On general theories on the momentum space

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Представлен краткий исторический обзор развития теорий искривленного импульсного пространства вместе с концептуальными вопросами, возникающими из-за неясностей, присущих формулировкам этих теорий. Дан краткий обзор недавней попытки систематической аксиоматизации, основанной на принципе минимального расширения.

A short historical review of the development of the theories on a curved momentum space is presented, together with the conceptual issues arising from the inherent ambiguities in formulation of these theories. A brief overview of a recent attempt at systematic axiomatization, following the principle of minimal extension, is provided.

1. Introduction

Our inability until today to unify the concepts of quantum mechanics with those of general relativity into a single consistent framework points towards a possible necessary modification of our understanding of the structure of space-time at the smallest scales. Some prominent examples of such modifications come from string theory, loop quantum gravity, doubly special relativity and noncommutative geometry.

The same effect could be reached by studying the corresponding modifications of the structure of momentum space. This idea was contemplated already by Heisenberg, while the first concrete model was published by Snyder in 1947 [1]. The motivation behind Heisenberg et al was an attempt to resolve the conceptual issues of quantum mechanics itself, which, despite its operational success, still plague it until today [2]. The motivation behind Snyder's work was to cure the infinities in the then emerging theory of quantum electrodynamics.

In this short note we review historical development of the ideas related to the generalization of the geometry of the momentum space, from its beginnings motivated by the infinities in the field theory until today where the principal motivation for the study of such deformations comes from quantum gravity considerations. We also highlight some key conceptual problems and the inherent ambiguities which arise when trying to define a modified theory that is far beyond our experimental reach.

2. Early history

With the development of quantum mechanics, the questions of the fundamental structure of space-time and the momentum started to emerge. Born, Heisenberg, Infeld and Wataghin were considering the lattice structure of space-time at the smallest scales as early as 1930 [3]. Pauli was thinking about the general geometry of the energy-momentum space. Snyder got aquinted with this idea while working with Pauli. In 1947 he published a first paper on the subject, with the next one following shortly after [4]. This was about the same time as the program of regularization and renormalization was being developed. Initiated by Stueckelberg and Petermann [5],

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Gell-Mann and Low [6], and further developed through the works, among others, of, Weinberg [7], t'Hooft and Veltman [8], and Wilson [9], it provides a systematic way of dealing with infinities leading to finite values of the amplitudes, as functions of small number of experimentally obtainable parameters, at all orders of the perturbation theory. This made QED one of the most succesful physical theories ever.

Subsequently, Snyder's work did not attract much attention [10]. In the period from early 1960-ies until the late 1980ies the theory on curved momentum space was further developed in the works of several prominent Soviet authors, including, among others, Gol'fand [11], Mir-Kasimov [12], Kadyshevsky [13] and Tamm [14]. Several very interesting proposals for the definition of such theories have been given. We point out to a very beautiful series of papers by Kadyshevsky et al [15] on one possible formulation of field theories in this context.

3. Modern days

Late 1980ies and early 1990ies witnessed the development of several new approaches in mathematical physics as well as pure mathematics. Noncommutative geometry is shown to arise in the string theory [16]. The study of the inverse scattering problem led to deformations of matrix groups and their corresponding Lie algebras [17, 18], and the same deformed structures appeared in the theory of quantum groups with the development of differential calculus on noncommutative spaces [19, 20]. Allthough they involve a much wider class of noncommutativity then the one proposed by Snyder, Snyder's work was duly recognized as a pioneering step in this direction, which led to a surge in the interest for it.

The predominant motivation today for the study of noncommutative geometry is the search for the theory of quantum gravity. One can speak of the three main classes (with a plethora of sub-classes) of noncommutativity models: the noncommutativity related to Snyder's model and its generalizations [21–23], the noncommutativity arising from diffeomorphisms of the flat space [24], and the so-called canonical noncommutativity [25]. Although in what follows we concider the Snyder model, many of the conceptual issues we present hold equally also for the other types of noncommutativity.

This also led to the development of concepts and ideas beyond the noncommutativity itself. Two prominent examples are the doubly special relativity and the relative locality. For a review on these approaches we refer to [26, 27].

4. Conceptual difficulties

The mathematical properties of the background on which a new physical theory is to be developed, although well established, are not enough to define the dynamics in a unique non-ambigous way. This comes from the fact that any dynamics necessarily incorporates a set of fundamental physical principles in addition to the set of physical parameters, both of which are rooted in the experimental observation. In any extension of a present theory outside the experimental scope, the necessary question emerges about which of the physical principles, symmetries and laws remain valid, and which get broken or deformed. For instance, the Lorentz symmetry, which is a cornerstone of our contemporary understanding of space-time and matter, may very well turn out to be a low energy-approximation of something more fundamental. Even if one assumes validity of all of symmetries and physical principles in the new extended theory, this is still not enough to single out one in the infinity of mathematically allowed possibilities.

Say, for example, we have a system that is standardly described by a Hamiltonian H_0. One then naturally asks what is this system described by in a new modified theory. Assuming that the new theory even allows for a Hamiltonian description, one introduces a constraint

$$\lim_{\mathcal{P}\to\infty} H[\mathcal{P}] = H_0, \tag{1}$$

where $\mathcal{P}$ denotes the momentum scale on which the effects of new physics become appriciable, usually considered to be many orders of magnitude above the typical scales of our best experiments.

Any function $H[\mathcal{P}]$ that satisfies the above constarint is a viable candidate for the description of new physics. Even if one chooses to maintain the symmetries of the original theory, it is still not restrictive enough. For instance, let us consider a concrete case of classical mechanics. The Hamiltonian of a free particle is given by

$$H_0 = \frac{p^2}{2m}, \tag{2}$$

fulfilling the demand of rotational invariance. For very high energies, maintaining the rotational invariance, this may generalize to

$$H[\mathcal{P}] = \frac{p^2}{2m}\left(1 + f(p^2/\mathcal{P}^2)\right), \tag{3}$$

where f is an arbitrary, suitably well-behaved function which vanishes for $\mathcal{P}\to\infty$. It can not be deduced a priori from some first principles, but rather must be postulated. One may try to argue that nonrelativistic classical mechanics is not the best candidate theory for the high energy generalization, since the Hamiltonian (2) is the kinetic energy which follows in the nonrelativistic limit of the more fundamental Lorentz-invariant mass-shell condition,

$$p^2 - \frac{E^2}{c^2} = m^2c^2. \tag{4}$$

But the dispersion relation can then be modified as well, in a Lorentz covariant way, so as to produce precisely the form (3) for the nonrelativistic kinetic energy. One example of such definition was given in [26], where it was postulated that the mass is the geodesic distance from the origin,

$$m^2 = d^2(0,p), \tag{5}$$

while the kinetic energy is given in terms of the geodesic distance between the point p ant the point p', which is the point which has the same geodesic distance to the origin as the point p, $d(0,p) = d(0,p')$ but whose momentum vanishes

$$K = \frac{d^2(p,p')}{2m} \tag{6}$$

It is then readily seen that one can construct a class of theories that produce precisely (3) for the kinetic energy in the nonrelativistic limit.

It is obvious from the above considerations that a number of things have to be postulated in a new theory. It is a certain mathematical ideal that the number of assumptions and postulates for the new theory should be minimal, although this need not have anything to do with physical reality. This ideal was introduced in [28] and dubbed the minimal extension principle. We proceed to describe its elements below.

5. Axioms and the consequences

We follow the line of reasoning put up in [26], where momentum space is assigned a fundamental role, while the space-time is in a sense an emerging concept, whose structure is completly determined by the structure of space-time.

A full set of postulates that the energy-momentum background of a new theory is required to satisfy is given in [29], together with the detailed analyses of its consequences. Here we review shortly the main points.

1. The momentum manifold is required to realise the full Poincare symmetry. This reduces the possible geometries of the momentum manifold to be that of the constant curvature space, with either positive, negative, or vanishing curvature. The case of the vanishing curvature represents the standard flat case. It can be viewed as the low energy limit of a fundamental case of nonvanishing curvature, where the radius of curvature is given by $\mathcal{P}$

2. There is a distinguished point on this manifold called the vacuum, and the physical momenta are measured with respect to it. This idea was first introduced by Tamm. For the values of the momenta whose distance to the origin is much smaller then $\mathcal{P}$, the assumed curvature of space is not observable.

3. No explicit mention of any coordinatization of the manifold is given in the set of postulates. This implies the covariance of the law of physics on the momentum space, in a similar way in which the covariance of space-time is implicit in GR. Otherwise, one would need to include a specification of a particular set of coordinates of the momentum manifold.

How these principles are to be incorporated into a construction of a generalized theory, has been considered specifically for the cases of classical field theory [28], nonrelativistic quantum mechanics [30] and quantum scalar field theory [29]. For the sake of the illustration, we review below the case of nonrelativistic quantum mechanics.

6. Quantum mechanics

The constant curvature space is conveneitly realised as a hypersurface in 4+1 dimensional Minkowskian space,

$$\left(\frac{\eta_0}{c}\right)^2 - (\eta^1)^2 - (\eta^2)^2 - (\eta^3)^2 - (\eta^4)^2 = -\mathcal{P}^2 \tag{7}$$

where η are the embedding space coordinates, and the physical energy-momentum coordinates are any function of the embedding coordinates. Since we are interested in non-relativistic case, the surface is the 3-momentum surface,

$$(\eta^1)^2 + (\eta^2)^2 + (\eta^3)^2 + (\eta^4)^2 = \mathcal{P}^2 \tag{8}$$

and the physical momenta p are related to the embedding coordinates via

$$\eta^i = h(p^2/\mathcal{P}^2)p^i, \qquad \eta^4 = \sqrt{\mathcal{P}^2 - h^2 p^2}, \tag{9}$$

with h an arbitrary function.

As a starting point we take Schrödinger's equation

$$(\hat{K}(\hat{p}^i) + \hat{V}(\hat{x}_i))\psi_n = E_n\psi_n, \tag{10}$$

where $\hat{K}$ is the kinetic energy operator expressed in terms of the momenta, and $\hat{V}$ is the potential energy operator in terms of the position operators. The latter are identified with the generators of infinitesimal translations on the sphere (8), namely

$$\hat{x}_i = \mathcal{P}^{-1}\hat{J}_{i3} = -i\hbar\mathcal{P}^{-1}\frac{\sqrt{1-\mathcal{P}^{-2}p^2h^2}}{h}\left[\frac{\partial}{\partial p^i} + \frac{2\mathcal{P}^{-2}h'}{h-2\mathcal{P}^{-2}p^2h'}\delta_{ik}p^kp^j\frac{\partial}{\partial p^j}\right], \tag{11}$$

which leads to a deformed Heisenberg algebra

$$[\hat{x}_i, \hat{p}^j] = i\hbar\frac{\sqrt{1-\mathcal{P}^{-2}p^2h^2}}{h}\left(\delta_i^j + \frac{2\mathcal{P}^{-2}h'}{h-2\mathcal{P}^{-2}p^2h'}\delta_{ik}p^kp^j\right), \ [\hat{x}_i, \hat{x}_j] = \mathcal{P}^{-2}\hat{J}_{ij}, \ [\hat{p}_i, \hat{p}_j] = 0. \tag{12}$$

It is readily seen that in the limit $\mathcal{P} \to \infty$, the position operator reduces to the standard form $\hat{x}_i = \partial/\partial p^i$ and the standard Heisenberg algebra is recovered,

$$[\hat{x}_i, \hat{p}^j] = i\hbar\delta_i^j, \qquad [\hat{x}_i, \hat{x}_j] = [\hat{p}^i, \hat{p}^j] = 0. \tag{13}$$

Next question is the definition of the kinetic and potential operators. For the kinetic operator, consistent with our above introduced requirements, and discussed in more detail in [30], we take the geometric invariant

$$\hat{K} = \frac{d^2(0,p)}{2m} = \frac{1}{2m}\mathcal{P}^2\arccos^2\sqrt{1-\mathcal{P}^{-2}p^2h^2}. \tag{14}$$

What concerns the potential energy operator, two specific choices of the potential were considered in [30], that illustrate nicely the general method. For the case of the harmonic oscillator potential, the choice of the potential operator

$$\hat{V}_{HO} = -\frac{m\omega^2\hbar^2}{2}\Delta = -\frac{m\omega^2\hbar^2}{2\sqrt{\det g}}\frac{\partial}{\partial p^i}\left(\sqrt{\det g}g^{ij}\frac{\partial}{\partial p^j}\right) \tag{15}$$

was argued, where g^{ij} is the metric tensor of the surface (8). This choice is again consistent with our general requirment of the covariance, that is, being expressed in terms of geometrical invariants. Any potential that can be expressed in terms of the power law expansion of the $\hat{x}^2$ terms is treated accordingly.

We point out that the quantum harmonic oscillator potential on Snyder space has been a topic of many studies, see the recent one in [31]. There a different choice of the kinetic and potential energy operators was proposed, leading to different eigenfunctions and eigenvalues then the ones that follow from our choice. This is a manifestation of the inherent ambiguities discussed in section 4. Even with the agreement on the geometry of the momentum background, the dynamical quantities that constitute a physical theory are still arbitrary. A particular benefit of our approach lies in that these are uniqly defined by the demand of covariance, which is the consequence of the introduced set of general postulates, and our demand of the minimal extension.

Another case considered in [30] was the potential that is not expandable in terms of the powers of $\hat{x}^2$, namely the Coulomb potential. In this case, we start with the canonical potential operator

$$\hat{V}_{Coul}\psi(p) = \frac{1}{\hbar}\int \frac{\psi(p')d^3p'}{|p-p'|^2}, \tag{16}$$

and preform the required "geometrization" of its form, namely

$$d^3p \to d\Omega_p = \sqrt{\det g}\,d^3p \tag{17}$$

$$|p-p'| \to d(p,p') = \mathcal{P}\arccos\left(\mathcal{P}^{-2}h_p h_{p'} p^i p'^i - \sqrt{1-\mathcal{P}^{-2}h_p^2 p^2}\sqrt{1-\mathcal{P}^{-2}h_{p'}^2 p'^2}\right), \tag{18}$$

where $h_p = h(p^2/\mathcal{P}^2)$ and $h_{p'} = h(p'^2/\mathcal{P}^2)$.

This, in combination with (14), leads to a full Schrödinger equation for the Coulomb potential that is completely covariant.

7. Outlook

Nonrelativistic quantum mechanics can be regarded as a toy model for the application of principles of the momentum space covariance argued in section 5. A more serious analyses was given for the case of a scalar QFT in [29], where it was demonstrated that the amplitudes at any order of the perturbation are finite, due to the finitness of the (Euclidean) momentum space. This result has already been obtained by Mir-Kasimov [32] in a different approach. This again confirms the validity of the original Snyder's hypothesis that the radius of curvature of the momentum space can serve as a cut-off in the field theory.

As a next logical step the spinor field theory is to be considered. The challenge is to demonstrate that the geometrical formulation on a momentum space of constant curvature leads again to finite Feynman amplitudes. Additionally, the question emerges of the realization of gauge symmetry on such spaces, as well as the interplay between gauge and diffeomorphism symmetries. This is the topic of an ongoing research.

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