Einstein's 1905 derivation of the mass-energy equivalence: is it valid? Is energy always equal to mass and vice versa?

G. D'Abramo[a,1]

[a] Ministero dell'Istruzione, dell'Università e della Ricerca, 00041, Albano Laziale, RM, Italy

To be checked В 1905 году Эйнштейн дал свой первый вывод эквивалентности массы и энергии, изучая в разных системах отсчета энергетический баланс тела, излучающего электромагнитное излучение, и принимая специальную теорию относительности в качестве предварительного условия. Здесь мы переоцениваем логическую обоснованность подхода Эйнштейна и справедливость одного допущения, имеющего решающее значение для его вывода. Это предположение не имеет ничего общего со специальной теорией относительности. Если мы примем это предположение как верное, сущность эквивалентности массы и энергии (но не ее точную формулу) можно будет понять без специальной теории относительности или какой-либо полноценной физической теории. Однако это предположение не подтверждается с точки зрения физики, и с его использованием у Эйнштейна возник вопрос. Мы также показываем, почему следствие широко распространенной интерпретации $E = mc^2$ (т. е. каждый вид энергии имеет массу) является проблематичным.

In 1905, Einstein gave his first derivation of the mass-energy equivalence by studying, in different reference frames, the energy balance of a body emitting electromagnetic radiation and assuming special relativity as a prerequisite. Here, we reassess the logical soundness of Einstein's approach and the validity of one assumption crucial for his derivation. That assumption has nothing to do with special relativity. If we accept that assumption as valid, the essence of the mass-energy equivalence (but not its exact formula) can be reached without special relativity or any full-fledged physical theory. However, that assumption is unsupported from a physics viewpoint, and with its use, Einstein was begging the question. We also show why a consequence of the widely received interpretation of $E = mc^2$ (i.e., every kind of energy has a mass) is problematic.

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1. Introduction

Mass-energy equivalence, known in the form of the celebrated equation $E = mc^2$, was first derived by Einstein in a three-page paper published at the end of 1905 [1]. In the literature, many different proofs of that equivalence followed. Over the years, Einstein himself presented some 18 proofs of it, the last one in 1946 [2]. Some of the most recent proofs do not require the machinery of special relativity (e.g., [3,4]). Together with Einstein's first derivation, which indeed requires special relativity, these proofs are generally considered to be valid as a special or limiting case. Today, the most

[1] E-mail: germano.dabramo@gmail.com

general and rigorous proofs[1] of the equivalence are considered to be those by Von Laue [5] and Klein [6]. In the literature, also criticism of Einstein's first derivation appeared soon. For instance, in 1907, Planck contended that Einstein's derivation was valid "under the assumption permissible only as a first approximation that the total energy of a body is composed additively of its kinetic energy and its energy referred to a system in which it is at rest" [7, 9]. Further criticism was raised by Ives in 1952 [8] and Jammer in 1961 [9]: they concluded that Einstein's derivation was but the result of a *petitio principii.*

In the following sections, we briefly review Einstein's 1905 derivation, go a bit into the most representative criticism advanced in the past and describe the results of our analysis of the logical structure, soundness, and validity of Einstein's approach and assumptions. We prove that it is possible to heuristically derive a general mass-energy relationship by following the logic behind Einstein's original derivation without the need of special relativity or any other full-fledged physical theory (with the sole exception of the principle of conservation of energy). This general mass-energy relationship, *per se*, does not express mass-energy equivalence. However, it conveys mass-energy equivalence when we apply it to the case of a body emitting energy in the form of electromagnetic waves. That reveals that special relativity plays no fundamental role in the validity of the equivalence. In fact, we show that, in Einstein's first derivation, the real core of the mass-energy equivalence result is the assumption that the difference between the total energy of an emitting body in its rest frame and its total energy measured in a uniformly moving frame is always equal to the kinetic energy of the body relative to the moving frame. That assumption has nothing to do with special relativity, and Einstein unwarrantedly took it as valid for electromagnetic emissions. We show why that assumption cannot hold in general. Finally, we prove why a consequence of the widely accepted interpretation of $E = mc^2$ (i.e., that every kind of energy has a mass) is problematic.

2. Einstein's 1905 derivation

In his first derivation, Einstein considered a body, at rest in an inertial frame S, that emits electromagnetic radiation of total energy L in two equal but oppositely directed amounts. He then considered the same emission process as seen from another inertial frame S', that of an observer moving in uniform parallel translation relative to the system S and having its origin of coordinates in motion along the x-axis with velocity v (Fig. 1).

Therefore, let there be a stationary body in the system S, and let its total energy referred to the system S be E_0. Let the total energy of the body relative to the system S' moving as above with velocity v, be H_0.

[1] These derivations dig deeply into the mathematics of special relativity, especially in its subsequent tensor formulation, and leave this author with the strong impression of being a lofty mathematical play for its own sake with little to no connection with the physics behind the problem.

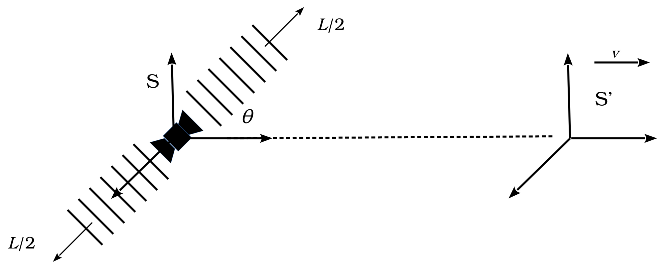

Fig. 1. Sketch of the light emission process described in Einstein's 1905 paper [1]

Let this body send out, in a direction making an angle θ with the x-axis, plane waves of light of energy $\frac{1}{2}$L measured relatively to S, and simultaneously an equal quantity of plane waves in the opposite direction, for a total emitted energy equal to L (see Fig. 1). During and after the emission, the body remains at rest in S. Einstein then showed that if the radiation is measured in S', its total energy L$'$ is equal to

$$\mathrm{L}' = \frac{\mathrm{L}}{\sqrt{1-\frac{v^2}{c^2}}}, \tag{1}$$

where c is the velocity of light. Equation (1) was established by using the law for the transformation of the energy of a plane light wave from one inertial frame to the other. It was derived in the 1905 paper on special relativity [10].

If we call the energy of the body after the emission of the plane light waves E_1 and H_1, measured relative to the system S and S', respectively, then, by making use of equation (1), we have

$$\begin{aligned} \mathrm{E}_0 &= \mathrm{E}_1 + \mathrm{L}, \\ \mathrm{H}_0 &= \mathrm{H}_1 + \frac{\mathrm{L}}{\sqrt{1-\frac{v^2}{c^2}}}. \end{aligned} \tag{2}$$

By subtraction, Einstein obtained the following relation

$$(\mathrm{H}_0 - \mathrm{E}_0) - (\mathrm{H}_1 - \mathrm{E}_1) = L\left\{\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1\right\}. \tag{3}$$

According to Einstein's reasoning, the two differences of the form H−E in equation (3) have the following simple physical meaning. H and E are the energy values of the same body referred to two reference frames in uniform relative motion, the body being at rest in S. Thus, the difference H−E can differ from the kinetic energy K of the body, relative to the system S', only by an additive constant C, which depends on the choice of the arbitrary additive constants of the energies H and E and does not change during the emission of light. Without loss of generality, this constant can be taken equal to zero, and the difference can be written simply as H−E=K.

From equation (3) we have,

$$K_0 - K_1 = L\left\{\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1\right\}. \tag{4}$$

What equation (4) tells us is that the kinetic energy of the body relative to S' diminishes as a result of the emission of the plane light waves, and the amount of diminution is independent of the properties of the body. Moreover, like the kinetic energy, it depends upon the relative velocity v. Neglecting quantities of the fourth and higher orders in v/c, equation (4) becomes,

$$K_0 - K_1 = \frac{1}{2}\left[\frac{L}{c^2}\right] v^2. \tag{5}$$

From equation (5), Einstein's mass-energy equivalence directly follows: if a body gives off the energy L (in the form of radiation), its mass diminishes by $\frac{L}{c^2}$.

3. Received interpretation of mass-energy equivalence

Although doubts and confusion remain about its correct interpretation, the widely accepted meaning of mass-energy equivalence can be summarized as follows:

1) A quantity of mass m can transform completely into energy E (mainly radiation) with $E = mc^2$;

2) An amount of energy E (every kind of energy) possesses an inertial/gravitational mass $m = E/c^2$. Therefore, if a body acquires an energy E, no matter what kind of energy it is, its mass increases by the amount $\Delta m = E/c^2$. To prove that, it is enough to apply Einstein's approach to the absorption of radiation instead of emission.

There is plenty of experimental proof of point 1 (for instance, see [11]). However, as we shall show later, we cannot be that sure of point 2.

4. Einstein's crucial assumption

The crucial assumption in Einstein's derivation is that the difference between the total energy H of the body in the reference frame S' and the total energy E of the body in the reference frame S is taken to be equal to the kinetic energy K of the body in frame S', i.e., H − E = K.

This assumption catalyzed the attention of most of the following literature on Einstein's mass-energy equivalence derivation and generated some controversy regarding its validity. For instance, Ives and Jammer generically asserted that the equality H − E = K is unwarranted. They claimed that, to have the precise equality H − E = K, one needs to use the formula for the relativistic kinetic energy derived by Einstein in his special relativity paper [10] with the further assumption $\frac{\mathrm{L}}{(m_0-m_1)c^2}=1$; namely, the assumption of the very same mass-energy equivalence [8, 9].

A careful discussion of this criticism is given in the paper by Stachel and Torretti [12]. In that paper, the authors also provide an allegedly formal and non-circular derivation of $\mathrm{H}-\mathrm{E}=\mathrm{K}$ from first principles. Their approach is presented as general. According to them, the equality $\mathrm{H}-\mathrm{E}=\mathrm{K}$ is logically sound and always true.

5. Intermezzo: the general mass-energy relationship

Suppose for a moment that Stachel and Torretti are right. Namely, let us accept that the relation $\mathrm{H}-\mathrm{E}=\mathrm{K}$ is always true.

Then, we shall show that it is possible to heuristically derive a general mass-energy relationship by applying $\mathrm{H}-\mathrm{E}=\mathrm{K}$ and the core logic behind Einstein's original derivation without special relativity or any other physical theory (with the sole exception of the principle of energy conservation) [13].

Consider a body, stationary in an inertial frame S, that emits a total amount of energy equal to L. The energy can be emitted in any imaginable form but, like in Einstein's derivation, always in equal amounts in opposite directions to maintain the symmetry of emission. That symmetry intuitively ensures the motionlessness of the body during the process. The equation of the energy balance in S is then $\mathrm{E}_0=\mathrm{E}_1+\mathrm{L}$, where E_0 and E_1 are the total energies of the body, respectively, before and after the emission referred to the system S.

Now, consider the same emission process seen from an inertial reference frame S' moving in uniform parallel translation relative to the system S and having its origin of coordinates in motion along the x-axis with velocity v. Then, it is reasonable to expect that the observed *total* emitted energy L' is different from L and greater than that. That is what we heuristically expect in real life simply because the observer moves relative to the emitter, and some energy adds to what he sees because of that motion. The equation of the energy balance in S' is then $\mathrm{H}_0=\mathrm{H}_1+\mathrm{L}'$, where H_0 and H_1 are the total energies of the body before and after the emission referred to the system S'. So far, we have used only the principle of energy conservation in any inertial frame.

Without loss of generality, we can write the mathematical relation that connects L' and L as follows

$$\mathrm{L}'=\mathcal{F}(\mathrm{L},v), \tag{6}$$

where $\mathcal{F}$ is a suitable mathematical function. Since the origin of reference frame S' moves along the x-axis, the functional dependence of equation (6) on velocity is by construction on scalar velocity v. Moreover, L' must necessarily be directly proportional to L. If the body emits energy equal to 2L, the energy observed in S' must be equal to $2\mathrm{L}'$. Thus, equation (6) becomes

$$\mathrm{L}' = \mathrm{L}f(v), \tag{7}$$

where $f(v)$ is a dimensionless function of the relative velocity.

In order to determine the approximate (low-velocity) mathematical form of the function $f(v)$, consider the Maclaurin expansion of $f(v)$ up to $O(v^3)$

$$f(v) = \alpha + \beta v + \delta v^2 + O(v^3), \tag{8}$$

where α, β, and δ are numerical coefficients.

Since $f(0) = 1$, α must be equal to 1. Furthermore, we must have that $f(-v) = f(v)$ since, for symmetry reasons, the *overall* energy L' observed by an observer in S' does not depend upon the arbitrary direction (towards the positive or the negative x-axis) of the velocity of S'. Consequently, the function $f(v)$ must be even, and β and all other terms with odd powers of v must be equal to zero. Therefore,

$$f(v) = 1 + \delta v^2 + O(v^4), \tag{9}$$

with constant δ having the physical units of an inverse square velocity. This velocity is the 'characteristic velocity' of the peculiar emission process.

Thus, we arrive at

$$\mathrm{L}' = \mathrm{L}(1 + \delta v^2 + O(v^4)). \tag{10}$$

Within the sphere of validity of the previous assumptions, equation (10) is very general and can be applied to all kinds of energy emission mechanisms. As a matter of fact, its derivation is completely independent of the specific energy emission process at play, except for the numerical value of the constant δ.

Now, the energy balance equations become

$$\begin{aligned} \mathrm{E}_0 &= \mathrm{E}_1 + \mathrm{L}, \\ \mathrm{H}_0 &= \mathrm{H}_1 + \mathrm{L}(1 + \delta v^2 + O(v^4)). \end{aligned} \tag{11}$$

Like in Einstein's 1905 derivation, we subtract the first equation from the second

$$\mathrm{H}_0 - \mathrm{E}_0 - (\mathrm{H}_1 - \mathrm{E}_1) = \mathrm{L}(\delta v^2 + O(v^4)), \tag{12}$$

and, with Einstein's assumption $\mathrm{H} - \mathrm{E} = \mathrm{K}$, we obtain

$$\mathrm{K}_0 - \mathrm{K}_1 = L(\delta v^2 + O(v^4)). \tag{13}$$

If, like in [12], we define the inertial mass for a body in translational motion (in keeping with the requirement that special relativistic dynamics have a Newtonian limit as $v \to 0$) by

$$m = \lim_{v \to 0} \frac{K}{v^2/2}, \tag{14}$$

then, from equation (13), it follows

$$-\Delta m = m_0 - m_1 = \lim_{v\to 0}\frac{(K_0 - K_1)}{v^2/2} = \lim_{v\to 0}\frac{L(\delta v^2 + O(v^4))}{v^2/2} = 2\delta \mathrm{L}. \qquad (15)$$

In short,

$$-\Delta m = 2\delta \mathrm{L}, \qquad (16)$$

and this is an exact, not an approximate, result. If a body gives off the energy L, its mass diminishes by 2δL.

Notice that equation (16) is not a mass-energy equivalence *per se*. If we apply equation (16) to a body releasing two projectiles of mass m in opposite directions with non-relativistic velocity v_0 (relative to the parent body), then it is possible to prove that $\delta = 1/v_0^2$. Since $\mathrm{L} = 2\cdot\frac{1}{2}mv_0^2$ (the emitted energy, in this case, is only kinetic), then $-\Delta m = 2m$. Namely, equation (16) simply gives the change of mass of the parent body due to the loss of two projectiles of mass m each. Thus, in this case, equation (16) does not express any mass-energy equivalence.

On the other hand, if we apply equation (16) to the emission of energy in the form of electromagnetic waves, we do have mass-energy equivalence: *radiation energy comes from mass reduction, and thus mass transforms into radiation energy*. Therefore, special relativity is not essential for the derivation of mass-energy equivalence: special relativity comes into play only in the numerical value of the constant δ. The constant δ has the physical units of an inverse square velocity, and in the case of electromagnetic phenomena, it must be heuristically proportional to $1/c^2$. In the case of Einstein's original derivation, we have that $\delta = 1/2c^2$.

In order to emphasize the implications of the derived general mass-energy relationship, consider that even within Maxwell's theory of light (and thus, without special relativity), one could have already come to mass-energy equivalence, albeit in the different form $E = \frac{1}{2}mc^2$. Within Maxwell's theory of light (pre-Lorentz, classical ether theory), it is possible to prove that $\delta = 1/c^2$ [13]. For the derivation of that result, we imagined the emission of two opposite electromagnetic plane waves and calculated the whole energy in the frame of the emitter by using the formula $u = \epsilon_0 E^2$ for the energy density associated with an electromagnetic wave (E is the amplitude of the electric field of the wave). Then, by using the transformation law for the electric field $\mathbf{E}' = \frac{\mathbf{F}}{q} = \mathbf{E} + \mathbf{v}\times\mathbf{B}$, obtained via the Lorentz force felt by a test charge q, we derived the total energy L$'$ of the waves measured in the frame S' moving with velocity v: $\mathrm{L}' = \mathrm{L}\left(1 + \frac{v^2}{c^2}\right)$. This formula has the same mathematical form of equation (10) with $\delta = 1/c^2$. We believe that if we adopt a more accurate, but still non-relativistic, transformation law for the electric field $\mathbf{E}$, we could derive the correct mass-energy formula even within Maxwell's theory of light. A possible approach to deriving that transformation law could be by using the 19th-century Liénard-Wiechert potentials to calculate the electric field generated by an oscillating point charge, first in a frame at rest with the oscillating center of the charge (frame S), and then in a frame moving with velocity v relative to the oscillating center of the charge (frame S'). In both cases, the field is calculated close to the oscillating center of the charge. Then, by comparing the two expressions for the field in S and S', we should obtain the actual transformation law for the electric field without special relativity.

6. Back to Einstein's crucial assumption

We now dwell on the general validity of Einstein's crucial assumption $\mathrm{H} - \mathrm{E} = \mathrm{K}$. At first sight, that equality may appear indisputable and even necessary. Upon deeper scrutiny, however, it is not. For the sake of derivation, let us rewrite it as the following sum

$$\mathrm{H} = \mathrm{E} + \mathrm{K}. \tag{17}$$

In the logic of the derivation, H and E are the total energies of the emitter with respect to reference frames S' and S, respectively, from which the energies of the electromagnetic emission L' and L come. With such premises, if we assume the validity of the strict equality in equation (17), we are implicitly assuming that solely the motion of the body, in the form of its kinetic energy K with respect to S', does contribute to the increase in the 'internal reservoir' of energy from which the electromagnetic emission originates in S'. This is not problematic in the classical case of emission of energy in mechanical form (e.g., emission of non-relativistic mass projectiles), where both L' and L are kinetic energies. In such cases, the increase in kinetic energy from L to L' directly derives from the increase in kinetic energy of the whole system from E to H.

On the other hand, with electromagnetic emissions, or any non-mechanical process, the consequences implied by assumption (17) are not unproblematic. In these cases, it is much like to take for granted that the kinetic energy of an electric battery in motion with respect to us can contribute for us to the increase in the electrical energy content of the battery. Or that the kinetic energy of a car in motion with respect to us can contribute for us to the increase in the energy content of the gasoline, and, ultimately, to the increase in the gasoline mass.

In [12], Stachel and Torretti formally derived $\mathrm{H} - \mathrm{E} = \mathrm{K}$ from first principles and presented their derivation as general and non-circular. However, their treatment does not explicitly address and answer the above questions more related to the physics behind the energy balance.

We are not claiming that the kinetic energy of the body does not contribute to the total energy of the body relative to S'. We are maintaining that if H is the total energy of the system from which the radiation energy L' comes in S', then H cannot be straightly equal to $\mathrm{E} + \mathrm{K}$. Suppose E is the internal metabolic energy of an arm wrestler seated before his contender, and K is his kinetic energy relative to a moving observer (the arm wrestler is at rest, and the observer is moving). In the observer rest frame, the arm wrestler is not more powerful simply because his total energy is now $E+K$. The fact that both E and K have the same physical units (joule) does not automatically imply that one kind of energy can flow into the other.

In the end, $\mathrm{H} = \mathrm{E} + \mathrm{K}$ is not true for all physical processes. It is not true *a priori*, and taking it as true for all physical processes is an *unjustified* and *arbitrary* step.

The crucial point, however, is that the assumption of the validity of $\mathrm{H} = \mathrm{E} + \mathrm{K}$ corresponds in Einstein's first derivation to pretty much assuming the equivalence between mass and energy from the outset. Special relativity has little to do with mass-energy equivalence. As a matter of fact, according to equation (9) and what has been said in section 5, it always heuristically holds that $\mathrm{L}' > \mathrm{L}$ and

$$|\Delta \mathrm{H}| = \mathrm{L}' > \mathrm{L} = |\Delta \mathrm{E}|, \tag{18}$$

and thus $|\Delta \mathrm{H}| > |\Delta \mathrm{E}|$. Therefore, by plugging this last inequality into equation (17), the variation of kinetic energy must necessarily be different from zero during the emission process, $\Delta \mathrm{K} \neq 0$. Then, since the velocity of the emitter does not change after the emission, $\Delta \mathrm{K} \neq 0$ implies a variation of the mass of the emitter, $\Delta m \neq 0$. Therefore, the unproven assumption that equation (17) also holds for the electromagnetic emission process necessarily leads to Einstein's mass-energy equivalence. In this very peculiar and involved sense, Einstein's original derivation is the result of a *petitio principii*. Our observation here is somewhat reminiscent of Planck's criticism cited in the introduction [7].

7. Indiscriminate energy-to-mass conversion is problematic

If and when mass transforms into energy, like, for instance, in nuclear reactions (fission, fusion, annihilation, etc.), mass and energy are indeed related according to the equation $E = mc^2$. However, the widely held belief that every form of energy (heat, electrical potential energy, etc.; see point 2 in section 3) does have an inertial/gravitational mass is problematic. Consider the following thought experiment by Misner, Thorne, and Wheeler [14] on the gravitational frequency shift derived from the conservation of energy:

> That a photon must be affected by a gravitational field Einstein (1911) showed from the law of conservation of energy, applied in the context of Newtonian gravitation theory. Let a particle of rest mass m start from rest in a gravitational field g at point $\mathcal{A}$ and fall freely for a distance h to point $\mathcal{B}$. It gains kinetic energy mgh. Its total energy, including rest mass, becomes
>
> $$m + mgh.$$

> Now, let the particle undergo an annihilation at $\mathcal{B}$, converting its total rest mass plus kinetic energy into a photon of the same energy. Let this photon travel upward in the gravitational field to $\mathcal{A}$. If it does not interact with gravity, it will have its original energy on arrival at $\mathcal{A}$. At this point it could be converted by a suitable apparatus into another particle of rest mass m (which could then repeat the whole process) plus an excess energy mgh that costs nothing to produce. To avoid this contradiction of the principal [*sic*] of conservation of energy, which can also be stated in purely classical terms, Einstein saw that the photon must suffer a red shift. [*The speed of light is set as* $c = 1$]

Unfortunately, Misner, Thorne, and Wheeler's argument is fallacious. If a particle of rest mass m starts from rest in a gravitational field g at point $\mathcal{A}$ and falls freely for a distance h to point $\mathcal{B}$, that particle posses also an energy equal to mgh already at point $\mathcal{A}$. It is called gravitational potential energy. Therefore, owing to the complete mass-energy equivalence, at point $\mathcal{A}$, that particle already has a mass equal to $m + mgh$[1]. Now, if the energy of the photon generated in the particle annihilation and traveling upward does not have its original value on arrival at $\mathcal{A}$ (i.e., $m + mgh$), the mass of the particle created by the suitable apparatus at the end of the process would not have the same mass as the original particle (again, $m+mgh$), and the total energy/mass would not be conserved. When Misner, Thorne, and Wheeler say that the particle "gains kinetic energy mgh" on arrival at point $\mathcal{B}$, and "its total energy, including rest mass, becomes $m + mgh$", they seem to forget that the particle already has a (gravitational potential) energy mgh just before starting to fall. That is demanded by the principle of conservation of energy.

Therefore, the widely held assumption that every energy has a mass is at odds with the conservation of energy and the existence of the gravitational frequency shift taken together. The thought experiment by Misner, Thorne and Wheeler clearly sets the first assumption against the simultaneous validity of the other two.

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[1] It can be shown that, in a uniform gravitational field g, the mass m_h of a particle at height h is $m_h = me^{\frac{gh}{c^2}}$, where m is the proper mass at a height taken as zero. The total energy E_{tot}, proper mass plus gravitational potential energy, at height h is given by $E_{tot} = mc^2 e^{\frac{gh}{c^2}}$. For small distance h, we have $m_h \simeq m + \frac{mgh}{c^2}$ and $E_{tot} \simeq mc^2 + mgh$. By assuming $c = 1$, like in [14], we have that the mass and the total energy of the particle at height h (point $\mathcal{A}$ in [14]) are $m + mgh$.

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