

Representation of the RG-invariant quantities in perturbative QCD through powers of the conformal anomaly

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В данной работе рассматривается возможность представления рядов теории возмущений ренорминвариантных величин в КХД в виде разложения по степеням конформной аномалии $\beta(\alpha_s)/\alpha_s$ в $\overline{\text{MS}}$ -схеме. Отмечается, что такое представление возможно для функции Адлера процесса e^+e^- аннигиляции в адроны и коэффициентной функции правила сумм Бьёркена глубоконеупругого рассеяния заряженных лептонов на поляризованных нуклонах, обе из которых связаны соотношением Крютера–Бродхарста–Катаева. Мы исследуем обсуждаемое разложение для кулоновской части статического потенциала взаимодействия тяжёлой кварк-антикварковой пары, его соотношения с величиной, определяемой угловой аномальной размерностью, а также коэффициентной функции правила сумм Бьёркена нейтрино-нуклонного рассеяния. В заключении мы также приводим результаты формального применения данного подхода к неренорминвариантному отношению полюсных и бегущих в $\overline{\text{MS}}$ -схеме масс тяжёлых кварков в КХД и сравниваем их с уже известными в литературе. Обсуждаются аргументы в пользу справедливости рассматриваемого представления для всех упомянутых ренорминвариантных величин, вычисленных по теории возмущений.

In this work we consider the possibility of representing the perturbative series for renormalization group invariant quantities in QCD in the form of their decomposition in powers of the conformal anomaly $\beta(\alpha_s)/\alpha_s$ in the $\overline{\text{MS}}$ -scheme. We remind that such expansion is possible for the Adler function of the process of e^+e^- annihilation into hadrons and the coefficient function of the Bjorken polarized sum rule for the deep-inelastic electron-nucleon scattering, which are both related by the Crewther-Broadhurst-Kataev relation. In addition, we study the discussed decomposition for the static quark-antiquark Coulomb-like potential, its relation with the quantity defined by the cusp anomalous dimension and the coefficient function of the Bjorken unpolarized sum rule of neutrino-nucleon scattering. In conclusion we also present the formal results of applying this approach to the non-renormalization invariant ratio between the pole and $\overline{\text{MS}}$ -scheme running mass of heavy quark in QCD and compare them with those already known in the literature. The arguments in favor of the validity of the considered representation in powers of $\beta(\alpha_s)/\alpha_s$ for all mentioned renorm-invariant perturbative quantities are discussed.

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1. Preliminaries

A long time ago, R.J. Crewther proved in his work [1] that in the Born approximation the product of the expressions for the Adler function of the process $e^+e^- \rightarrow \gamma^* \rightarrow \text{hadrons}$ annihilation in massless limit and the coefficient function of the Bjorken polarized sum rule of deep-inelastic lepton-nucleon scattering is proportional to amplitude of $\pi^0 \rightarrow \gamma\gamma$ decay, which contains the number of quark colors N_c . In the normalization of the Adler function by unity, this product also becomes equal to unity. However, when higher orders of the perturbation theory (PT) in powers of the running strong coupling constant $\alpha_s(Q^2)$, defined in the Euclidean region in the MS-like renormalization scheme, are taken into account, the Crewther relation is modified. Indeed, starting from $\mathcal{O}(\alpha_s^2)$ level in the product of the flavor non-singlet (NS) perturbative contributions to the Adler function $D_{NS}(Q^2)$ and the coefficient function $C_{NS}(Q^2)$ of the Bjorken polarized sum rule instead of unity an extra term appears. It reflects the violation of the symmetry with respect to conformal transformations of the triangle Green function, composed from axial-vector-vector (AVV) quark currents [2, 3]. Application of these

transformations and the operator product expansion approach to the AVV diagram enabled Crewther to obtain his original result [1].

In the $\overline{\text{MS}}$ -scheme (and generally speaking in the class of the gauge-invariant MS-like schemes) at $Q^2 = \mu^2$ this extra conformal symmetry breaking term is proportional to the renormalization group (RG) β -function:

$$\beta^{(M)}(a_s(\mu^2)) = \mu^2 \frac{\partial a_s(\mu^2)}{\partial \mu^2} = - \sum_{i=0}^{M-1} \beta_i a_s^{i+2}(\mu^2). \quad (1)$$

The solution of this RG equation allows to obtain the expression for $a_s(Q^2) = \alpha_s(Q^2)/\pi$ in terms of the renormalized strong coupling $a_s(\mu^2) = \alpha_s(\mu^2)/\pi$ depending on the scale μ . The index $M \geq 1$ in Eq.(1) is introduced for convenience and throughout this work it will stand for the order of approximation in which the concrete quantity will be considered. Note here that one-, two- and three-loop coefficients of β -function, used by us in this manuscript, were calculated in [4, 5], [6–8], [9, 10] correspondingly.

For the first time the modification form of the Crewther relation in the perturbative QCD was discovered at $M = 3$ in [11] and confirmed at $M = 4$ later on in [12]. At least at this level its generalization, called now by the Crewther-Broadhurst-Kataev (CBK) relation, reads

$$D_{NS}^{(M)}(a_s)C_{NS}^{(M)}(a_s) = 1 + \left(\frac{\beta^{(M-1)}(a_s)}{a_s} \right) K^{(M-1)}(a_s) + \mathcal{O}(a_s^{M+1}), \quad (2)$$

where $a_s = a_s(\mu^2 = Q^2)$ and therefore all RG logarithms are nullified.

Thus, in higher orders of PT the radiative corrections violates the simple Crewther identity and leads to the factorization of the conformal anomaly term $\beta(a_s)/a_s$ in Eq.(2). Vice versa, in the conformal-invariant limit [2, 3] (and in the scale-invariant one in particular) the β -function is vanished and the simple Crewther identity is restored.

The conformal symmetry breaking (CSB) term in the right side of Eq.(2) contains the flavor n_f -dependent coefficient function $K^{(M-1)}(a_s)$, which is the $(M - 1)$ -degree polynomial in a_s . The explicit expressions for this function were obtained at $M = 3$ in [11] and at $M = 4$ in [12].

One should mention that as was recently shown in Ref. [13] the analog of the CBK-relation also holds in the extended QCD model with arbitrary number of fermion representations at $M = 4$. This fact demonstrates that the factorization of the β -function in Eq.(2) is not accidental at least at this level.

At the next stage, it was demonstrated in [14] at $M = 3$ and in [15] at $M = 4$ that in the case of a generic simple gauge group the CSB term can be represented in the expansion form in powers of the conformal anomaly. Then, the CBK relation is rewritten as:

$$D_{NS}^{(M)}(a_s)C_{NS}^{(M)}(a_s) = 1 + \sum_{n=1}^{M-1} \left(\frac{\beta^{(M-n)}(a_s)}{a_s} \right)^n P_n^{(M-n)}(a_s) + \mathcal{O}(a_s^{M+1}). \quad (3)$$

At the 4-loop level the polynomials $P_n^{(r)}(a_s)$ in Eq.(3) read:

$$P_n^{(r)}(a_s) = \sum_{k=1}^r P_{n,k}^{(r)} a_s^k = \sum_{p=1}^{4-n} a_s^p \sum_{k=1}^p P_n^{(r)}[k, p-k] C_F^k C_A^{p-k}, \quad (4)$$

where $r = 4 - n$ when $M = 4$ and C_F , C_A are the quadratic Casimir operator in the fundamental and adjoint representation of the gauge group respectively. Coefficients $P_{n,k}^{(r)}$ are defined unambiguously [15]. An important point here is that at this level of PT *all dependence on n_f* in r.h.s. of Eq.(3) is contained in the coefficients of the β -function. Thus, in contrast to the coefficients of the polynomial $K^{(M-1)}(a_s)$ in Eq.(2), the terms of $P_n^{(r)}(a_s)$ in Eq.(3) are independent on the number of quark flavors.

The double sum representation (3), (4) for the CSB term motivated the authors of the work [16] to consider the similar one for the NS contributions to the Adler function and the coefficient function of the Bjorken polarized sum rule at least at the analytically available $\mathcal{O}(a_s^4)$ level. For instance, according to this paper at $M = 4$ the PT expression for the NS Adler function, calculated in the $\overline{\text{MS}}$ -scheme for the non-abelian gauge theory with a simple compact Lie group, may be presented in the following form:

$$D_{NS}^{(M)}(a_s) = 1 + D_0^{(M)}(a_s) + \sum_{n=1}^{M-1} \left(\frac{\beta^{(M-n)}(a_s)}{a_s} \right)^n D_n^{(M-n)}(a_s), \quad (5)$$

where polynomials $D_n^{(r)}(a_s)$ in the coupling constant a_s are:

$$D_n^{(r)}(a_s) = \sum_{k=1}^r D_{n,k}^{(r)} a_s^k. \quad (6)$$

In a more detailed form Eq.(5) may be written down as:

$$\begin{aligned} D_{NS}^{(M=4)}(a_s) = & 1 + D_{0,1}^{(4)} a_s + \left(D_{0,2}^{(4)} - \beta_0 D_{1,1}^{(3)} \right) a_s^2 + \left(D_{0,3}^{(4)} - \beta_0 D_{1,2}^{(3)} \right. \\ & \left. - \beta_1 D_{1,1}^{(3)} + \beta_0^2 D_{2,1}^{(2)} \right) a_s^3 + \left(D_{0,4}^{(4)} - \beta_0 D_{1,3}^{(3)} - \beta_1 D_{1,2}^{(3)} - \beta_2 D_{1,1}^{(3)} \right. \\ & \left. + \beta_0^2 D_{2,2}^{(2)} + 2\beta_0 \beta_1 D_{2,1}^{(2)} - \beta_0^3 D_{3,1}^{(1)} \right) a_s^4, \end{aligned} \quad (7)$$

where at the fixed number of n and k , $D_{n,k}^{(r)} \equiv D_{n,k}^{(r+1)} \equiv D_{n,k}^{(r+2)} \equiv \dots$, e.g. the terms $D_{1,2}^{(2)} \equiv D_{1,2}^{(3)}$.

The coefficients $D_{n,k}^{(r)}$ including in Eq.(7) are determined by an unambiguous way as solutions of a system of linear equations, analogous to those presented in [15]. Herewith, the full dependence on n_f (except for the light-by-light scattering effects – see explanations below) is absorbed into the coefficients of β -function and their combinations (7). For the first time at the four-loop level the values of $D_{n,k}^{(r)}$ were obtained in Ref. [16]. It is clear that

the representation (5) and its counterpart for the coefficient function of the Bjorken polarized sum rule is a *sufficient condition* for the validity of the generalized Crewther relation in the form of Eq.(3).

Note here two interesting facts. First of them is that the representation (5), whose detailed form at the $\mathcal{O}(a_s^4)$ level is given in Eq.(7), is in full agreement with its $\{\beta\}$ -expansion structure proposed in Ref. [17] much earlier than this expansion in powers of conformal anomaly. Indeed, accordingly to this work the coefficients d_M ($1 \leq M \leq 4$) of the NS Adler function

$$D_{NS}^{(M=4)}(a_s) = 1 + d_1 a_s + d_2 a_s^2 + d_3 a_s^3 + d_4 a_s^4 \quad (8)$$

may be decomposed into the coefficients of the β -function and have the following form:

$$d_1 = d_1[0], \quad (9a)$$

$$d_2 = \beta_0 d_2[1] + d_2[0], \quad (9b)$$

$$d_3 = \beta_0^2 d_3[2] + \beta_1 d_3[0, 1] + \beta_0 d_3[1] + d_3[0], \quad (9c)$$

$$d_4 = \beta_0^3 d_4[3] + \beta_1 \beta_0 d_4[1, 1] + \beta_2 d_4[0, 0, 1] + \beta_0^2 d_4[2] + \beta_1 d_4[0, 1] + \beta_0 d_4[1] + d_4[0]. \quad (9d)$$

Relations (9a-9d) are in full compliance with representation (7).

The second interesting fact lies in observation that certain β -dependent terms $D_{n,k}^{(r)}$ in (7) *coincide* in different orders of PT. For instance, $D_{1,1}^{(3)}$ term is the same for $\beta_0 a_s^2$, $\beta_1 a_s^3$ and $\beta_2 a_s^4$ contributions. Matching this fact with results of $\{\beta\}$ -expansion (9b-9d), one can obtain the following equalities:

$$D_{1,1}^{(1)} = -d_2[1] = -d_3[0, 1] = -d_4[0, 0, 1], \quad (10a)$$

$$D_{1,2}^{(2)} = -d_3[1] = -d_4[0, 1], \quad (10b)$$

$$D_{2,1}^{(1)} = d_3[2] = d_4[1, 1]/2. \quad (10c)$$

The $\{\beta\}$ -expansion terms (9a-9d) were obtained in [16] (see also [18], where the misprints made in [16] were found and several $\{\beta\}$ -expanded coefficients of d_5 were fixed as well). As a result, the application of Eqs.(10a-10c) enables to determine all terms of $\{\beta\}$ -expansion, except for ones proportional to β_0^0 and higher powers of β_0 in concrete order of PT. Indeed, all other terms proportional to higher coefficients of β -function will be uniquely fixed from $\{\beta\}$ -expansion pattern obtained at the previous order of PT. In the case, when the coefficient d_M is known explicitly, the terms proportional to powers of β_0 are defined unambiguously.

It is worth pointing out that the same relations between some $\{\beta\}$ -dependent terms were also observed and used e.g. in Refs. [19–21] in the process of application of the \mathcal{R}_δ -scheme (or class of the MS-like schemes) to the study of the $\{\beta\}$ -pattern of the perturbative series for different observables in QCD. As was explained there, the application of the \mathcal{R}_δ -scheme exposes a special degeneracy of the $\{\beta\}$ -expanded terms in various orders of

PT. Now we see that this degeneracy may be clearly understood from the decomposition of the PT series for physical quantities in powers of $\beta(\alpha_s)/\alpha_s$.

Let us raise the following issue: is this decomposition valid for arbitrary observables that are not related by the CBK relation? In order to shed light on this question, we will consider the higher order PT corrections to various physical quantities and apply to them the $\{\beta\}$ -expansion technique in powers of $\beta(\alpha_s)/\alpha_s$ in next sections. Further, we will compare some of our results with those already known in literature, obtained by independent methods.

2. The case of the static potential

2.1. The expansion in powers of the conformal anomaly

Let us start our consideration from the case of the static interaction potential of the heavy quark and antiquark in QCD in color-singlet state. The perturbative Coulomb-like part of this quantity is defined through the vacuum expectation value of the gauge-invariant Wilson loop $W[C]$:

$$V_{Q\bar{Q}}(r) = - \lim_{T \rightarrow \infty} \frac{1}{iT} \log W[C] = \int \frac{d^3 \vec{q}}{(2\pi)^3} e^{i\vec{q}\vec{r}} V_{Q\bar{Q}}(\vec{q}^2), \quad (11)$$

$$\text{where } W[C] = \frac{\langle 0 | \text{Tr } \hat{P} \exp \left(ig \oint_C dx^\mu A_\mu^a t^a \right) | 0 \rangle}{\langle 0 | \text{Tr } 1 | 0 \rangle} \quad (12)$$

and \vec{q}^2 is the square of the Euclidean three-dimensional momentum, C is a closed rectangular contour, T and r are the time and three-dimensional spatial variables, \hat{P} is the path-ordering operator, A_μ^a is a gluon field and t^a are the generators of the Lie algebra of the generic simple gauge group in fundamental representation. In our study we are primarily interested in the case of the $SU(N_c)$ gauge color group.

The limit $T \rightarrow \infty$ formally leads to $q_0 \rightarrow 0$ and the square of the Euclidean four-dimensional transferred momentum $Q^2 \rightarrow \vec{q}^2$. Thus, technically we carry out the transition from the Euclidean four-dimensional space to its three-dimensional subspace. The $\overline{\text{MS}}$ -scheme β -function is also considered in this subspace and the coupling $a_s(\vec{q}^2)$ starts to depend on \vec{q}^2 .

The perturbative expression for the Fourier image $V_{Q\bar{Q}}(\vec{q}^2)$ is known in analytic form in the $\overline{\text{MS}}$ renormalization scheme at the three-loop level (see [22, 23], [24, 25] and [26, 27] respectively). It is written as:

$$V_{Q\bar{Q}}(\vec{q}^2) = - \frac{4\pi C_F \alpha_s(\vec{q}^2)}{\vec{q}^2} \left[1 + a_1 a_s(\vec{q}^2) + a_2 a_s^2(\vec{q}^2) + \left(a_3 + \frac{\pi^2 C_A^3 L}{8} \right) a_s^3(\vec{q}^2) + \mathcal{O}(a_s^4) \right]. \quad (13)$$

Here $L = \log(\mu^2/\vec{q}^2)$ and the term $\pi^2 C_A^3 L/8$ [28] arises due to the infrared divergences.

Let us introduce the next notation:

$$\mathcal{V}_{Q\bar{Q}}(a_s(\bar{q}^2)) = 1 + a_1 a_s(\bar{q}^2) + a_2 a_s^2(\bar{q}^2) + a_3 a_s^3(\bar{q}^2) + \mathcal{O}(a_s^4). \quad (14)$$

Following our idea of decomposition of the perturbative coefficients into powers of the conformal anomaly, which is valid for the ones of the Adler function and the coefficient function of the Bjorken polarized sum rule at least at $M = 4$ in QCD, we can write down the expression for $\mathcal{V}_{Q\bar{Q}}(a_s(\bar{q}^2 = \mu^2))$ in the M -th order of PT ($1 \leq M \leq 3$) in the following form:

$$\mathcal{V}_{Q\bar{Q}}^{(M)}(a_s) = 1 + \mathcal{V}_0^{(M)}(a_s) + \sum_{n=1}^M \left(\frac{\beta^{(M-n+1)}(a_s)}{a_s} \right)^n \mathcal{V}_n^{(M-n+1)}(a_s), \quad (15)$$

$$\text{where } \mathcal{V}_0^{(M)}(a_s) = \sum_{k=1}^M \mathcal{V}_{0,k}^{(M)} a_s^k, \quad \mathcal{V}_n^{(r)}(a_s) = \sum_{k=1}^r \mathcal{V}_{n,k}^{(r)} a_s^{k-1}. \quad (16)$$

In a more detailed way, Eq.(15) may be rewritten as:

$$\begin{aligned} \mathcal{V}_{Q\bar{Q}}^{(M=3)}(a_s) = & 1 + \left(\mathcal{V}_{0,1}^{(3)} - \beta_0 \mathcal{V}_{1,1}^{(3)} \right) a_s + \left(\mathcal{V}_{0,2}^{(3)} - \beta_0 \mathcal{V}_{1,2}^{(3)} - \beta_1 \mathcal{V}_{1,1}^{(3)} \right. \\ & \left. + \beta_0^2 \mathcal{V}_{2,1}^{(2)} \right) a_s^2 + \left(\mathcal{V}_{0,3}^{(3)} - \beta_0 \mathcal{V}_{1,3}^{(3)} - \beta_1 \mathcal{V}_{1,2}^{(3)} - \beta_2 \mathcal{V}_{1,1}^{(3)} \right. \\ & \left. + \beta_0^2 \mathcal{V}_{2,2}^{(2)} + 2\beta_0 \beta_1 \mathcal{V}_{2,1}^{(2)} - \beta_0^3 \mathcal{V}_{3,1}^{(1)} \right) a_s^3, \end{aligned} \quad (17)$$

where at the fixed numbers n and k , $\mathcal{V}_{n,k}^{(r)} \equiv \mathcal{V}_{n,k}^{(r+1)} \equiv \mathcal{V}_{n,k}^{(r+2)} \equiv \dots$. As we have anticipated, the representation (17) is in full agreement with its $\{\beta\}$ -expansion pattern

$$a_1 = \beta_0 a_1[1] + a_1[0], \quad (18a)$$

$$a_2 = \beta_0^2 a_2[2] + \beta_1 a_2[0, 1] + \beta_0 a_2[1] + a_2[0], \quad (18b)$$

$$\begin{aligned} a_3 = & \beta_0^3 a_3[3] + \beta_1 \beta_0 a_3[1, 1] + \beta_2 a_3[0, 0, 1] + \beta_0^2 a_3[2] + \beta_1 a_3[0, 1] \\ & + \beta_0 a_3[1] + a_3[0]. \end{aligned} \quad (18c)$$

Moreover, the representation (17) provides us equalities similar to Eqs.(10a-10c), namely:

$$a_1[1] = a_2[0, 1] = a_3[0, 0, 1] = -\mathcal{V}_{1,1}^{(1)}, \quad (19a)$$

$$a_2[1] = a_3[0, 1] = -\mathcal{V}_{1,2}^{(2)}, \quad (19b)$$

$$a_2[2] = a_3[1, 1]/2 = \mathcal{V}_{2,1}^{(2)}. \quad (19c)$$

Using the three-loop explicit analytic results [27] for the static potential in the perturbative QCD and taking into account Eqs.(19a-19c), we obtain all terms of $\{\beta\}$ -expansion included in Eqs.(18a-18c). The corresponding results are given in Table 1.

| | Group structures | Numbers |
|------------------------------------|--|--|
| $a_1[1]$ | — | $\frac{5}{3}$ |
| $a_1[0]$ | C_A | $-\frac{2}{3}$ |
| $a_2[2]$ | — | $\frac{25}{9}$ |
| $a_2[1]$ | C_F | $\frac{35}{16} - 3\zeta_3$ |
| | C_A | $-\frac{217}{72} + \frac{7}{2}\zeta_3$ |
| $a_2[0]$ | $C_F C_A$ | $-\frac{385}{192} + \frac{11}{4}\zeta_3$ |
| | C_A^2 | $\frac{133}{144} - \frac{11}{4}\zeta_3 + \frac{\pi^2}{4} - \frac{\pi^4}{64}$ |
| $a_3[3]$ | — | $\frac{125}{27}$ |
| $a_3[2]$ | C_F | $\frac{5471}{288} - \frac{39}{2}\zeta_3$ |
| | C_A | $-\frac{7943}{576} + \frac{69}{4}\zeta_3 + \frac{\pi^4}{15}$ |
| $a_3[1]$ | C_F^2 | $-\frac{571}{192} - \frac{19}{8}\zeta_3 + \frac{15}{2}\zeta_5$ |
| | $C_F C_A$ | $-\frac{70069}{3456} + \frac{49}{2}\zeta_3 - \frac{15}{4}\zeta_5$ |
| | C_A^2 | $\frac{2491}{288} - \frac{309}{16}\zeta_3 - \frac{1091}{128}\zeta_5 - \frac{171}{128}\zeta_3^2 + \frac{9}{4}s_6 - \frac{761}{53760}\pi^6$ |
| | | $+\pi^4\left(\frac{9}{640} + \frac{5}{192}\log 2 - \frac{3}{64}\log^2 2\right)$ $+\pi^2\left(-\frac{17}{576} + \frac{19}{64}\zeta_3 + \frac{1}{16}\log 2 + \frac{21}{32}\zeta_3\log 2 + \frac{3}{2}\alpha_4\right)$ |
| $a_3[0]$ | $C_F^2 C_A$ | $\frac{6281}{2304} + \frac{209}{96}\zeta_3 - \frac{55}{8}\zeta_5$ |
| | $C_F C_A^2$ | $\frac{3709}{3456} - \frac{379}{96}\zeta_3 + \frac{55}{16}\zeta_5$ |
| | C_A^3 | $-\frac{19103}{27648} + \frac{181}{48}\zeta_3 + \frac{1431}{512}\zeta_5 + \frac{55}{512}\zeta_3^2 + \frac{3}{16}s_6 - \frac{21097}{1935360}\pi^6$ |
| | | $+\pi^4\left(\frac{211}{23040} - \frac{15}{256}\log 2 - \frac{61}{2304}\log^2 2\right)$ $+\pi^2\left(-\frac{191}{768} + \frac{841}{768}\zeta_3 - \frac{955}{576}\log 2 + \frac{203}{384}\zeta_3\log 2 + \frac{5}{3}\alpha_4\right)$ |
| | $\frac{d_F^{abcd}d_A^{abcd}}{N_A}n_f$ | $\frac{5}{96}\pi^6 + \pi^4\left(-\frac{23}{24} + \frac{1}{6}\log 2 - \frac{1}{2}\log^2 2\right)$ $+\pi^2\left(\frac{79}{36} - \frac{61}{12}\zeta_3 + \log 2 + \frac{21}{2}\zeta_3\log 2\right)$ |
| $\frac{d_F^{abcd}d_A^{abcd}}{N_A}$ | $\frac{1511}{2880}\pi^6 + \pi^4\left(-\frac{39}{16} + \frac{35}{12}\log 2 + \frac{31}{12}\log^2 2\right)$ $+\pi^2\left(\frac{929}{72} - \frac{827}{24}\zeta_3 - 74\alpha_4 + \frac{461}{6}\log 2 - \frac{217}{4}\zeta_3\log 2\right)$ | |

Table 1: All terms in the $\{\beta\}$ -expansion of coefficients a_1 , a_2 and a_3 .

Note that there are transcendent numbers $\alpha_4 = \text{Li}_4(1/2) + \log^4 2/4!$ with polylogarithmic function $\text{Li}_n(x) = \sum_{k=1}^{\infty} x^k k^{-n}$, $s_6 = \zeta_6 + \zeta_{-5,-1}$ with $\zeta_6 = \pi^6/945$ and multiple zeta value $\zeta_{-5,-1} = \sum_{k=1}^{\infty} \sum_{i=1}^{k-1} (-1)^{i+k}/ik^5$ in Table 1. They appear from scalar master-integrals [27], which are different from those arising in the calculation of corrections to $D_{NS}(Q^2)$ and $C_{NS}(Q^2)$ (see e.g. [29]).

Excepting two reservations, the presented coefficients of $\{\beta\}$ -expansion mostly coincide with ones defined in [20] by independent way using the \mathcal{R}_δ -scheme approach. The first of them lies in the small difference between our rational numbers, highlighted in blue in Table 1, and those found in [20]. Indeed, instead of coefficients $5171/288$ and $-2981/192 \approx -15.526$, contained in $a_3[2]$ -term in Ref. [20], we get slightly different ones, namely $5471/288$ and $-7943/576 \approx -13.789$. Instead of the $C_F C_A$ -term $-66769/3456 \approx -19.320$ included in $a_3[1]$ in [20], we obtain $-70069/3456 \approx -20.275$.

The second clause concerns the light-by-light scattering effects, first occurring at the three-loop level for the static potential [27]. In our opinion, despite the n_f -dependence of these effects, they have to be included in the scale-invariant coefficient $a_3[0]$. This fact is reflected in Table 1 by the incorporation of the terms proportional to $d_F^{abcd} d_F^{abcd} n_f / N_A$ and $d_F^{abcd} d_A^{abcd} / N_A$ group structures in this “ n_f -independent” coefficient. Here N_A is the number of group generators, $d_F^{abcd} = \text{Tr}(t^a t^b t^c t^d)/6$ and $d_A^{abcd} = \text{Tr}(C^a C^b C^c C^d)/6$, where the symbol $\{\dots\}$ stands for the full symmetrization procedure of elements $t^b t^c t^d$ by superscripts b, c and d [30, 31]; $(C^a)_{bc} = -if^{abc}$ are the generators of the adjoint representation with the antisymmetric structure constants f^{abc} of the Lie algebra: $[t^a, t^b] = if^{abc} t^c$. In the particular case of the $SU(N_c)$ color gauge group the contractions of the symmetric invariant tensors considered above read:

$$\frac{d_F^{abcd} d_F^{abcd}}{N_A} = \frac{N_c^4 - 6N_c^2 + 18}{96N_c^2}, \quad \frac{d_F^{abcd} d_A^{abcd}}{N_A} = \frac{N_c(N_c^2 + 6)}{48}. \quad (20)$$

Although the term $d_F^{abcd} d_F^{abcd} n_f / N_A$ in the correction a_3 (14) is proportional to the number of flavors n_f , which formally enters the β_0 -coefficient, we will not include it into $\mathcal{V}_{1,3}^{(3)}$ -coefficient in Eq.(17), since such rearrangement will not be supported by the QED limit [16, 18]. Indeed, in the QED limit of the QCD-like theory with the $SU(N_c)$ group $d_F^{abcd} d_F^{abcd} / N_A = 1$ and $n_f = N$, where N is the number of the charged leptons (structures with d_A^{abcd} are nullified). This term arises from the three-loop Feynman diagram with light-by-light scattering internal subgraphs (see Fig.1). However, in QED the sum of these subgraphs are convergent and does not give extra β_0 -dependent (or N -dependent) contribution to the coefficient $\mathcal{V}_{1,3}^{(3)}$ [16, 18]. Therefore, to get a smooth transition from the case of $SU(N_c)$ to $U(1)$ gauge group, these light-by-light scattering terms should be included into the β -independent coefficient $\mathcal{V}_{0,3}^{(3)}$ (or $a_3[0]$) at the three-loop level.

In accordance with the foregoing, at the three-loop level the static Coulomb

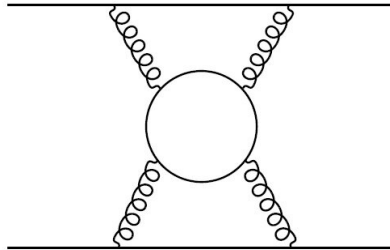


Fig. 1: Light-by-light scattering type diagram in the static potential

potential of QED can be expressed through the invariant charge as:

$$V_{\text{QED}}(\vec{q}^2) = -\frac{4\pi}{\vec{q}^2} \frac{\alpha(\mu^2)}{1 + \Pi_{\text{QED}}(L, \alpha(\mu^2))} \left(1 + N \cdot \mathcal{C} \left(\frac{\alpha(\mu^2)}{\pi} \right)^3 \right) + \mathcal{O}(\alpha^5), \quad (21)$$

where the $\overline{\text{MS}}$ -scheme photon vacuum polarization function $\Pi_{\text{QED}}(L, \alpha(\mu^2))$ was calculated in α^3 order in Ref. [32]. The constant \mathcal{C} originates from the light-by-light scattering effects and may be extracted from the results of work [27]:

$$\mathcal{C} = \frac{5}{96}\pi^6 - \pi^4 \left(\frac{23}{24} - \frac{1}{6}\log 2 + \frac{1}{2}\log^2 2 \right) + \pi^2 \left(\frac{79}{36} - \frac{61}{12}\zeta_3 + \log 2 \right) + \frac{21}{2}\zeta_3 \log 2. \quad (22)$$

2.2. Special representations of the relation between $\mathcal{V}_{Q\bar{Q}}$ and $\mathcal{V}_{\text{cusp}}$

It is interesting to note that there is a quantity $\mathcal{V}_{\text{cusp}}$ associated with the static potential by the relation where, as in the CBK relation, the effect of conformal symmetry breaking is explicitly manifested in the form of the factorized β -function. This relation was first obtained at two-loop level (at $M = 2$) in Ref. [33] and reads:

$$\mathcal{V}_{\text{cusp}}^{(M)}(a_s) - \mathcal{V}_{Q\bar{Q}}^{(M)}(a_s) = \frac{\beta^{(M-1)}(a_s)}{a_s} C^{(M-1)}(a_s). \quad (23)$$

Here $C^{(M-1)}(a_s) = \sum_{k=1}^{M-1} C_k a_s^k$ with $C_1 = \frac{47}{27}C_A - \frac{28}{27}T_F n_f$.

In Eq.(23) the function $\mathcal{V}_{\text{cusp}}(a_s)$ is related with the cusp anomalous dimension $\Gamma(\phi, a_s)$ of the vacuum expectation value of the Wilson loop $W[\tilde{C}]$ (12), but unlike the case of the static potential, \tilde{C} is now a closed contour formed by two segments along directions v_1^μ and v_2^μ with Euclidean cusp angle ϕ : $\cos \phi = (v_1, v_2)$ (see investigations on this topic [33–38] and references therein).

The quantity $\log W[\tilde{C}]$ is expressed through the logarithm of the corresponding $\overline{\text{MS}}$ -scheme cusp renormalization constant Z_{cusp} [33–38]:

$$\log W = \log Z_{\text{cusp}} + \mathcal{O}(\varepsilon^0), \quad (24)$$

where Z_{cusp} contains the ultraviolet divergences and it is written in terms of the renormalized coupling constant. In its turn, the cusp anomalous dimension is defined as:

$$\Gamma(\phi, a_s) = \frac{\partial \log Z_{\text{cusp}}}{\partial \log \mu^2}. \quad (25)$$

The quantity $\mathcal{V}_{\text{cusp}}(a_s)$ is a δ -independent coefficient function of $\Gamma(\phi, a_s)$ in the anti-parallel lines limit, i.e. when $\phi = \pi - \delta$ and $\delta \ll 1$ [33, 39]:

$$\Gamma(\pi - \delta, \alpha_s) = -C_F \alpha_s \frac{\mathcal{V}_{\text{cusp}}(\alpha_s)}{2\delta} + \mathcal{O}\left(\frac{\alpha_s^4 \log \delta}{\delta}\right). \quad (26)$$

The first order corrections to $\mathcal{V}_{\text{cusp}}(\alpha_s)$ and $\mathcal{V}_{Q\bar{Q}}(\alpha_s)$ coincide [39] and the differences between them are observed only from the second order of PT and they are proportional to the first coefficient of the RG β -function [33] (see Eq.(23)).

At $M = 3$ the convincing arguments in favor of validity of the expression (23) were given in [35]. Except for the pure non-abelian part proportional to C_A^2 -structure, all other terms to coefficient C_2 (proportional to terms $T_F^2 n_f^2$, $C_F T_F n_f$ and $C_A T_F n_f$) were defined there in analytical form.

Since the function $\mathcal{V}_{\text{cusp}}(\alpha_s)$ has the same dependence on n_f and the same group structures like the quantity $\mathcal{V}_{Q\bar{Q}}(\alpha_s)$, it is obvious that its series of PT can be decomposed into powers of the conformal anomaly as well:

$$\mathcal{V}_{\text{cusp}}^{(M)}(a_s) = 1 + \mathcal{V}_{\text{cusp}, 0}^{(M)}(a_s) + \sum_{n=1}^M \left(\frac{\beta^{(M-n+1)}(a_s)}{a_s} \right)^n \mathcal{V}_{\text{cusp}, n}^{(M-n+1)}(a_s). \quad (27)$$

But representation of the functions $\mathcal{V}_{Q\bar{Q}}(\alpha_s)$ and $\mathcal{V}_{\text{cusp}}(\alpha_s)$ in the form of double sum (15) and (27) is a sufficient condition for the CSB term in the difference $\mathcal{V}_{\text{cusp}}(a_s) - \mathcal{V}_{Q\bar{Q}}(a_s)$ can also be decomposed in powers of $\beta(\alpha_s)/\alpha_s$:

$$\mathcal{V}_{\text{cusp}}^{(M)}(a_s) - \mathcal{V}_{Q\bar{Q}}^{(M)}(a_s) = \sum_{n=1}^M \left(\frac{\beta^{(M-n+1)}(a_s)}{a_s} \right)^n T_n^{(M-n+1)}(a_s), \quad (28)$$

$$\text{where } T_n^{(r)} = \sum_{k=1}^r T_{n,k}^{(r)} a_s^{k-1}. \quad (29)$$

One should emphasize that since the first order corrections to $\mathcal{V}_{\text{cusp}}(a_s)$ and to $\mathcal{V}_{Q\bar{Q}}(a_s)$ coincide, then $T_{1,1}^{(1)} = T_{1,1}^{(2)} = \dots = 0$. Taking this fact into account and using the representation (28) and the results of Refs. [33, 35], we obtain:

$$\begin{aligned} \mathcal{V}_{\text{cusp}}^{(3)}(a_s) - \mathcal{V}_{Q\bar{Q}}^{(3)}(a_s) &= \left(-\frac{28}{9}\beta_0^2 + \frac{10}{9}C_A\beta_0 \right) a_s^2 + \left(\beta_0^3 \left(2\zeta_3 - \frac{134}{27} \right) \right. \\ &+ \frac{10}{9}C_A\beta_1 - 2 \cdot \frac{28}{9}\beta_0\beta_1 + \beta_0^2 \left(C_F \left(\frac{19}{2}\zeta_3 - \frac{1487}{96} + \frac{\pi^4}{20} \right) + C_A \left(\frac{1879}{144} \right. \right. \\ &\left. \left. - \frac{89}{12}\zeta_3 - \frac{\pi^4}{8} \right) \right) + \beta_0 \left(C_F C_A \left(\frac{16357}{1152} - \frac{209}{24}\zeta_3 - \frac{11}{240}\pi^4 \right) - C_A^2 T_{1,3}^{AA} \right) \Big) a_s^3, \end{aligned} \quad (30)$$

where $T_{1,3}^{AA}$ is the still unknown coefficient, contained in term $T_{1,3}^{(3)}$ and proportional to C_A^2 group factor. Its explicit form may be found from the coefficient C_2 [35], where C_A^2 -term C_2^{AA} was fixed in part. They are related to each other in the following way: $T_{1,3}^{AA} = C_2^{AA} + 86893/15552 - 737/144\zeta_3 - 11/96\pi^4$. It is important to emphasize that the system of linear equations, arising in process of application of the decomposition into powers of the conformal anomaly, has a unique solution at any value of $T_{1,3}^{AA}$. This means that the representation (28) is always possible at least at the three-loop level.

Note that the absence of terms $\beta_1 a_s^2$ and $\beta_2 a_s^3$ in Eq.(30) is explained by fact that $T_{1,1}^{(2)} = T_{1,1}^{(3)} = 0$.

3. The case of the unpolarized Bjorken sum rule

Let us now consider a quantity that, according to current data, is not included in any of known relations that would reflect the effect of violation of the conformal symmetry. As an example, we examine the unpolarized Bjorken sum rule for deep-inelastic neutrino-nucleon scattering [40]. The detailed theoretical study of this process may be still of interest in view of its possible investigation from the potential future DIS νN data, which may be collected at SAND detector (see e.g. [41]) of the DUNE Collaboration.

So, consider the aforementioned unpolarized Bjorken sum rule:

$$F_{Bjump}(Q^2) = \int_0^1 dx (F_1^{\bar{\nu}p}(x, Q^2) - F_1^{\nu p}(x, Q^2)), \quad (31)$$

where $F_1^{\bar{\nu}p}(\nu p)(x, Q^2)$ is the structure function of νN DIS process that arises in the general decomposition of the hadronic tensor into possible Lorentz structures. At large $Q^2 = -q^2$ the dependence of F_{Bjump} on Q^2 is absorbed into the coupling constant.

In the Born approximation $F_{Bjump} = 1$. Deviation from unity is already observed in the leading order of PT [42]. Analytical two- and three-loop corrections to the coefficient function of the unpolarized Bjorken sum rule in the $\overline{\text{MS}}$ -scheme for the case of the generic simple gauge group were computed in [43] and [44] correspondingly. In particular case of the $SU(3)$ color group the analytical four-loop correction to F_{Bjump} is known due to the unpublished results [45]. One should mention that the general formula for the leading renormalon contributions to this considered perturbative quantity was obtained in [46]. Its predictions are in full agreement with the results of the papers cited above.

Applying the idea about the representation of the perturbative series for observables in QCD-like theories with the generic simple gauge group in form of their decomposition into powers of $\beta(\alpha_s)/\alpha_s$, we obtain that at $1 \leq M \leq 4$ the NS contribution to the Bjorken unpolarized sum rule may be rewritten

as:

$$F_{Bj unp}^{(M)}(a_s) = 1 + F_0^{(M)}(a_s) + \sum_{n=1}^{M-1} \left(\frac{\beta^{(M-n)}(a_s)}{a_s} \right)^n F_n^{(M-n)}(a_s). \quad (32)$$

In more detailed form at the four-loop level we have:

$$\begin{aligned} F_{Bj unp}^{(M=4)}(a_s) = & 1 + F_{0,1}^{(4)} a_s + \left(F_{0,2}^{(4)} - \beta_0 F_{1,1}^{(3)} \right) a_s^2 + \left(F_{0,3}^{(4)} - \beta_0 F_{1,2}^{(3)} \right. \\ & - \beta_1 F_{1,1}^{(3)} + \beta_0^2 F_{2,1}^{(2)} \left. \right) a_s^3 + \left(F_{0,4}^{(4)} - \beta_0 F_{1,3}^{(3)} - \beta_1 F_{1,2}^{(3)} \right. \\ & \left. - \beta_2 F_{1,1}^{(3)} + \beta_0^2 F_{2,2}^{(2)} + 2\beta_0 \beta_1 F_{2,1}^{(2)} - \beta_0^3 F_{3,1}^{(1)} \right) a_s^4. \end{aligned} \quad (33)$$

Taking into account the results of Refs. [42–44], one can get:

$$F_{0,1} = -\frac{C_F}{2}, \quad F_{1,1} = \frac{4}{3} C_F, \quad (34a)$$

$$F_{0,2} = \frac{11}{16} C_F^2 - \frac{1}{24} C_F C_A, \quad F_{2,1} = -\frac{155}{36} C_F, \quad (34b)$$

$$F_{1,2} = \left(\frac{431}{96} - \frac{1}{2} \zeta_3 \right) C_F^2 + \left(-\frac{35}{144} + \frac{7}{2} \zeta_3 - 5\zeta_5 \right) C_F C_A, \quad (34c)$$

$$\begin{aligned} F_{0,3} = & \left(-\frac{313}{64} - \frac{47}{4} \zeta_3 + \frac{35}{2} \zeta_5 \right) C_F^3 + \left(\frac{687}{128} + \frac{125}{8} \zeta_3 - \frac{95}{4} \zeta_5 \right) C_F^2 C_A \\ & + \left(-\frac{463}{288} - \frac{137}{24} \zeta_3 + \frac{115}{2} \zeta_5 \right) C_F C_A^2. \end{aligned} \quad (34d)$$

Using now the results of four-loop calculations [45] made for particular case of the SU(3) color gauge group, we arrive to the following expressions:

$$F_{3,1} = \frac{1780}{81}, \quad F_{2,2} = \frac{78155}{2592} + \frac{87}{2} \zeta_3 + 12\zeta_3^2 - 110\zeta_5, \quad (34e)$$

$$F_{1,3} = -\frac{97247}{2592} - \frac{61153}{648} \zeta_3 - \frac{965}{54} \zeta_5 + \frac{296}{27} \zeta_3^2 - \frac{49}{4} \zeta_7 - 6B, \quad (34f)$$

$$\begin{aligned} F_{0,4} = & -\frac{8139161}{124416} - \frac{308489}{1296} \zeta_3 + \frac{239665}{1296} \zeta_5 - \frac{2927}{108} \zeta_3^2 + \frac{253757}{2592} \zeta_7 \\ & - \frac{33}{2} B + B \cdot n_f, \end{aligned} \quad (34g)$$

where $B \sim d_F^{abcd} d_F^{abcd} / d_R$ -term is the light-by-light scattering type contribution to the Bjorken unpolarized sum rule. Note that the nonabelian flavor-independent correction of this effect is already contained in the coefficient $F_{0,4}$. To get expressions (34e–34g) in an analytical form for the case of the generic simple gauge group, it would be interesting to evaluate the 4-loop correction to the coefficient function of the Bjorken unpolarized sum rule for the generic simple gauge group as well.

4. Conclusion

The proposed decomposition procedure of the PT series for observables into powers of the conformal anomaly in QCD allows us not only to reproduce the known structure of $\{\beta\}$ -expansion, but to predict its definite terms in higher orders of PT as well. For instance, application of this technique to the non-singlet contribution of the coefficient function of the Bjorken unpolarized sum rule gives

$$\begin{aligned}
F_{Bj unp}^{(M=4)}(a_s) = & 1 + F_{0,1}^{(4)} a_s + \left(F_{0,2}^{(4)} - \beta_0 F_{1,1}^{(3)} \right) a_s^2 + \left(F_{0,3}^{(4)} - \beta_0 F_{1,2}^{(3)} - \beta_1 F_{1,1}^{(3)} \right. \\
& + \left. \beta_0^2 F_{2,1}^{(2)} \right) a_s^3 + \left(F_{0,4}^{(4)} - \beta_0 F_{1,3}^{(3)} - \beta_1 F_{1,2}^{(3)} - \beta_2 F_{1,1}^{(3)} + \beta_0^2 F_{2,2}^{(2)} \right. \\
& + \left. 2\beta_0\beta_1 F_{2,1}^{(2)} - \beta_0^3 F_{3,1}^{(1)} \right) a_s^4.
\end{aligned}$$

We can see that the knowledge of $F_{1,1}^{(3)}$ -coefficient of β_0 -dependent term in a_s^2 order enables to predict values of β_1 - and β_2 -dependent ones in a_s^3 and a_s^4 orders. In its turn, the knowledge of β_0 -dependent coefficient $F_{1,2}^{(3)}$ and β_0^2 -dependent one $F_{2,1}^{(2)}$ in a_s^3 order gives values of the β_1 - and $\beta_0\beta_1$ -dependent coefficients in a_s^4 order correspondingly. Thus, the utilization of the expansion in powers of the conformal anomaly allows to determine all terms of $\{\beta\}$ -expansion, except for ones proportional to β_0^0 and higher powers of β_0 in concrete order of PT. Indeed, all other terms proportional to higher coefficients of β -function will be uniquely fixed from $\{\beta\}$ -expansion pattern obtained at the previous order of PT. In the case, when the correction of the M -th order to observable quantity is known explicitly, the terms proportional to powers of β_0 may be defined unambiguously.

Application of this procedure to the static Coulomb-like potential, to the relation between it and cusp anomalous dimension in anti-parallel lines limit and to the Bjorken unpolarized sum rule enables to determine all their $\{\beta\}$ -expanded terms unambiguously. The arguments in favor of validity of this decomposition in powers of $\beta(a_s)/a_s$ gives not only from Refs. [19–21] and [47], [18], but also from work [48], where the ratio of the pole mass M to $\overline{\text{MS}}$ -scheme scale-dependent running $\overline{m}(\mu^2)$ mass of the heavy quark was obtained at the three-loop level analytically. The results of this work were also presented there in the following numerical form

$$\begin{aligned}
\frac{M}{\overline{m}(\overline{m}^2)} = & 1 + 1.333a_s + (6.248\beta_0 - 3.739)a_s^2 + (23.497\beta_0^2 + 6.248\beta_1 \\
& + 1.019\beta_0 - 29.94)a_s^3.
\end{aligned} \quad (35)$$

It reproduces the $\{\beta\}$ -expansion structure and the relations between its definite coefficients, which we have obtained within procedure of decomposition in powers of $\beta(a_s)/a_s$ (see e.g. Eqs.(10a-10c) or (19a-19c)). Indeed, the coefficient before $\beta_0 a_s^2$ -term is the same as the coefficient before $\beta_1 a_s^3$ -term.

The same feature we have observed above in the cases of the NS Adler function, the NS coefficient functions of the Bjorken polarized and unpolarized sum rules and the static Coulomb-like potential. However, unlike the previous cases, the scale dependence of ratio $M/\overline{m}(\mu^2)$ is governed now by two RG functions: β -function and anomalous mass dimension γ_m . Note one interesting fact. The utilization of *the formal* $\{\beta\}$ -*expansion in powers of* $\beta(a_s)/a_s$ of the two- [49, 50] and three-loop [51, 52] coefficients of γ_m -function, namely

$$\gamma_1 = \frac{5}{8}C_F\beta_0 + \frac{3}{32}C_F^2 + \frac{7}{16}C_FC_A, \quad (36a)$$

$$\begin{aligned} \gamma_2 = & -\frac{35}{48}C_F\beta_0^2 + \beta_0 \left(C_F^2 \left(\frac{27}{16} - \frac{9}{4}\zeta_3 \right) + C_FC_A \left(-\frac{679}{96} + \frac{9}{4}\zeta_3 \right) \right) \\ & + \frac{5}{8}C_F\beta_1 + \frac{129}{128}C_F^3 + C_F^2C_A \left(-\frac{525}{256} + \frac{33}{16}\zeta_3 \right) + C_FC_A^2 \left(\frac{1063}{128} - \frac{33}{16}\zeta_3 \right), \end{aligned} \quad (36b)$$

and the subsequent substitution of these decompositions into NLO and NNLO general expressions, given in Ref. [53] and obtained there within the \mathcal{R}_δ -scheme motivated approach, allows us to reproduce the numerical form of Eq.(35). This finding may be related to the fact that the QCD trace anomaly $T_\mu^\mu = (\beta(a_s)/2a_s)F^{a\mu\nu}F_{\mu\nu}^a + (1 + \gamma_m)m\bar{\psi}\psi$ [54–56] contains not only β -term, but also the anomalous mass dimension γ_m -term. However, as was noted in [47], it is necessary to treat with caution to the $\{\beta\}$ -expansion of the non-RG-invariant quantities with other non-zero anomalous dimension functions.

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