end{pmatrix} = \begin{pmatrix} 1, & x_A, & y_A, & z_A \\ 1, & x_B, & y_B, & z_B \end{pmatrix}$$

$$\forall\ \overrightarrow{AB} \in \{t = 0\}\ \exists\ \ \overrightarrow{OA} \wedge \overrightarrow{OB}$$

However, if $C, D \in \{t = 0\}$ then NO force corresponds to $\overrightarrow{OC} \wedge \overrightarrow{OD}$

Sum of forces maps into sum of bivectors

$\Longrightarrow$ $\overrightarrow{CD} \in \{t = 0\}$ correponds to pure torque

Plücker coordinates

of the 2-plane of the bivector $a \wedge b$,

$$\begin{array}{cc} \mathbf{a} & \mathbf{b} \\ \end{array}$$

$$\begin{pmatrix} 1 & 1 \\ x_A & x_B \\ y_A & y_B \\ z_A & z_B \end{pmatrix}$$

$$p_{01} = \begin{vmatrix} 1 & 1 \\ x_A & x_B \end{vmatrix} \qquad p_{12} = \begin{vmatrix} x_A & x_B \\ y_A & y_B \end{vmatrix}$$

$$p_{02} = \begin{vmatrix} 1 & 1 \\ y_A & y_B \end{vmatrix} \qquad p_{13} = \begin{vmatrix} x_A & x_B \\ z_A & z_B \end{vmatrix}$$

$$p_{03} = \begin{vmatrix} 1 & 1 \\ z_A & z_B \end{vmatrix} \qquad p_{23} = \begin{vmatrix} y_A & y_B \\ z_A & z_B \end{vmatrix}$$

involve both the the force and its moment with respect to the origin:
$F_i = p_{0i}, M_i = \epsilon_{ijk} p_{jk}$

Antisymmetric tensors and bivectors

Antisymmetric tensor p_{ij} in 4D can be represented by bivector

$$\mathbf{p} = \sum_{0 \le i < j \le 3} V_{ij} e_i \wedge e_j = \mathbf{a} \wedge \mathbf{b}$$

provided that

$$p_{01} p_{23} - p_{02} p_{13} + p_{03} p_{12} = 0 \; .$$

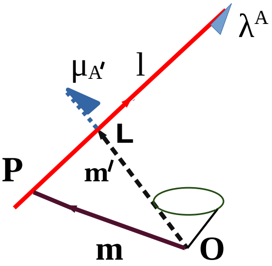
(Plücker condition). In the case under consideration, this condition can be recast in the form $\overrightarrow{M} \cdot \overrightarrow{F} = 0$. In the general case,

$$\mathbf{p} = \mathbf{a} \wedge \mathbf{b} + \mathbf{c} \wedge \mathbf{d}$$

where $\mathbf{a} \wedge \mathbf{b}$ is relared to a force and $\mathbf{c} \wedge \mathbf{d}$ — to pure torque.
2-planes determined by $\mathbf{a} \wedge \mathbf{b}$ and $\mathbf{c} \wedge \mathbf{d}$ form Grassmann manifold.

Forces and torques can be considered on equal foot

Properties of 5-dimensional manifold of light rays in Minkowski space. [R.Penrose (J.Math.Phys. (1967))]



Light ray:

1) Light-like vector ℓ

2) Its moment $m \wedge \ell$ about origin

$m \wedge l$ can be represented in terms of spinors μ, λ

The vector m pointing to the light ray can be chosen light-like: $m \to m'$. Then

$$\ell^{A\dot{A}} = \lambda^A \bar{\lambda}^{\dot{A}}, \quad m'_{A\dot{A}} = \imath \frac{\bar{\mu}_A \mu_{\dot{A}}}{\lambda^B \bar{\mu}_B}$$

provided that $\mu_{\dot{A}} = -\imath \lambda^A m'_{A\dot{A}}$. The pair of spinors $Z_\alpha = (\lambda^A, \mu_{\dot{A}})$

represent the light ray if $\mathrm{Re}(\lambda^A \bar{\mu}_A) = 0$.

Twistors

$$Z_\alpha = (\lambda^A, \mu_{\dot{A}})$$

is called a twistor

The Penrose incidence relation

$$\mu_{\dot{A}} = -\imath \lambda^A x_{A\dot{A}}$$

maps a point x in Minkowski space into 2-plane in twistor space:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -\imath(x_0 + x_3) & -\imath x_1 - x_2 \\ -\imath x_1 + x_2 & -\imath(x_0 - x_3) \end{pmatrix} \begin{pmatrix} \lambda^1 \\ \lambda^2 \end{pmatrix} = \begin{pmatrix} \lambda^1 \\ \lambda^2 \\ \mu_{\dot{1}} \\ \mu_{\dot{2}} \end{pmatrix}$$

CONCLUSIONS

- Representations of Poincare group have an interesting analog in rigid body dynamics.
- The ideas of Grassmann manifolds, odd numbers and the like may stem from applications in machinery.
- Mathematical formalism of screws proved to be fruitful in different fields: from computations in robotics to generation of new ideas in elementary particle physics.

Thank you for attention.