Inclusive Decays of Heavy Quark Hybrids

XXXIII International (Online) Workshop on High Energy Physics
"Hard Problems of Hadron Physics: Non-perturbative QCD & Related Quests"

Logunov Institute, Moscow
Nov 09, 2021

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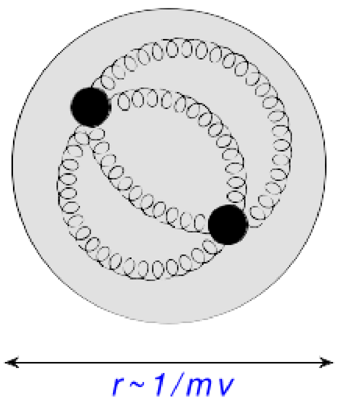
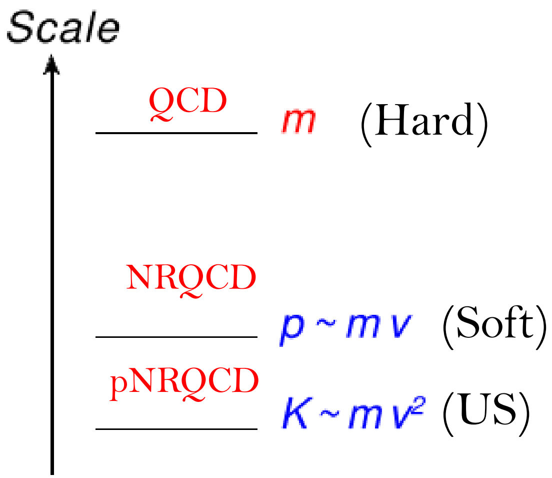
Outline

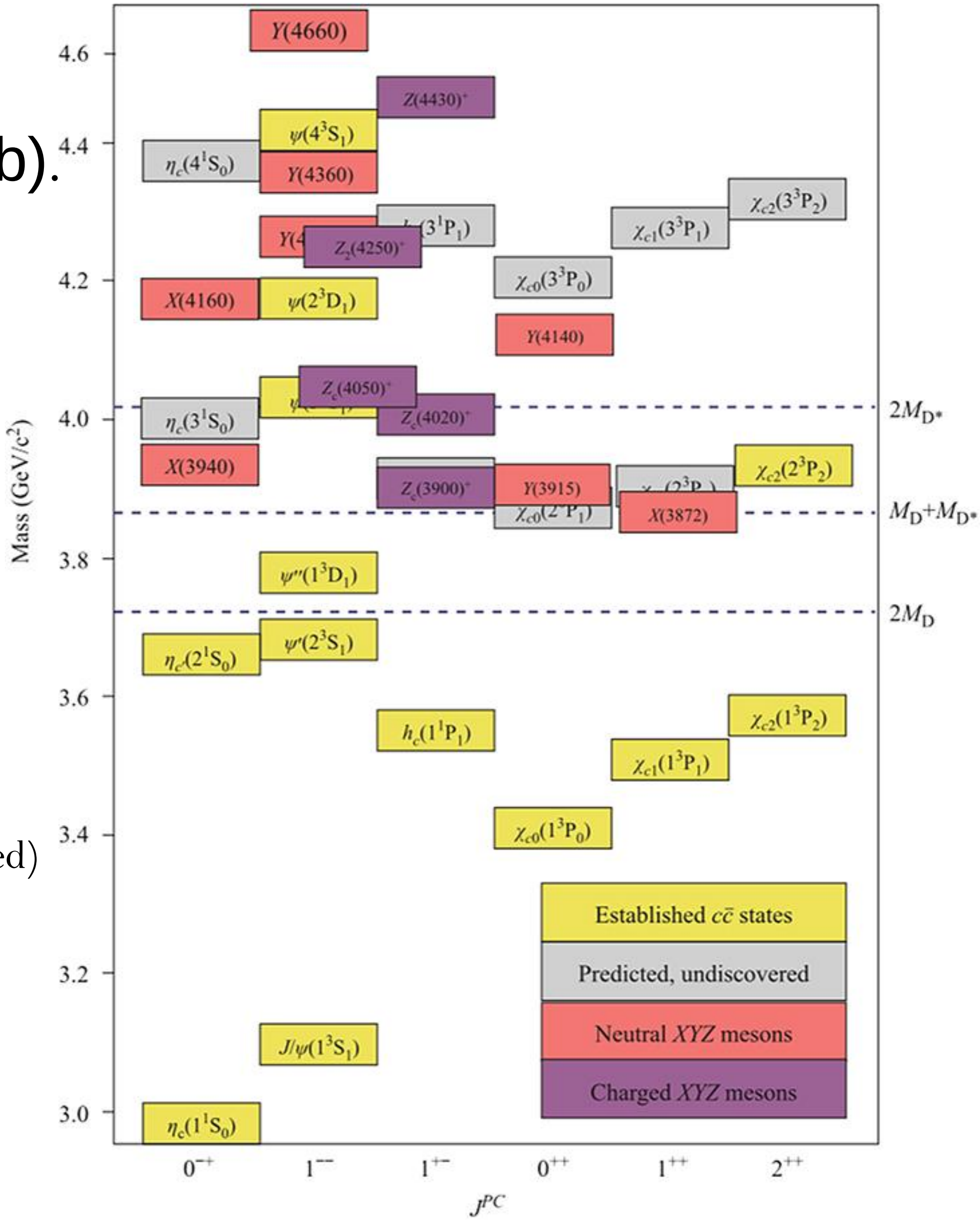
- Introduction to X Y Z mesons
- EFT for Quarkonium Hybrids
  - BO-EFT effective theory
  - Quarkonium Hybrid Spectrum
- Inclusive Decay Rates for hybrid to quarkonium
- Summary and Outlook

Introduction

- Quarkonium: Color singlet bound state of $Q\bar{Q}$ (Q = c, b).







- Hierarchy of Energy Scales in $Q\bar{Q}$:

$m \gg mv \gg mv^2, \Lambda_{QCD}$ (perturbative dynamics: Weakly Coupled)

$m \gg mv, \Lambda_{QCD} \gg mv^2$ (nonperturbative dynamics: Strongly Coupled)

- pNRQCD: Relevant EFT for Quarkonium.

Bodwin, Braaten & Lepage (1994), Mehen and Fleming, Phys. Rev. D73, 034502 (2005)

Brambilla, Vairo, and Rosch, Phys. Rev. D72, 034021 (2005) Luke, Manohar & Rothstein (2000)

Front. Phys. 10 101401 (2015)

Introduction

- Quark Model:

  Mesons: quark-antiquark states   Baryons: 3-quark states

- QCD spectrum also allows for more complex structures called as Exotics.

- Exotic states: XYZ mesons

  ✓ Quarkonium-like states that don't fit traditional $Q\bar{Q}$ spectrum.

  ✓ In some cases exotic quantum numbers (charged Z_c and Z_b states)

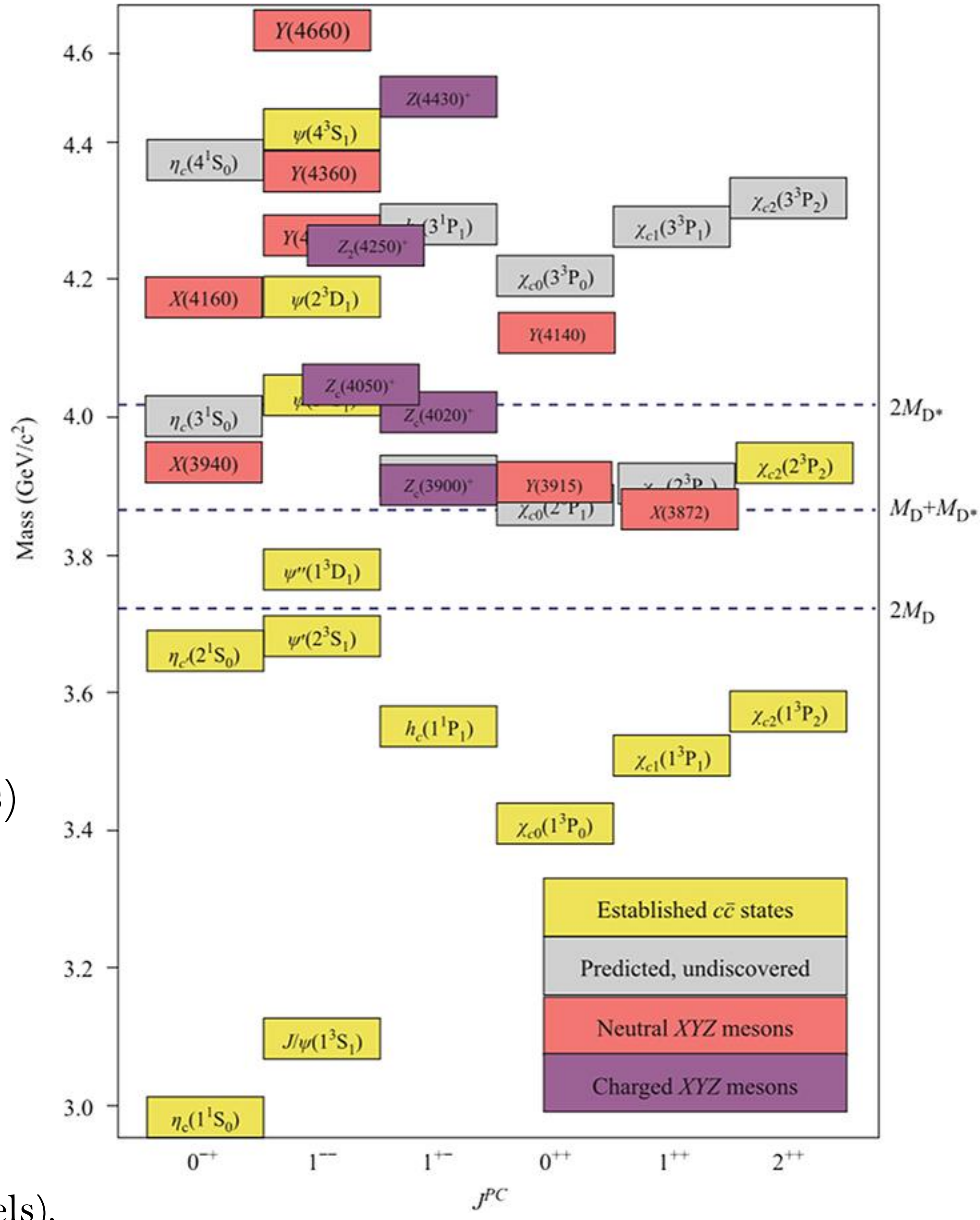
  For review see Brambilla et al. Phys. Reports. 873 (2020)

- X(3872): First exotic state discovered in 2003 by Belle.

  Phys. Rev. Lett. 91, 262001 (2003)

- Several **new** heavy quark exotic states have been discovered since 2003 (masses & decay rates measured in various channels).

  PDG 2021



Front. Phys. 10 101401 (2015)

Introduction

- Exotics broadly classified as
  - ❖ Structures with active gluons ❖ Multiquark states
- Several interpretations of Exotics:

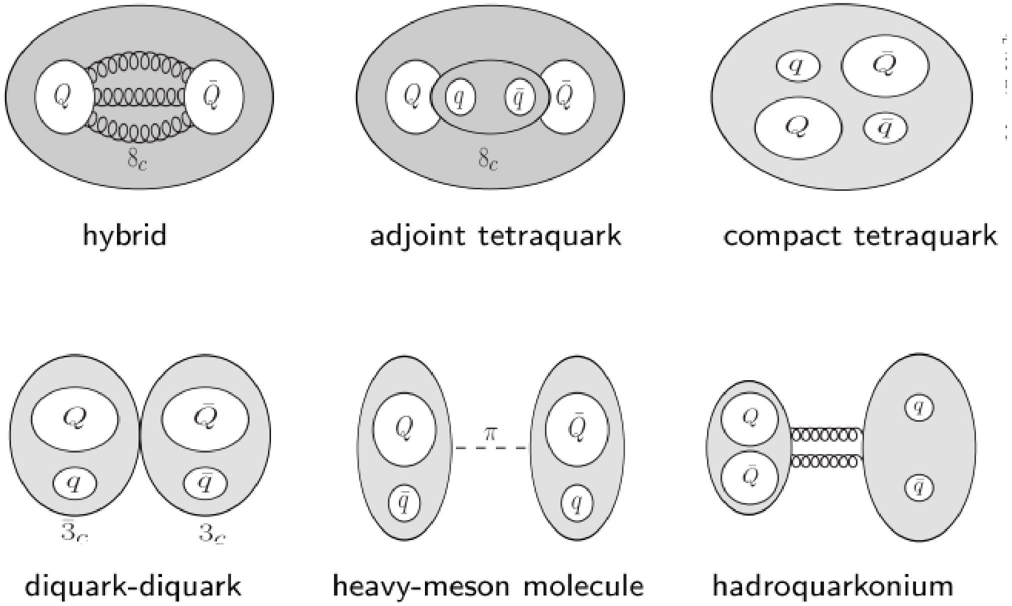
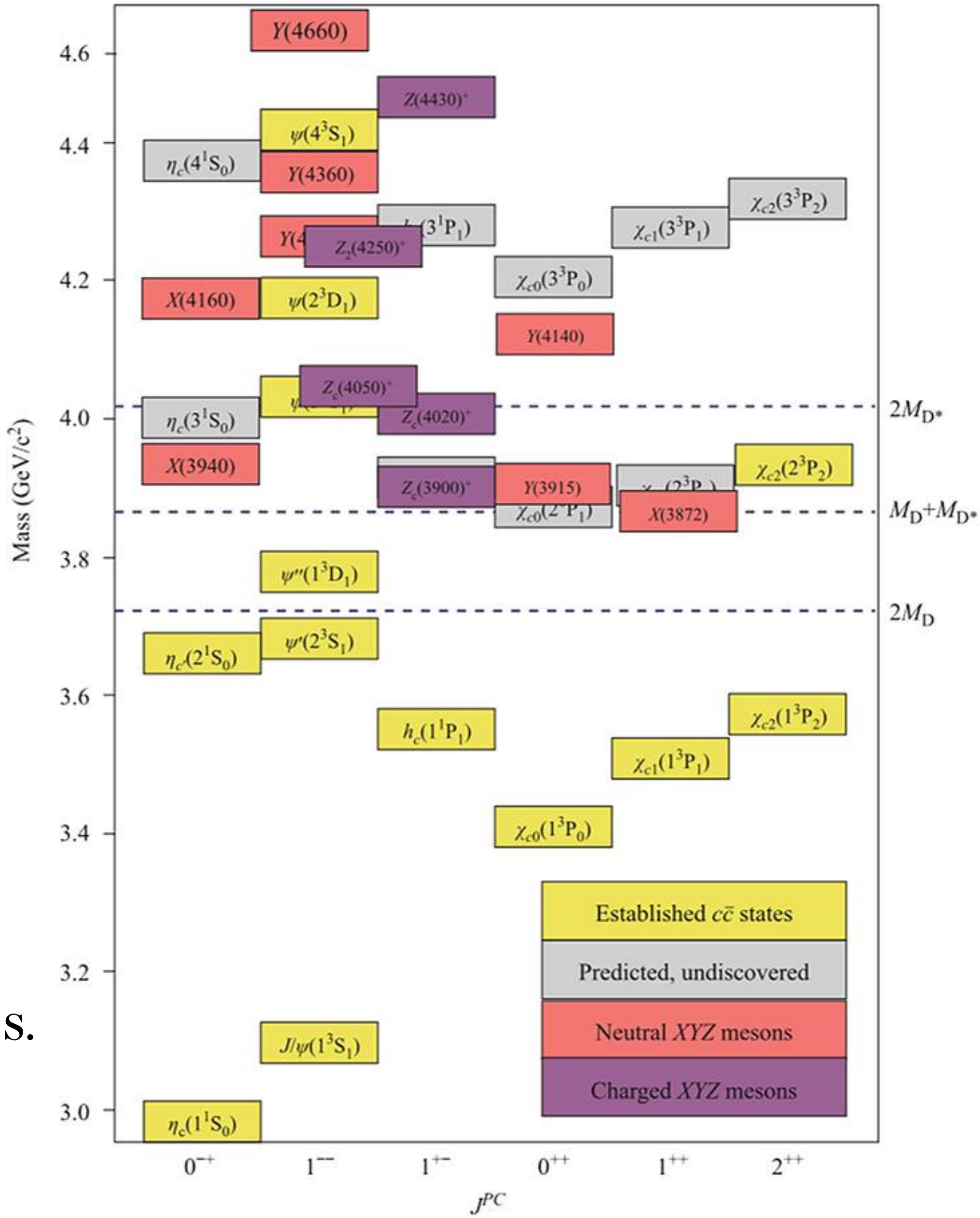


Figure from W.K.Lai talk

- No single model completely describes all the XYZ states.
- Hybrids ($Q\bar{Q}g$): Focus of this talk. Use EFT + lattice to have model independent description.



Front. Phys. 10 101401 (2015)

Quarkonium hybrids: EFT

- Hybrids ($Q\bar{Q}g$): Color singlet combination of color octet $Q\bar{Q}$ + gluonic excitations.

- Separation of scales in hybrids:

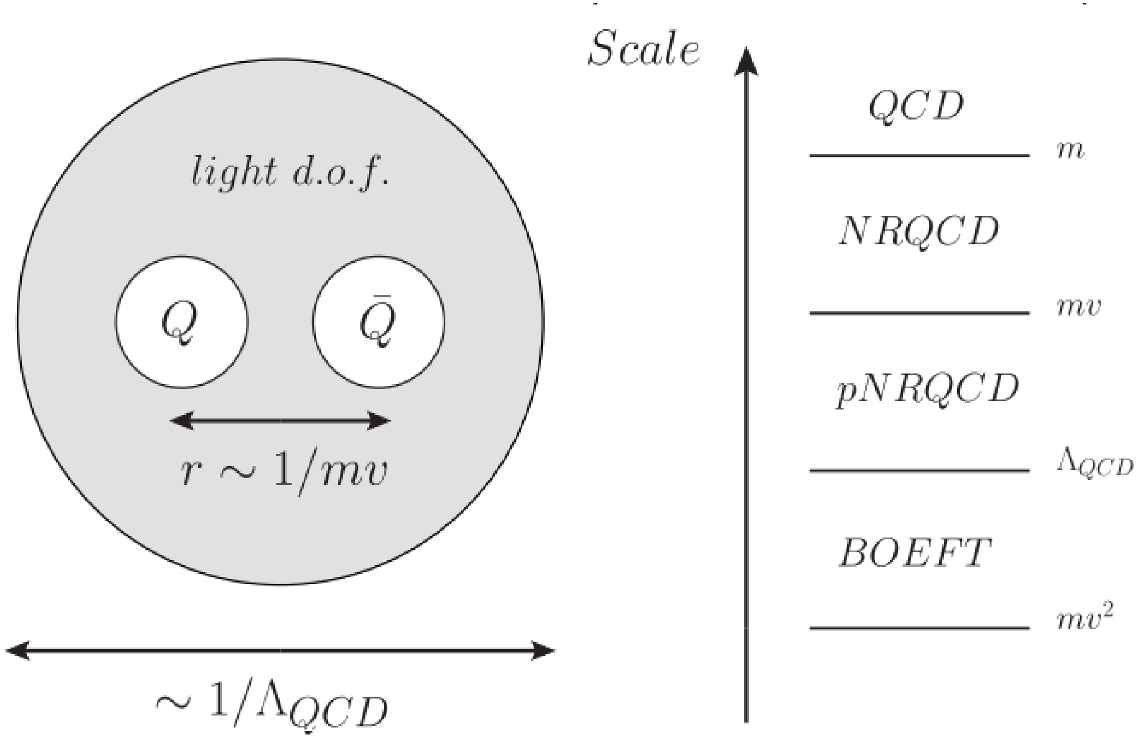
$$m \gg mv \gg \Lambda_{QCD} \gg mv^2$$

  - light d.o.f: Λ_{QCD}
  - Relative separation between heavy quarks: $r \sim 1/mv$
  - Heavy Quark K.E scale: mv^2

- Time-scale for dynamics of $Q\bar{Q}$: $\sim \frac{1}{mv^2} \gg \frac{1}{\Lambda_{QCD}}$

  Born-Oppenheimer Approximation

  Braaten, Langmack, Smith Phys. Rev. D. 90, 014044 (2014)



- Appropriate EFT framework for Hybrids: **Born-Oppenheimer EFT (BOEFT)**

$$\text{QCD} \rightarrow \text{NRQCD} \rightarrow \text{pNRQCD} \rightarrow \text{BOEFT}$$

Brambilla, Krein, Castellà , Vairo Phys. Rev. D. 97, (2018)
Berwein, Brambilla, Castellà , Vairo Phys. Rev. D. 92, (2015)
R. Oncala, J. Soto, Phys. Rev. D96 (2017)

Quarkonium hybrids: BOEFT

- Static limit (m→ ∞): Quantum #'s for hybrid

Irreducible representations of $D_{\infty h}$

- $\boldsymbol{K}$: angular momentum of light d.o.f.
  $\lambda = \hat{\boldsymbol{r}} \cdot \boldsymbol{K} = 0, \pm 1, \pm 2, \pm 3, \ldots$
  $\Lambda = |\lambda| = 0, 1, 2, 3, \ldots \; (\Sigma, \Pi, \Delta, \Phi, \ldots)$
- Eigenvalue of CP: $\eta = +1\,(g), \; -1\,(u)$
- σ: eigenvalue of relfection about a plane containing $\hat{\boldsymbol{r}}$ (only for Σ states)

- Static Energies (Σ, Π, Δ): Eigenvalue of NRQCD Hamiltonian in the static limit.

- For r→ 0: static energies are degenerate. Characterized by O(3)×C symmetry group.

Labelled by: $(K^{PC}, \Lambda_\eta^\sigma)$

Berwein, Brambilla, Castellà , Vairo Phys. Rev. D. 92, (2015)

Gluonic static energies

M. Foster and C. Michael, Phys. Rev. D59 (1999)

K. Juge, J. Kuti, C. Morningstar, Phys. Rev. Lett. 90 (2003)

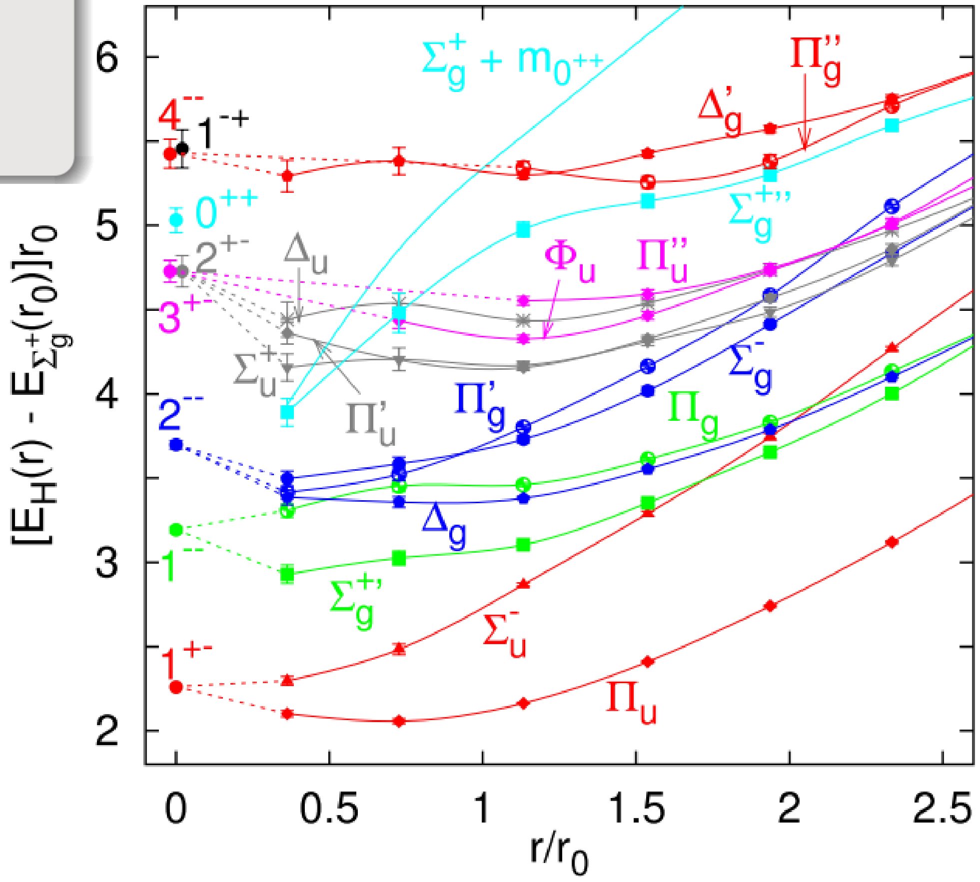


Fig from G. S. Bali and A. Pineda, Phys. Rev. D69 (2004)

Quarkonium hybrids: BOEFT

- Static limit (m→ ∞): Quantum #'s for hybrid

Irreducible representations of $D_{\infty h}$

- $\boldsymbol{K}$: angular momentum of light d.o.f.
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- Eigenvalue of CP: $\eta = +1\,(g),\ -1\,(u)$
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Gluonic operators characterizing Hybrids in Wilson loop

- Static Energies (Σ, Π, Δ): Eigenvalue of NRQCD Hamiltonian in the static limit.

Λ_η^σ	K^{PC}	O_n
Σ_u^-	1^{+-}	$\hat{\boldsymbol{r}} \cdot \boldsymbol{B},\ \hat{\boldsymbol{r}} \cdot (\boldsymbol{D} \times \boldsymbol{E})$
Π_u	1^{+-}	$\hat{\boldsymbol{r}} \times \boldsymbol{B},\ \hat{\boldsymbol{r}} \times (\boldsymbol{D} \times \boldsymbol{E})$
$\Sigma_g^{+\prime}$	1^{--}	$\hat{\boldsymbol{r}} \cdot \boldsymbol{E},\ \hat{\boldsymbol{r}} \cdot (\boldsymbol{D} \times \boldsymbol{B})$
Π_g	1^{--}	$\hat{\boldsymbol{r}} \times \boldsymbol{E},\ \hat{\boldsymbol{r}} \times (\boldsymbol{D} \times \boldsymbol{B})$
Σ_g^-	2^{--}	$(\hat{\boldsymbol{r}} \cdot \boldsymbol{D})(\hat{\boldsymbol{r}} \cdot \boldsymbol{B})$
Π_g'	2^{--}	$\hat{\boldsymbol{r}} \times ((\hat{\boldsymbol{r}} \cdot \boldsymbol{D})\boldsymbol{B} + \boldsymbol{D}(\hat{\boldsymbol{r}} \cdot \boldsymbol{B}))$
Δ_g	2^{--}	$(\hat{\boldsymbol{r}} \times \boldsymbol{D})^i (\hat{\boldsymbol{r}} \times \boldsymbol{B})^j + (\hat{\boldsymbol{r}} \times \boldsymbol{D})^j (\hat{\boldsymbol{r}} \times \boldsymbol{B})^i$
Σ_u^+	2^{+-}	$(\hat{\boldsymbol{r}} \cdot \boldsymbol{D})(\hat{\boldsymbol{r}} \cdot \boldsymbol{E})$
Π_u'	2^{+-}	$\hat{\boldsymbol{r}} \times ((\hat{\boldsymbol{r}} \cdot \boldsymbol{D})\boldsymbol{E} + \boldsymbol{D}(\hat{\boldsymbol{r}} \cdot \boldsymbol{E}))$
Δ_u	2^{+-}	$(\hat{\boldsymbol{r}} \times \boldsymbol{D})^i (\hat{\boldsymbol{r}} \times \boldsymbol{E})^j + (\hat{\boldsymbol{r}} \times \boldsymbol{D})^j (\hat{\boldsymbol{r}} \times \boldsymbol{E})^i$

- For r→ 0: static energies are degenerate. Characterized by O(3)×C symmetry group.

Labelled by: $(K^{PC}, \Lambda_\eta^\sigma)$

Berwein, Brambilla, Castellà , Vairo Phys. Rev. D. 92, (2015)

Focus on these two for low lying hybrids

Quarkonium hybrids: BOEFT

- BOEFT d.o.f involve color singlet fields $\hat{\Psi}_{\kappa\lambda}(\boldsymbol{r},\boldsymbol{R},t) \propto P^{i}_{\kappa\lambda} O^{a\dagger}(\boldsymbol{r},\boldsymbol{R},t) G^{ia}_{\kappa}(\boldsymbol{R},t)$
  - $O^{a\dagger}(\boldsymbol{r},\boldsymbol{R},t) G^{ia}_{\kappa}(\boldsymbol{R},t)$: Gluelump operator. Eigenvector of NRQCD Hamiltonian in (m⟶ ∞):

$$H^{(0)} O^{a\dagger}(\boldsymbol{r},\boldsymbol{R},\boldsymbol{t}) G^{ia}_{\kappa}(\boldsymbol{R},t)|0\rangle = (V_0(r)+\Lambda_\kappa)\, O^{a\dagger}(\boldsymbol{r},\boldsymbol{R},t) G^{ia}_{\kappa}(\boldsymbol{R},t)|0\rangle$$

Λ_κ : Gluelump energy

  - $P^{i}_{\kappa\lambda}$: Projection operators of light d.o.f along heavy quark-antiquark axis.

- BOEFT Lagrangian:

$$L_{\mathrm{BOEFT}} = \int d^3R\, d^3r \sum_{\kappa} \sum_{\lambda\lambda'} \hat{\Psi}^{\dagger}_{\kappa\lambda}(\boldsymbol{r},\boldsymbol{R},t) \left\{ i\partial_t - V_{\kappa\lambda\lambda'}(r) + P^{i\dagger}_{\kappa\lambda} \frac{\boldsymbol{\nabla}^2_r}{m} P^{i}_{\kappa\lambda'} \right\} \hat{\Psi}_{\kappa\lambda'}(\boldsymbol{r},\boldsymbol{R},t) + \ldots$$

- Schrödinger Eq: Dynamics of $Q\bar{Q}$ at scale $mv^2 \ll \Lambda_{QCD}$

Schrödinger equation

$$\left[-P^{i\dagger}_{\kappa\lambda} \frac{\boldsymbol{\nabla}^2_r}{m} P^{i}_{\kappa\lambda'} + V_{\kappa\lambda\lambda'}(r) \right] \Psi^{n}_{\kappa\lambda'}(\boldsymbol{r}) = E_n \Psi^{n}_{\kappa\lambda}(\boldsymbol{r})$$

Hybrid wf

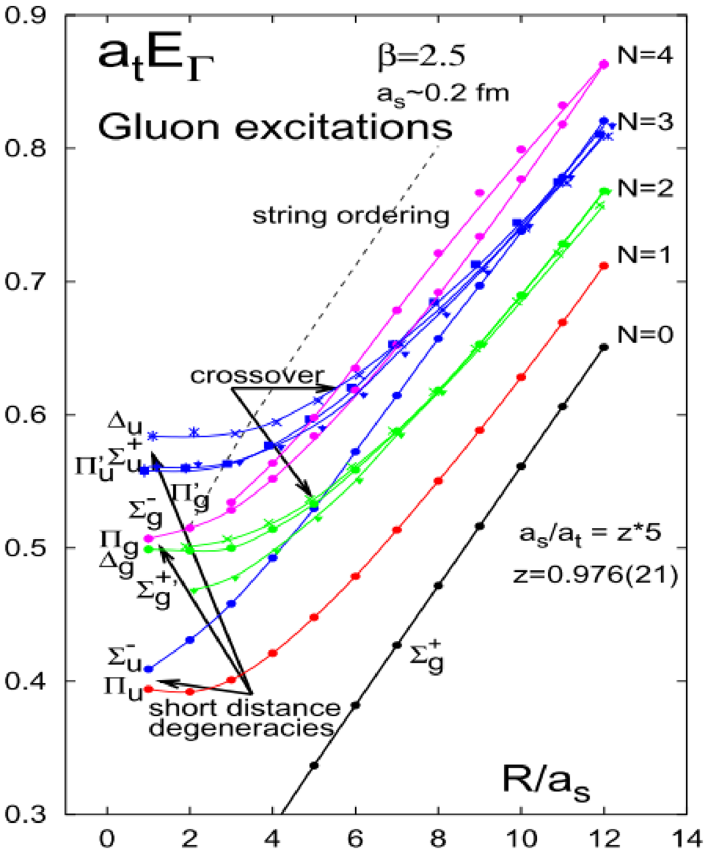
- Coupled Eq. due to projection operators. Mixes Σ_u and Π_u states.

Brambilla, Krein, Castellà , Vairo Phys. Rev. D. 97, (2018) Berwein, Brambilla, Castellà , Vairo Phys. Rev. D. 92, (2015)

Quarkonium hybrids: Spectrum

- Lattice potentials for solving the Schrödinger Eq:

Gluonic Static energies from lattice:



K. Juge, J. Kuti, C. Morningstar, Phys. Rev. Lett. 90 (2003)

Potential after fitting lattice data:

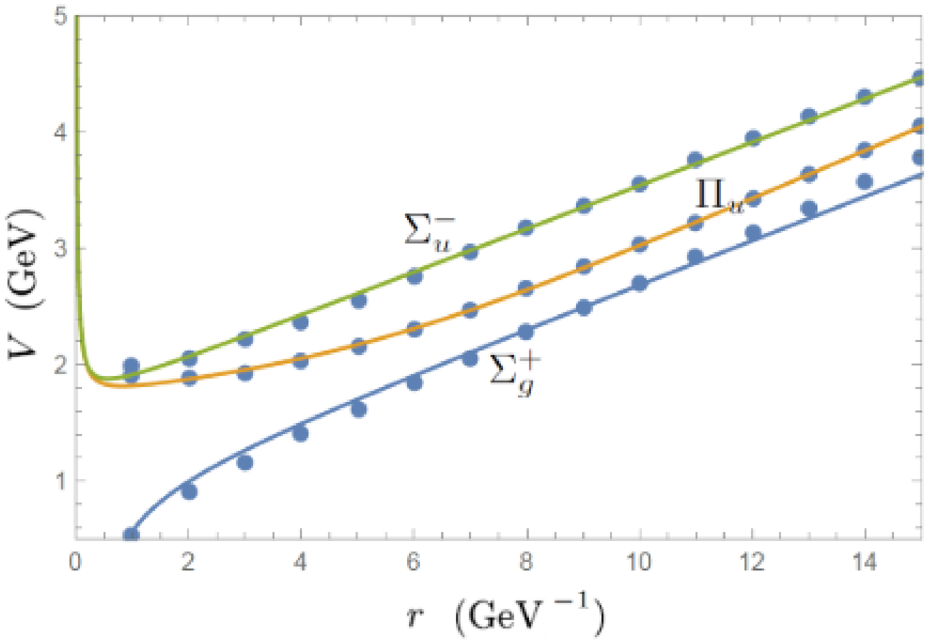


Fig from R. Oncala, J. Soto, Phys. Rev. D96 (2017)

Quarkonium hybrids: Spectrum

- Lattice potentials for solving the Schrödinger Eq:

N. Brambilla, W.K. Lai, AM, A. Vairo (in progress)

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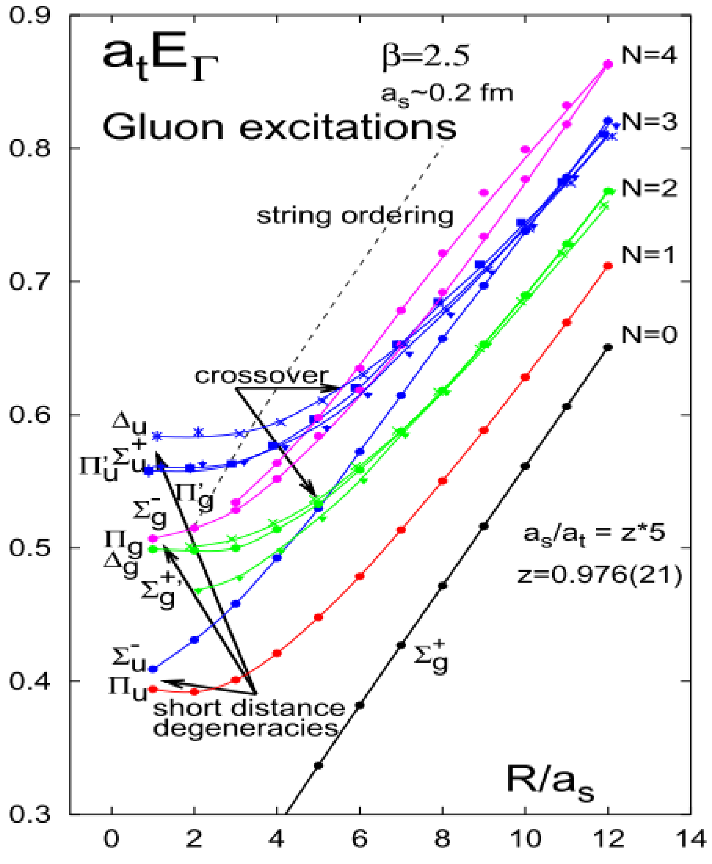


K. Juge, J. Kuti, C. Morningstar, Phys. Rev. Lett. 90 (2003)

RS-Scheme Quarkonium Potential:

$$V_{\Sigma_g^+}(r) = \begin{cases} V_s^{\rm RS}(r,\mu,\nu_f), & r < 0.38\, fm \\ \sigma r + V_0', & r \geq 0.38\, fm \end{cases}$$

$$\sigma = 0.18\,\mathrm{GeV}^2, V_0' = -0.581\,\mathrm{GeV}$$

Cheng et al Phys. Rev. D. 77, (2008)

✓ RS-scheme: reproduce $\Upsilon(1b)$ mass: 9460 MeV

Pineda JHEP 06, (2001)

RS-Scheme Hybrid Potential:

$$E_n^{(0)}(r) = \begin{cases} V_o^{RS}(\nu_f) + \Lambda_H^{RS}(\nu_f) + b_n r^2, & r < 0.25\, fm \\ \frac{a_1}{r} + \sqrt{a_2 r^2 + a_3} + a_4, & r > 0.25\, fm \end{cases}$$

$a_1^\Sigma = 0.000\,\mathrm{GeV fm}$, $a_2^\Sigma = 1.543\,\mathrm{GeV}^2/\mathrm{fm}^2$, $a_3^\Sigma = 0.599\,\mathrm{GeV}^2$, $a_4^\Sigma = 0.154\,\mathrm{GeV}$,

$a_1^\Pi = 0.023\,\mathrm{GeV fm}$, $a_2^\Pi = 2.716\,\mathrm{GeV}^2/\mathrm{fm}^2$, $a_3^\Pi = 11.091\,\mathrm{GeV}^2$, $a_4^\Pi = -2.536\,\mathrm{GeV}$,

$b_\Sigma = 1.246\,\mathrm{GeV/fm}^2$, $b_\Pi = 0.000\,\mathrm{GeV/fm}^2$

✓ Perturbative RS-scheme potentials $V_s^{\rm RS}$ and $V_o^{\rm RS}$ upto order α_s^3.

Berwein, Brambilla, Castellà , Vairo Phys. Rev. D. 92, (2015)   Bali and Pineda Phys. Rev. D. 69, (2004)

Quarkonium hybrids: Spectrum

- Results for Hybrids from Berwein, Brambilla, Castellà , Vairo Phys. Rev. D. 92, (2015)

$$m_c^{RS} = 1.477(40)\ \text{GeV}$$
$$m_b^{RS} = 4.863(55)\ \text{GeV}$$

multiplet	J^{PC}	$c\bar{c}$				$b\bar{c}$				$b\bar{b}$			
		m_H	$\langle 1/r \rangle$	E_{kin}	P_Π	m_H	$\langle 1/r \rangle$	E_{kin}	P_Π	m_H	$\langle 1/r \rangle$	E_{kin}	P_Π
H_1	$\{1^{--}, (0,1,2)^{-+}\}$	4.15	0.42	0.16	0.82	7.48	0.46	0.13	0.83	10.79	0.53	0.09	0.86
H_1'		4.51	0.34	0.34	0.87	7.76	0.38	0.27	0.87	10.98	0.47	0.19	0.87
H_2	$\{1^{++}, (0,1,2)^{+-}\}$	4.28	0.28	0.24	1.00	7.58	0.31	0.19	1.00	10.84	0.37	0.13	1.00
H_2'		4.67	0.25	0.42	1.00	7.89	0.28	0.34	1.00	11.06	0.34	0.23	1.00
H_3	$\{0^{++}, 1^{+-}\}$	4.59	0.32	0.32	0.00	7.85	0.37	0.27	0.00	11.06	0.46	0.19	0.00
H_4	$\{2^{++}, (1,2,3)^{+-}\}$	4.37	0.28	0.27	0.83	7.65	0.31	0.22	0.84	10.90	0.37	0.15	0.87
H_5	$\{2^{--}, (1,2,3)^{-+}\}$	4.48	0.23	0.33	1.00	7.73	0.25	0.27	1.00	10.95	0.30	0.18	1.00
H_6	$\{3^{--}, (2,3,4)^{-+}\}$	4.57	0.22	0.37	0.85	7.82	0.25	0.30	0.87	11.01	0.30	0.20	0.89
H_7	$\{3^{++}, (2,3,4)^{+-}\}$	4.67	0.19	0.43	1.00	7.89	0.22	0.35	1.00	11.05	0.26	0.24	1.00

Other notation of hybrid states

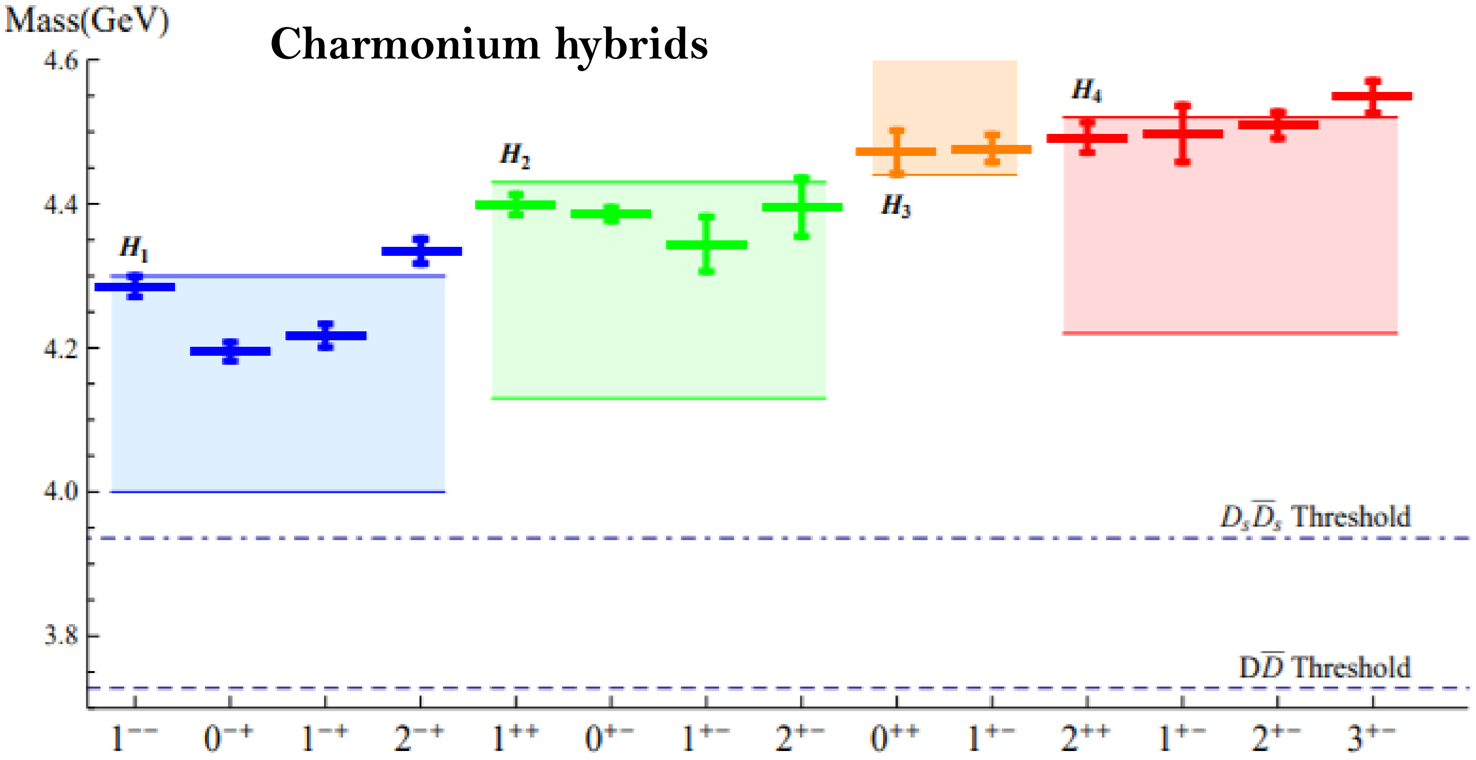
		l	$J^{PC}\{s=0, s=1\}$	$E_n^{(0)}$
$N(s/d)_1$ →	H_1	1	$\{1^{--}, (0,1,2)^{-+}\}$	Σ_u^-, Π_u
Np_1 →	H_2	1	$\{1^{++}, (0,1,2)^{+-}\}$	Π_u
Np_0 →	H_3	0	$\{0^{++}, 1^{+-}\}$	Σ_u^-
$N(p/f)_2$ →	H_4	2	$\{2^{++}, (1,2,3)^{+-}\}$	Σ_u^-, Π_u
Nd_2 →	H_5	2	$\{2^{--}, (1,2,3)^{-+}\}$	Π_u

Braaten, Langmack, Smith Phys. Rev. D. 90, 014044 (2014)

R. Oncala, J. Soto, Phys. Rev. D96 (2017)

Quarkonium hybrids: Spectrum

- Comparison with lattice results:



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$N(p/f)_2$ →	H_4	2	$\{2^{++}, (1,2,3)^{+-}\}$	Σ_u^-, Π_u
Nd_2 →	H_5	2	$\{2^{--}, (1,2,3)^{-+}\}$	Π_u

- Λ- doubling: opposite parity states non-degenerate. Also confirmed by lattice

Berwein, Brambilla, Castellà , Vairo Phys. Rev. D. 92, (2015)

Braaten, Langmack, Smith Phys. Rev. D. 90, 014044 (2014)

R. Oncala, J. Soto, Phys. Rev. D96 (2017)

Lattice data from
Hadron Spectrum collaboration JHEP 1207 (2012) 126

Inclusive Decays

- Dozens of XYZ states have been discovered (mass and decay rates measured) but physics still unknown.

- Several theoretical models for exotic states but no general consensus.

- Most of the exotic states discovered from decays to quarkonium. So, decays might provide information on the structure of XYZ.

- Consider the process: $\boldsymbol{H_m \to Q_n + X}$; H_m: low-lying hybrid, Q_n: low-lying quarkonium.

  ✓ $\Delta \equiv \boldsymbol{m_H - m_Q} \gtrsim \boldsymbol{1}$ **GeV**. For low-lying states, we observe that $\Delta \gg \Lambda_{QCD} \gg mv^2$

  ✓ Hierarchy of Scales: $\mathbf{mv \gg \Delta \gg \Lambda_{QCD} \gg mv^2}$

- Start with **pNRQCD** effective theory and **obtain BOEFT** by **matching**: Integrate out modes of scale $\sim \Delta$ and $\sim \Lambda_{QCD}$.

N. Brambilla, W.K. Lai, AM, A. Vairo (in progress)

Inclusive Decays

- pNRQCD Lagrangian:

Weakly-coupled pNRQCD Lagrangian

$$L_{\text{pNRQCD}} = \int d^3R \Big\{ \int d^3r \Big(\text{Tr}\left[\text{S}^\dagger (i\partial_0 - h_s) \text{S} + \text{O}^\dagger (iD_0 - h_o) \text{O} \right]$$

$$+ g\text{Tr}\left[\text{S}^\dagger \boldsymbol{r} \cdot \boldsymbol{E}\, \text{O} + \text{O}^\dagger \boldsymbol{r} \cdot \boldsymbol{E}\, \text{S} + \frac{1}{2} \text{O}^\dagger \boldsymbol{r} \cdot \{\boldsymbol{E}, \text{O}\} \right] + \frac{g}{4m} \text{Tr}\left[\text{O}^\dagger \boldsymbol{L}_{Q\bar{Q}} \cdot [\boldsymbol{B}, \text{O}] \right]$$

$$+ \frac{gc_F}{m} \text{Tr}\left[\text{S}^\dagger (\boldsymbol{S}_1 - \boldsymbol{S}_2) \cdot \boldsymbol{B}\, \text{O} + \text{O}^\dagger (\boldsymbol{S}_1 - \boldsymbol{S}_2) \cdot \boldsymbol{B}\, \text{S} + \text{O}^\dagger \boldsymbol{S}_1 \cdot \boldsymbol{B}\, \text{O} - \text{O}^\dagger \boldsymbol{S}_2 \text{O} \cdot \boldsymbol{B} \right] - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu\, a} \Big\}$$

- BOEFT:

BOEFT Hamiltonian

$$H_{\text{BOEFT}} = \int d^3x \int d^3R\, \text{Tr}\left[H^{i\dagger} \left(h_o \delta^{ij} + V^{ij}_{soft} \right) H^j \right]$$

Potential term in BOEFT

- Decays are computed from local imaginary terms in the BOEFT Lagrangian.

- Imaginary term in V^{ij}_{soft} from 1-loop diagram in pNRQCD and then **matching** to BOEFT .

N. Brambilla, W.K. Lai, AM, A. Vairo (in progress)

Inclusive Decays

- pNRQCD Lagrangian:

N. Brambilla, W.K. Lai, AM, A. Vairo (in progress)

Weakly-coupled pNRQCD Lagrangian

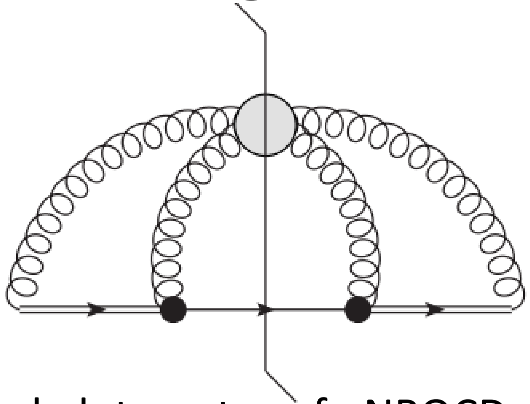
$$L_{\rm pNRQCD} = \int d^3R \Big\{ \int d^3r \Big(\mathrm{Tr}\left[\mathrm{S}^\dagger \left(i\partial_0 - h_s \right) \mathrm{S} + \mathrm{O}^\dagger \left(iD_0 - h_o \right) \mathrm{O} \right]$$

$$+ g\mathrm{Tr}\left[\mathrm{S}^\dagger \boldsymbol{r}\cdot \boldsymbol{E}\,\mathrm{O} + \mathrm{O}^\dagger \boldsymbol{r}\cdot\boldsymbol{E}\,\mathrm{S} + \frac{1}{2}\mathrm{O}^\dagger \boldsymbol{r}\cdot\{\boldsymbol{E},\mathrm{O}\}\right] + \frac{g}{4m}\mathrm{Tr}\left[\mathrm{O}^\dagger \boldsymbol{L}_{Q\bar{Q}}\cdot[\boldsymbol{B},\mathrm{O}]\right]$$

$$+ \frac{gc_F}{m}\mathrm{Tr}\left[\mathrm{S}^\dagger (\boldsymbol{S}_1 - \boldsymbol{S}_2)\cdot\boldsymbol{B}\,\mathrm{O} + \mathrm{O}^\dagger(\boldsymbol{S}_1 - \boldsymbol{S}_2)\cdot\boldsymbol{B}\,\mathrm{S} + \mathrm{O}^\dagger \boldsymbol{S}_1\cdot\boldsymbol{B}\,\mathrm{O} - \mathrm{O}^\dagger\boldsymbol{S}_2\mathrm{O}\cdot\boldsymbol{B}\right] - \frac{1}{4}G^a_{\mu\nu}G^{\mu\nu\,a}\Big\}$$

- Spin preserving decays
- Spin flipping decays

- 1-loop diagram in pNRQCD contributing to Im V^{ij}_{soft} in BOEFT:



Black dot: vertex of pNRQCD

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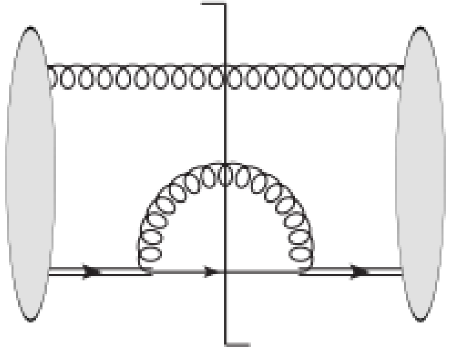

Spectator gluon approximation

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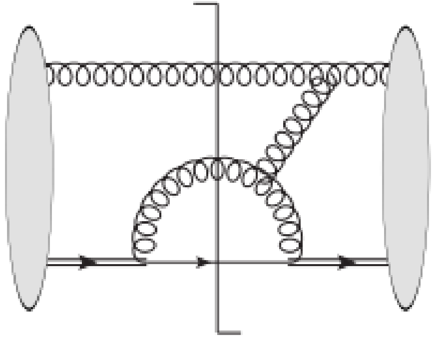


example of α_s correction

Inclusive Decays

N. Brambilla, W.K. Lai, AM, A. Vairo (in progress)

Spin-preserving inclusive decay rate for $H_m \to Q_n + X$

$$\Gamma(H_m \to Q_n + X) = \frac{4\alpha_s T_F}{3N_c} \sum_{n'} |h_{nn'}|^2 \sum_{q,q'} \int dE \int dE' \, f^i_{mq}(E) g^j_{qn}(E) \times g^{j\dagger}_{q'n}(E') f^{i\dagger}_{mq'}(E') (\Lambda + E/2 + E'/2 - E^s_n)^3$$

Depends on several Overlap functions:

$$f^i_{mq}(E) = \left[\int d^3r \, \Psi^{i\dagger}_m(\boldsymbol{r}) \Phi^o_{E,q}(\boldsymbol{r})\right]$$

$$g^j_{qn}(E) = \left[\int d^3r \, \Phi^{o\dagger}_{E,q}(\boldsymbol{r}) r^j \Phi^s_n(\boldsymbol{r})\right]$$

$$h_{nn'} = \int d^3r \, \Phi^{s\dagger}_{n'}(\boldsymbol{r}) \Phi^Q_n(\boldsymbol{r})$$

- Double integral over the energies E, E' of the octet wave function $\Phi^{o}_{E,q}$.
- Depends on overlap function of singlet w.f Φ^{s}_{n} and quarkonium w.f Φ^{Q}_{n}.
- Last factor $\left(\Lambda + E/2 + E'/2 - E^s_n\right)^3 \sim \Delta^3$ parametrically. $(\Delta \equiv \boldsymbol{m_H} - \boldsymbol{m_Q})$
- Above result different from the one in R. Oncala, J. Soto, Phys. Rev. D96, 014004 (2017).

Inclusive Decays

N. Brambilla, W.K. Lai, AM, A. Vairo
(in progress)

Spin-preserving inclusive decay rate for $H_m \to Q_n + X$

$$\Gamma(H_m \to Q_n + X) = \frac{4\alpha_s T_F}{3N_c} \sum_{n'} |h_{nn'}|^2 \sum_{q,q'} \int dE \int dE' \, f^i_{mq}(E) g^j_{qn}(E) \times g^{j\dagger}_{q'n}(E') f^{i\dagger}_{mq'}(E') (\Lambda + E/2 + E'/2 - E^s_n)^3$$

Assumption:
$f^i_{mq}(E) \neq 0$ only for $E_m \approx E + \Lambda$

$h_{nn'} \approx 1$ and $E^Q_m \approx E^s_m$

Spin-preserving inclusive decay rate for $H_m \to Q_n + X$

$$\Gamma(H_m \to Q_n + X) = \frac{4\alpha_s T_F}{3N_c} (E_m - E^Q_n)^3 \, T^{ij} \, (T^{ij})^*$$

$$T^{ij} \equiv \int d^3r \, \Psi^{i\dagger}_m(\boldsymbol{r}) r^j \, \Phi^Q_n(\boldsymbol{r})$$

- Above result looks similar to the one in R. Oncala, J. Soto, Phys. Rev. D96, 014004 (2017). In general has **tensor structure T^{ij}** that agrees with J. Castellà, E. Passemar, arXiv:2104.03975.

Inclusive Decays

N. Brambilla, W.K. Lai, AM, A. Vairo
(in progress)

Spin-preserving inclusive decay rate for $H_m \to Q_n + X$

$$\Gamma(H_m \to Q_n + X) = \frac{4\alpha_s T_F}{3N_c} (E_m - E_n^Q)^3 \, T^{ij} \, (T^{ij})^*$$

$$T^{ij} \equiv \int d^3r \, \Psi_m^{i\dagger}(\boldsymbol{r}) r^j \, \Phi_n^Q(\boldsymbol{r})$$

- R. Oncala, J. Soto, Phys. Rev. D96, 014004 (2017): only **diagonal elements** $\boldsymbol{T^{ii}}$ are considered

Inclusive decay rate for $H_m \to Q_n + X$ computed by Oncala and Soto

$$\Gamma^{\text{Oncala}}(H_m \to Q_n + X) = \frac{4\alpha_s T_F}{3N_c} (E_m - E_n^Q)^3 \, T^{ii} \, (T^{jj})^*$$

- ~~$\boldsymbol{T^{ii}}$ leads to **selection rule**: Hybrids such as Np_1 (H_2) where L=J **don't decay to quarkonium**.~~
- $\boldsymbol{T^{ij}}$: allows for the decay of Np_1 (H_2) hybrid decays.

Preliminary Results

N. Brambilla, W.K. Lai, AM, A. Vairo (in progress)

$m_c^{RS} = 1.477(40)$ GeV

$m_b^{RS} = 4.863(55)$ GeV

- **Spin conserving** decay rates for **Charm hybrids**:

Decays not allowed in R. Oncala, J. Soto, Phys. Rev. D96, 014004 (2017) (bracket: rows $1p_1 \to 1s$ through $3p_1 \to 2s$)

$ML_J \to NL$	ΔE (GeV)	$\alpha_s(\Delta E)$	$\sqrt{T^{ij}(T^{ij})^*}$ (GeV^{-1})	Γ (MeV)
charmonium hybrid decay				
$1p_0 \to 1s$	1.598	0.291	1.432	543.03 ± 114.91
$1p_0 \to 2s$	1.101	0.346	1.550	246.33 ± 61.34
$2p_0 \to 1s$	2.063	0.266	0.137	9.68 ± 1.90
$2p_0 \to 2s$	1.566	0.294	2.101	1108.97 ± 236.71
$3p_0 \to 1s$	2.482	0.249	0.005	4.62 ± 0.86
$3p_0 \to 2s$	1.984	0.269	0.009	0.04 ± 0.01
$1p_1 \to 1s$	1.295	0.320	2.366	863.911 ± 200.122
$2p_1 \to 1s$	1.675	0.287	0.705	148.85 ± 31.05
$2p_1 \to 2s$	1.178	0.335	2.843	983.00 ± 237.41
$3p_1 \to 1s$	2.044	0.267	0.311	48.77 ± 9.60
$3p_1 \to 2s$	1.547	0.296	0.711	123.15 ± 26.42
$1(s/d)_1 \to 1p$	0.844	0.398	5.268	1475.13 ± 416.36
$2(s/d)_1 \to 1p$	1.196	0.332	3.051	1174.56 ± 281.83
$4(s/d)_1 \to 1p$	1.563	0.295	1.633	667.04 ± 142.47

- Error bars from higher order corrections in α_s.

Preliminary Results

N. Brambilla, W.K. Lai, AM, A. Vairo

(in progress)

- **Spin conserving** decay rates for **Bottom hybrids**:

$ML_J \to NL$	ΔE (GeV)	$\alpha_s\,(\Delta E)$	$\sqrt{T^{ij}(T^{ij})^*}$ (GeV^{-1})	Γ (MeV)
bottomonium hybrid decay				
$1p_0 \to 1s$	1.605	0.292	0.732	143.56 ± 30.34
$1p_0 \to 2s$	1.218	0.329	1.295	221.67 ± 53.75
$2p_0 \to 1s$	1.892	0.274	0.245	24.78 ± 4.99
$2p_0 \to 2s$	1.505	0.299	1.029	240.13 ± 52.12
$2p_0 \to 3s$	1.219	0.329	1.822	439.98 ± 104.66
$3p_0 \to 1s$	2.156	0.262	0.132	10.18 ± 1.97
$3p_0 \to 2s$	1.769	0.281	0.202	14.04 ± 2.88
$4p_0 \to 1s$	2.403	0.252	0.084	5.46 ± 1.02
$4p_0 \to 2s$	2.017	0.268	0.117	6.70 ± 1.32
$4p_0 \to 3s$	1.731	0.283	0.124	5.02 ± 1.04
$1p_1 \to 1s$	1.386	0.310	1.082	215.01 ± 48.07
$2p_1 \to 1s$	1.600	0.292	0.645	110.48 ± 23.37
$2p_1 \to 2s$	1.213	0.330	0.764	76.33 ± 18.20
$3p_1 \to 1s$	1.810	0.278	0.395	57.39 ± 11.70
$3p_1 \to 2s$	1.424	0.307	0.417	75.48 ± 16.78
$2(s/d)_1 \to 1p$	1.214	0.330	1.358	241.76 ± 57.62
$3(s/d)_1 \to 1p$	1.410	0.308	2.125	866.35 ± 193.37
$3(s/d)_1 \to 2p$	1.126	0.342	1.827	362.30 ± 89.30
$4(s/d)_1 \to 1p$	1.445	0.305	3.406	2371.45 ± 523.723
$4(s/d)_1 \to 2p$	1.161	0.337	3.864	1750.64 ± 425.58
$5(s/d)_1 \to 1p$	1.620	0.290	0.467	41.51 ± 8.75
$5(s/d)_1 \to 2p$	1.336	0.316	1.347	303.21 ± 69.28

$m_c^{RS} = 1.477(40)$ GeV

$m_b^{RS} = 4.863(55)$ GeV

Decays not allowed in R. Oncala, J. Soto, Phys. Rev. D96, 014004 (2017)

- Error bars from higher order corrections in α_s.

Inclusive Decays

N. Brambilla, W.K. Lai, AM, A. Vairo
(in progress)

- Result for **spin-flipping** decays due to $\boldsymbol{S}.\boldsymbol{B}$ term:

Spin-flipping inclusive decay rate for $H_m \to Q_n + X$

$$\Gamma(H_m \to Q_n + X) = \frac{4\alpha_s T_F c_F^2}{3N_c m_Q^2} T^{ij} \left(T^{ij}\right)^* (E_m - E_n^Q)^3$$

$$T^{ij} \equiv \int d^3r \, \Psi_m^{i\dagger}(\boldsymbol{r}) S^j \Phi_n^Q(\boldsymbol{r})$$

- Spin-flipping decays implies $|S_H = 1 > \dashrightarrow |S_Q = 0 >$ & $|S_H = 0 > \dashrightarrow |S_Q = 1 >$
- Above result agrees with J. Castellà, E. Passemar, arXiv:2104.03975.

- $Q_m \to Q_n + X$ spin-flipping decays: Decay rate suppressed by additional $(\boldsymbol{r} \cdot \boldsymbol{E})^2 \sim v^2$ vertex factor.

Preliminary Results

N. Brambilla, W.K. Lai, AM, A. Vairo (in progress)

- **Spin flipping** decay rates for **Charm hybrids**:

$m_c^{RS} = 1.477(40)$ GeV

$m_b^{RS} = 4.863(55)$ GeV

$ML_J \to NL$	ΔE (GeV)	$\alpha_S\,(\Delta E)$	Γ(MeV) $(1 \to 0)$	Γ(MeV) $(0 \to 1)$
Charmonium hybrid decay				
$1p_0 \to 1p$	0.993	0.365	35.60 ± 9.30	106.81 ± 27.90
$2p_0 \to 1p$	1.426	0.307	1.44 ± 0.32	4.31 ± 0.96
$2p_0 \to 2p$	0.952	0.373	31.34 ± 8.35	94.03 ± 25.04
$3p_0 \to 1p$	1.804	0.279	0.41 ± 0.08	1.22 ± 0.25
$3p_0 \to 2p$	1.330	0.316	1.52 ± 0.35	4.56 ± 1.04
$1p_1 \to 1p$	0.653	0.467	46.57 ± 14.75	139.72 ± 44.24
$2p_1 \to 1p$	1.018	0.360	14.51 ± 3.75	43.53 ± 11.24
$2p_1 \to 2p$	0.544	0.531	26.13 ± 9.03	78.39 ± 27.08
$3p_1 \to 1p$	1.369	0.312	5.69 ± 1.29	17.07 ± 3.86
$3p_1 \to 2p$	0.895	0.385	10.97 ± 3.01	32.90 ± 9.03
$1(s/d)_1 \to 1s$	0.944	0.374	74.98 ± 20.05	224.95 ± 60.16
$2(s/d)_1 \to 1s$	1.288	0.321	32.51 ± 7.55	97.53 ± 22.65
$2(s/d)_1 \to 2s$	0.678	0.455	18.86 ± 5.87	56.57 ± 17.61
$3(s/d)_1 \to 1s$	1.624	0.290	7.73 ± 1.63	23.18 ± 4.88
$3(s/d)_1 \to 2s$	1.014	0.360	1.04 ± 0.27	3.11 ± 0.81
$4(s/d)_1 \to 1s$	1.650	0.288	7.35 ± 1.54	22.06 ± 4.62
$4(s/d)_1 \to 2s$	1.040	0.356	38.86 ± 9.93	116.58 ± 29.80

- Error bars from higher order corrections in α_s.

Preliminary Results

N. Brambilla, W.K. Lai, AM, A. Vairo (in progress)

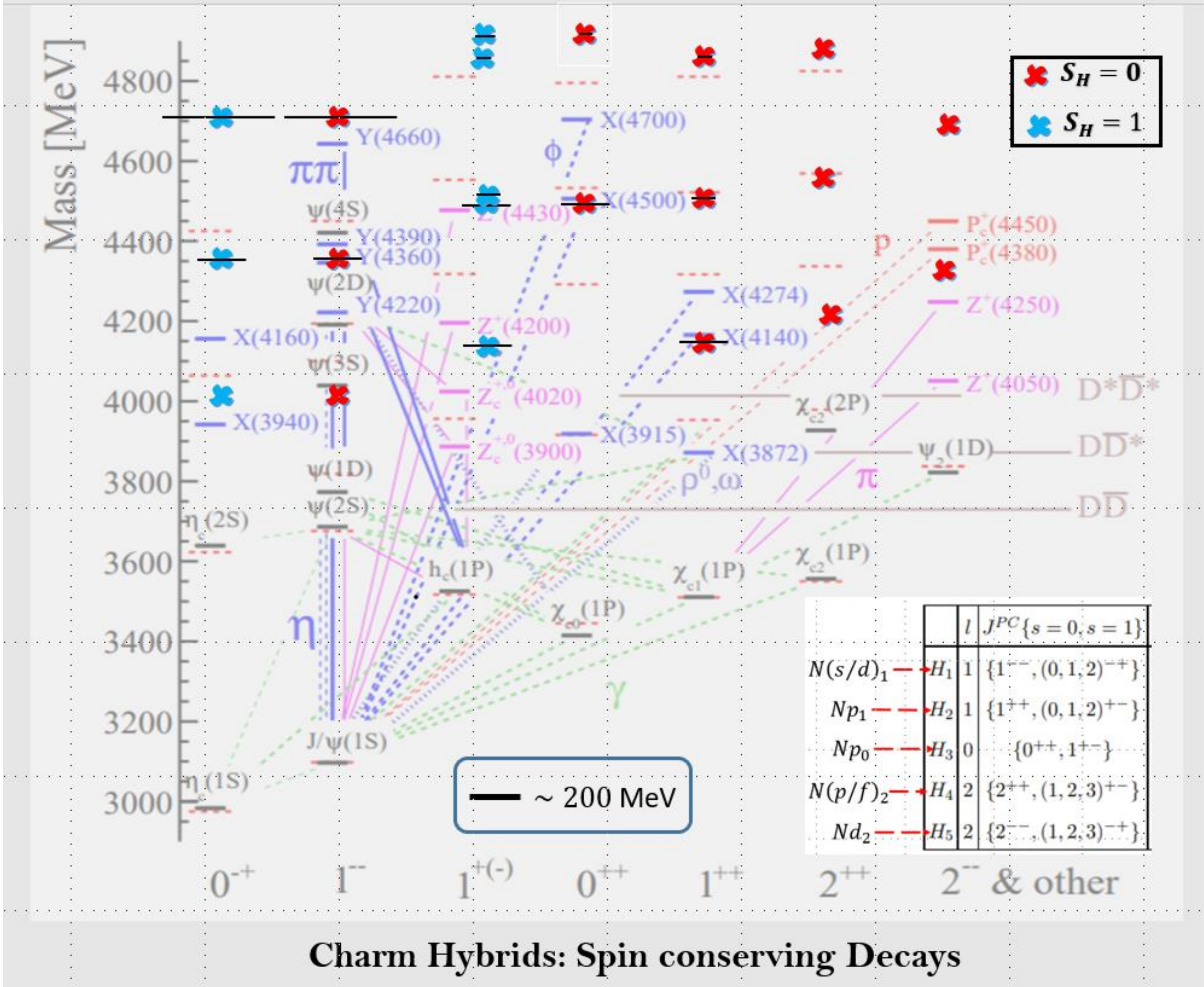
- **Spin flipping** decay rates for **Bottom hybrids**:

$ML_J \to NL$	ΔE (GeV)	$\alpha_S(\Delta E)$	$\Gamma^{\rm sim}$(MeV) $(1 \to 0)$	$\Gamma^{\rm sim}$(MeV) $(0 \to 1)$
Bottomonium hybrid decay				
$1p_0 \to 1p$	1.103	0.346	3.94 ± 0.98	11.81 ± 2.94
$2p_0 \to 1p$	1.391	0.310	0.54 ± 0.12	1.61 ± 0.36
$2p_0 \to 2p$	1.034	0.357	2.95 ± 0.76	8.84 ± 2.27
$3p_0 \to 1p$	1.643	0.289	0.19 ± 0.04	0.56 ± 0.12
$3p_0 \to 2p$	1.286	0.321	0.50 ± 0.12	1.49 ± 0.35
$4p_0 \to 1p$	1.872	0.275	0.09 ± 0.02	0.28 ± 0.06
$4p_0 \to 2p$	1.515	0.299	0.19 ± 0.04	0.57 ± 0.12
$4p_0 \to 3p$	1.226	0.328	0.45 ± 0.11	1.35 ± 0.32
$1p_1 \to 1p$	0.853	0.396	6.32 ± 1.77	18.96 ± 5.32
$2p_1 \to 1p$	1.062	0.352	3.22 ± 0.82	9.67 ± 2.45
$2p_1 \to 2p$	0.705	0.444	1.99 ± 0.61	5.97 ± 1.83
$3p_1 \to 1p$	1.266	0.323	1.62 ± 0.38	4.85 ± 1.13
$3p_1 \to 2p$	0.909	0.382	2.16 ± 0.59	6.47 ± 1.76
$1(s/d)_1 \to 1s$	1.247	0.326	8.62 ± 2.03	25.86 ± 6.09
$2(s/d)_1 \to 1s$	1.443	0.305	5.95 ± 1.31	17.84 ± 3.94
$2(s/d)_1 \to 2s$	0.876	0.390	0.56 ± 0.16	1.68 ± 0.47
$3(s/d)_1 \to 1s$	1.642	0.289	3.56 ± 0.75	10.67 ± 2.24
$3(s/d)_1 \to 2s$	1.075	0.350	0.95 ± 0.24	2.84 ± 0.71
$4(s/d)_1 \to 1s$	1.713	0.284	0.50 ± 0.10	1.51 ± 0.31
$4(s/d)_1 \to 2s$	1.146	0.339	2.56 ± 0.63	7.68 ± 1.88
$4(s/d)_1 \to 3s$	0.799	0.411	0.37 ± 0.11	1.12 ± 0.32

$m_c^{RS} = 1.477(40)$ GeV

$m_b^{RS} = 4.863(55)$ GeV

- Error bars from higher order corrections in α_s.



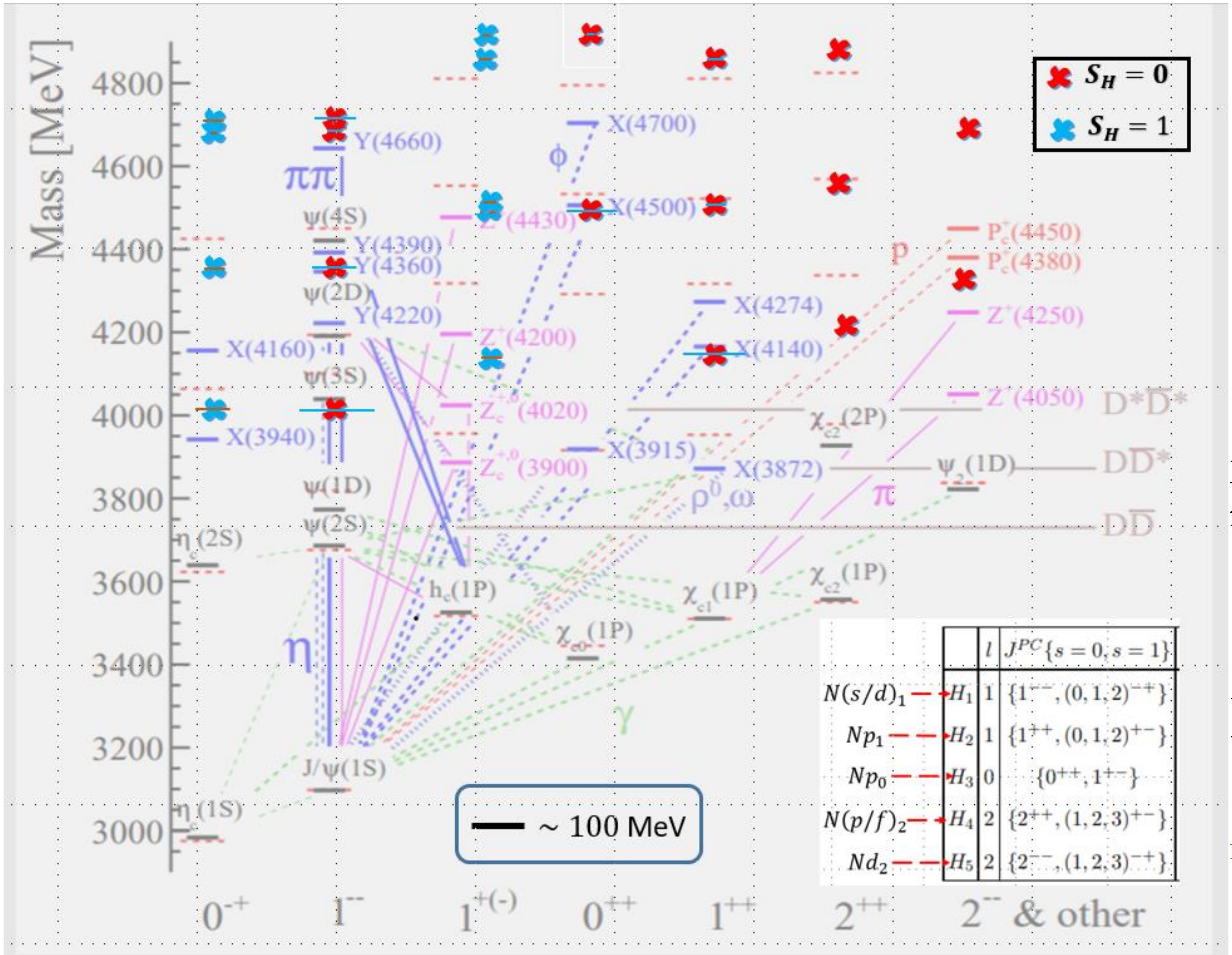
		l	$J^{PC}\{s=0, s=1\}$
$N(s/d)_1$	H_1	1	$\{1^{--}, (0,1,2)^{-+}\}$
Np_1	H_2	1	$\{1^{++}, (0,1,2)^{+-}\}$
Np_0	H_3	0	$\{0^{++}, 1^{+-}\}$
$N(p/f)_2$	H_4	2	$\{2^{++}, (1,2,3)^{+-}\}$
Nd_2	H_5	2	$\{2^{--}, (1,2,3)^{-+}\}$

Background Fig from
S.L. Olsen, T. Skwarnicki, D. Zieminska
Rev. Mod. Phys. 90, 015003 (2018)

Decay rates not computed till now for $J^{PC} = 2^{++}$ and 2^{--} states.

Preliminary Results

N. Brambilla, W.K. Lai, AM, A. Vairo (in progress)



		l	$J^{PC}\{s=0, s=1\}$
$N(s/d)_1$	H_1	1	$\{1^{--}, (0,1,2)^{-+}\}$
Np_1	H_2	1	$\{1^{++}, (0,1,2)^{+-}\}$
Np_0	H_3	0	$\{0^{++}, 1^{+-}\}$
$N(p/f)_2$	H_4	2	$\{2^{++}, (1,2,3)^{+-}\}$
Nd_2	H_5	2	$\{2^{--}, (1,2,3)^{-+}\}$

Charm Hybrids: Spin flipping Decays

Background Fig from

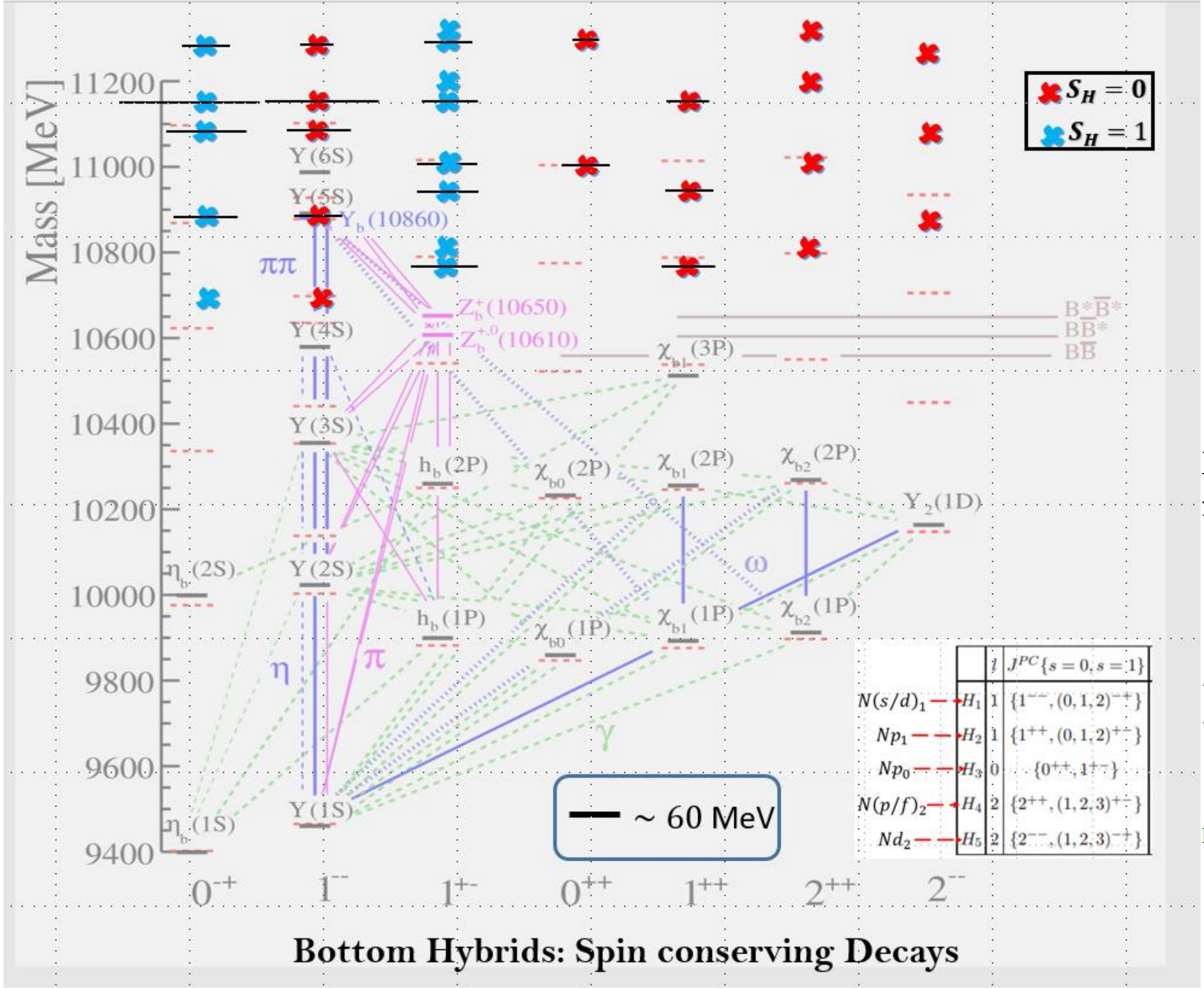
S.L. Olsen, T. Skwarnicki, D. Zieminska

Rev. Mod. Phys. 90, 015003 (2018)

Decay rates not computed till now for $J^{PC} = 2^{++}$ and 2^{--} states.

Preliminary Results

N. Brambilla, W.K. Lai, AM, A. Vairo (in progress)



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Rev. Mod. Phys. 90, 015003 (2018)

Decay rates not computed till now for $J^{PC} = 2^{++}$ and 2^{--} states.

Preliminary Results

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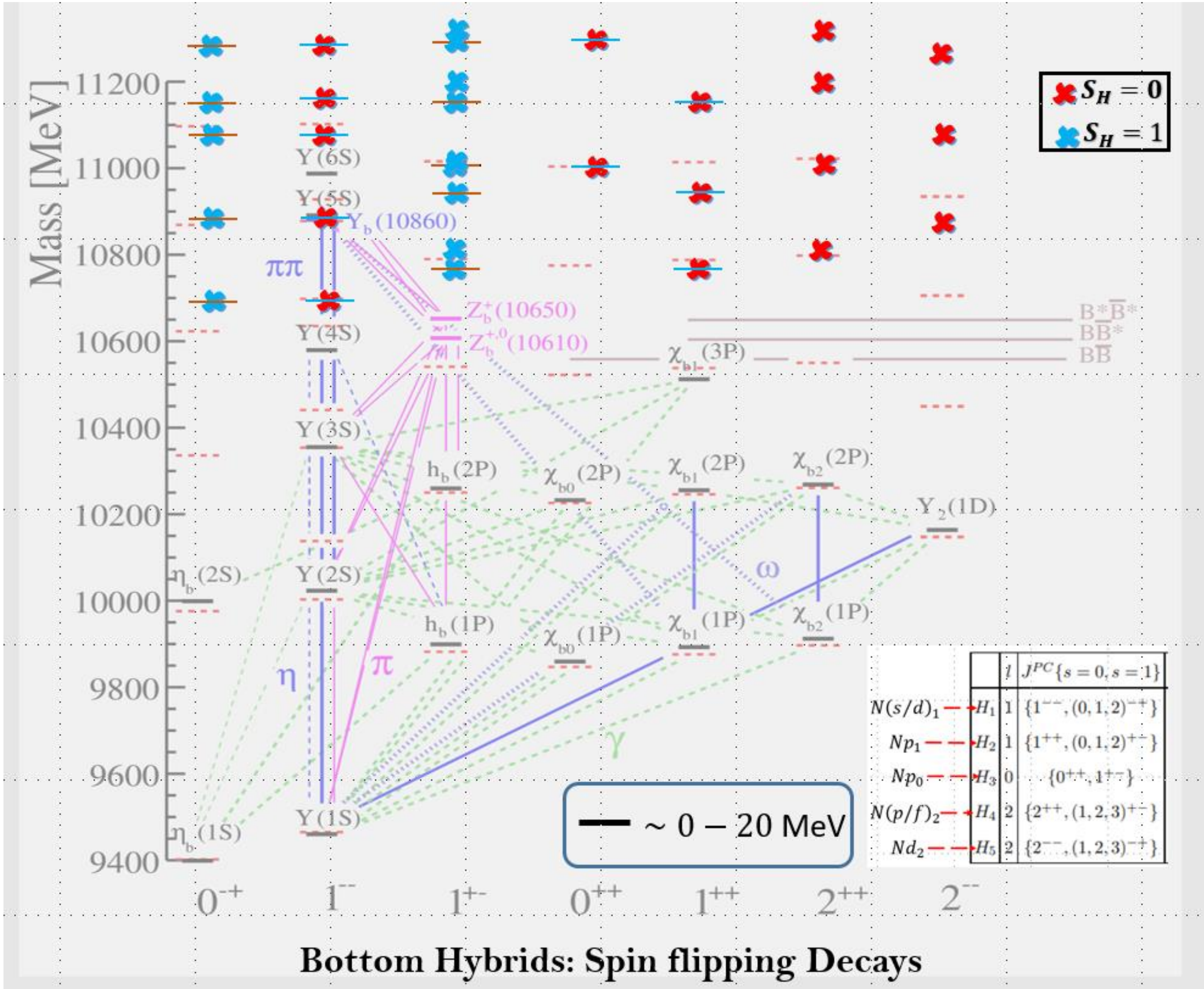

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Decay rates not computed till now for $J^{PC} = 2^{++}$ and 2^{--} states.

Preliminary Results

N. Brambilla, W.K. Lai, AM, A. Vairo (in progress)

Preliminary Results

N. Brambilla, W.K. Lai, AM, A. Vairo (in progress)

Spin-preserving inclusive decay rate for $H_m \to Q_n + X$

$$\Gamma(H_m \to Q_n + X) = \frac{4\alpha_s T_F}{3N_c} \sum_{n'} |h_{nn'}|^2 \sum_{q,q'} \int dE \int dE' \, f^i_{mq}(E) g^j_{qn}(E) \times g^{j\dagger}_{q'n}(E') f^{i\dagger}_{mq'}(E') (\Lambda + E/2 + E'/2 - E^s_n)^3$$

Depends on several Overlap functions:

$$f^i_{mq}(E) = \left[\int d^3r \, \Psi^{i\dagger}_m(\boldsymbol{r}) \Phi^o_{E,q}(\boldsymbol{r}) \right]$$

$$g^j_{qn}(E) = \left[\int d^3r \, \Phi^{o\dagger}_{E,q}(\boldsymbol{r}) r^j \Phi^s_n(\boldsymbol{r}) \right]$$

$$h_{nn'} = \int d^3r \, \Phi^{s\dagger}_{n'}(\boldsymbol{r}) \Phi^Q_n(\boldsymbol{r})$$

- Double integral over the energies E, E′ of the octet wave function $\Phi^o_{E,q}$.
- Depends on overlap function of singlet w.f Φ^S_n and quarkonium w.f Φ^Q_n.
- Last factor $\left(\Lambda + {}^E/_2 + {}^{E'}/_2 - E^S_n\right)^3 \sim \Delta^3$ parametrically. $(\Delta \equiv \boldsymbol{m_H} - \boldsymbol{m_Q})$
- Above result different from the one in R. Oncala, J. Soto, Phys. Rev. D96, 014004 (2017).

Preliminary Results

N. Brambilla, W.K. Lai, AM, A. Vairo (in progress)

- **Spin conserving** decay rates from general decay rate:

$$m_c^{RS} = 1.477(40)\ \text{GeV}$$
$$m_b^{RS} = 4.863(55)\ \text{GeV}$$

$ML_J \to NL$	ΔE (GeV)	$\alpha_s\,(\Delta E)$	Γ^{over} (MeV)
charmonium hybrid decay			
$1p_0 \to 1s$	1.598	0.291	305.64 ± 64.68
$1p_0 \to 2s$	1.101	0.346	29.04 ± 7.23
$2p_0 \to 1s$	2.063	0.266	2.07 ± 0.41
$2p_0 \to 2s$	1.566	0.294	43.18 ± 9.22
$3p_0 \to 1s$	2.482	0.249	26.59 ± 4.93
$3p_0 \to 2s$	1.984	0.269	30.94 ± 6.14
$1p_1 \to 1s$	1.295	0.320	993.15 ± 230.06
$2p_1 \to 1s$	1.675	0.287	17.82 ± 3.72
$2p_1 \to 2s$	1.178	0.335	204.18 ± 49.31
$3p_1 \to 1s$	2.044	0.267	80.74 ± 15.89
$3p_1 \to 2s$	1.547	0.296	92.86 ± 19.92
$2(s/d)_1 \to 1p$	1.196	0.332	125.1 ± 24.7
$4(s/d)_1 \to 1p$	1.563	0.295	336.72 ± 42.0

- Error bars from higher order corrections in α_s.
- Lower value of decay rate due to overlap functions that depends on octet & singlet wave-functions.

Preliminary Results

N. Brambilla, W.K. Lai, AM, A. Vairo
(in progress)

- **Spin conserving** decay rates for **Bottom hybrids**:

$ML_J \to NL$	ΔE (GeV)	$\alpha_s(\Delta E)$	$\Gamma^{\rm over}$ (MeV)
bottomonium hybrid decay			
$1p_0 \to 1s$	1.605	0.291	108.59 ± 22.95
$1p_0 \to 2s$	1.218	0.329	50.33 ± 11.98
$2p_0 \to 1s$	1.892	0.274	18.84 ± 3.79
$2p_0 \to 2s$	1.505	0.299	154.66 ± 33.57
$2p_0 \to 3s$	1.219	0.329	42.19 ± 10.04
$3p_0 \to 1s$	2.156	0.262	11.13 ± 2.16
$3p_0 \to 2s$	1.769	0.281	24.96 ± 5.12
$4p_0 \to 1s$	2.403	0.252	5.87 ± 1.10
$4p_0 \to 2s$	2.017	0.268	34.13 ± 6.74
$4p_0 \to 3s$	1.731	0.283	16.17 ± 3.34
$1p_1 \to 1s$	1.386	0.310	78.14 ± 17.57
$2p_1 \to 1s$	1.600	0.292	31.15 ± 6.59
$2p_1 \to 2s$	1.213	0.330	124.88 ± 29.77
$3p_1 \to 1s$	1.810	0.278	17.85 ± 3.64
$3p_1 \to 2s$	1.424	0.307	27.11 ± 6.03
$2(s/d)_1 \to 1p$	1.214	0.330	43.0 ± 10.87
$3(s/d)_1 \to 2p$	1.126	0.342	13.9 ± 3.54
$4(s/d)_1 \to 2p$	1.161	0.337	236.5 ± 43.20
$5(s/d)_1 \to 1p$	1.620	0.290	52.69 ± 12.72

$m_c^{RS} = 1.477(40)$ GeV
$m_b^{RS} = 4.863(55)$ GeV

- Error bars from higher order corrections in α_s.
- Lower value of decay rate due to overlap functions that depends on octet & singlet wave-functions.

Summary/Outlook

- BOEFT provides a model-independent & systematic way to study heavy quark hybrids (exotic) and decays.
- General formula for the H_m inclusive decay based on overlap functions:

Inclusive decay rate for $H_m \to Q_n + X$

$$\Gamma(H_m \to Q_n + X) = \frac{4\alpha_s T_F}{3N_c} \sum_{n'} |h_{nn'}|^2 \sum_{q,q'} \int dE \int dE' f^i_{mq}(E) g^j_{qn}(E) \times g^{j\dagger}_{q'n}(E') f^{i\dagger}_{mq'}(E') (\Lambda + E/2 + E'/2 - E^s_n)^3$$

- Computed preliminary results on decay rates for spin-flipping and spin preserving decays.

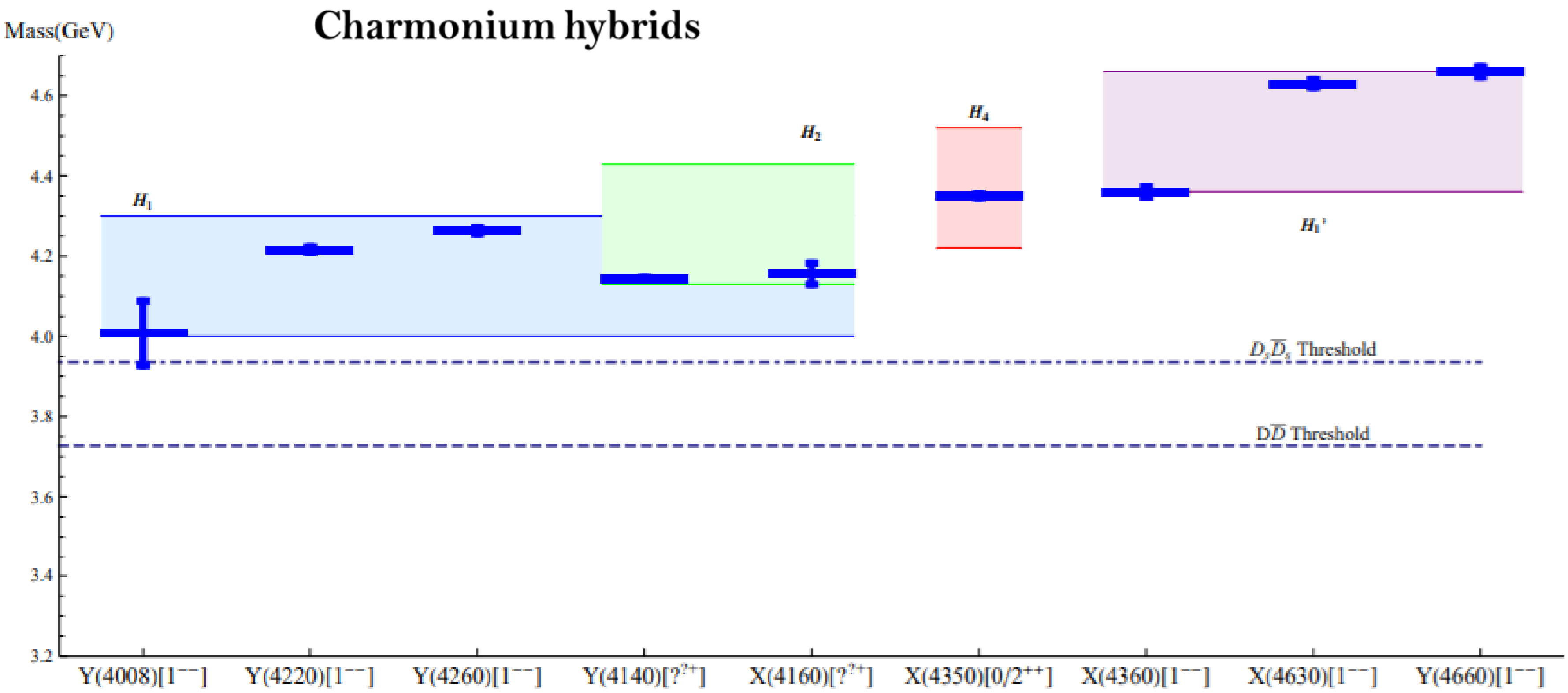
- Future and ongoing work includes:
  - Computing decay rates using the formulae based on overlap functions.
  - Quantifying errors in the decay rates & comparing with the PDG data for observed exotic states.
  - Exclusive decays: $H \to Q\pi\pi$ and include effect of mixing with excited quarkonia $Q' \to Q + X$.
  - Extending this analysis to study Quarkonium tetraquarks.

Thank you!!

Backup Slides

Quarkonium hybrids: Spectrum

- Comparison with Experimental results:



	l	$J^{PC}\{s=0, s=1\}$	$E_n^{(0)}$
H_1	1	$\{1^{--}, (0,1,2)^{-+}\}$	Σ_u^-, Π_u
H_2	1	$\{1^{++}, (0,1,2)^{+-}\}$	Π_u
H_3	0	$\{0^{++}, 1^{+-}\}$	Σ_u^-
H_4	2	$\{2^{++}, (1,2,3)^{+-}\}$	Σ_u^-, Π_u
H_5	2	$\{2^{--}, (1,2,3)^{-+}\}$	Π_u

- Uncertainty in hybrid result (the bands) from gluelump masses

Berwein, Brambilla, Castellà , Vairo Phys. Rev. D. 92, (2015)

BOEFT Hamiltonian

$$H_{\text{BOEFT}} = \int d^3x \int d^3R \, \mathrm{Tr}\left[H^{i\dagger}\left(h_o\delta^{ij} + V^{ij}_{soft}\right)H^j\right]$$

$V^{ij}_{soft} = \Lambda + b^{ij}r^2 + \cdots$

Λ = gluelump mass ($= 0.87(15)$ GeV for lowest lying $\kappa = 1^{+-}$ gluelump)

For two insertions of the $\boldsymbol{r}\cdot\boldsymbol{E}$ vertex, the contribution to the two-point function is

$$I^{(2)}_{ij}(\boldsymbol{r},\boldsymbol{R},\boldsymbol{r}',\boldsymbol{R}')$$

$$= -\lim_{T\to\infty} g^2\frac{T_F}{N_c}\int_{-T/2}^{T/2}dt\int_{-T/2}^{t}dt'\, e^{-ih_o(T/2-t)}r^k e^{-ih_s(t-t')}r^l e^{-ih_o(t'+T/2)}$$

$$\times \langle 0|G^{ib}(T/2)\phi^{ab}(T/2,t)E^{kb}(t)E^{lc}(t')\phi^{cd}(t',-T/2)G^{jd}(-T/2)|0\rangle \mathbb{I}\delta^3(\boldsymbol{r}-\boldsymbol{r}')\delta^3(\boldsymbol{R}-\boldsymbol{R}')$$

To seperate the scales Δ and Λ_{QCD}, write $\boldsymbol{E} = \boldsymbol{E}_h + \boldsymbol{E}_s$, $\boldsymbol{E}_h \sim \Delta$, $\boldsymbol{E}_s \sim \Lambda_{\text{QCD}}$.
Replace $\boldsymbol{E}$ by $\boldsymbol{E}_h$ to get the leading contribution.

$$\langle 0|G^{ib}(T/2)\phi^{ab}(T/2,t)E^{kb}_h(t)E^{lc}_h(t')\phi^{cd}(t',-T/2)G^{jd}(-T/2)|0\rangle$$

$$= \langle 0|G^{ib}(T/2)\phi^{ab}(T/2,t)\phi^{cd}(t',-T/2)G^{jd}(-T/2)|0\rangle\langle 0E^{kb}_h(t)E^{lc}_h(t')|0\rangle$$

$$= \langle 0|G^{ib}(T/2)\phi^{ab}(T/2,t)\phi^{bd}(t',-T/2)G^{jd}(-T/2)|0\rangle\frac{\delta^{kl}}{3}\int\frac{d^3k}{(2\pi)^3}|\boldsymbol{k}|e^{-i|\boldsymbol{k}|(t-t')}$$

$$\approx \frac{\delta^{kl}}{3}e^{i\Lambda(t-t')}\langle 0|G^{ib}(T/2)\phi^{ab}(T/2,-T/2)G^{jb}(-T/2)|0\rangle\int\frac{d^3k}{(2\pi)^3}|\boldsymbol{k}|e^{-i|\boldsymbol{k}|(t-t')}$$

$$= \delta^{ij}\frac{\delta^{kl}}{3}e^{i\Lambda(t-t'-T)}\int\frac{d^3k}{(2\pi)^3}|\boldsymbol{k}|e^{-i|\boldsymbol{k}|(t-t')}.$$

Decay Rate: Calculation Details.

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Inclusive Decays

It is interesting to see how $f^i_{mq}(E) = \left[\int d^3r\, \Psi^{i\dagger}_m(\boldsymbol{r})\Phi^o_{E,q}(\boldsymbol{r})\right]$ looks like as a function of E:

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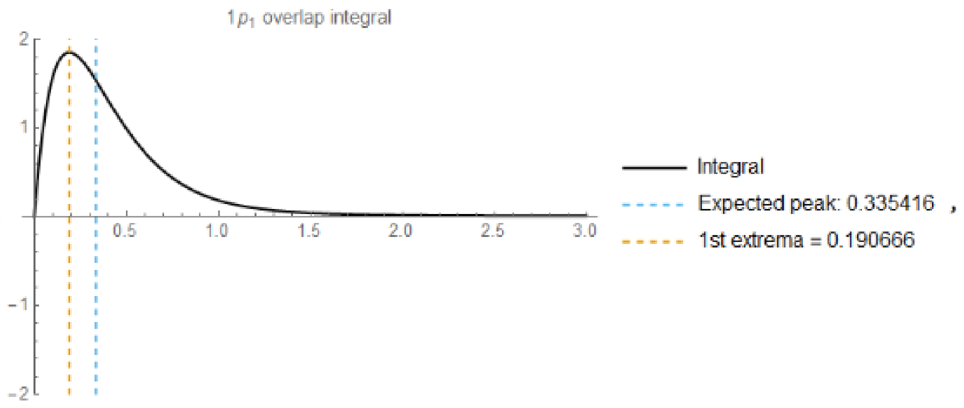
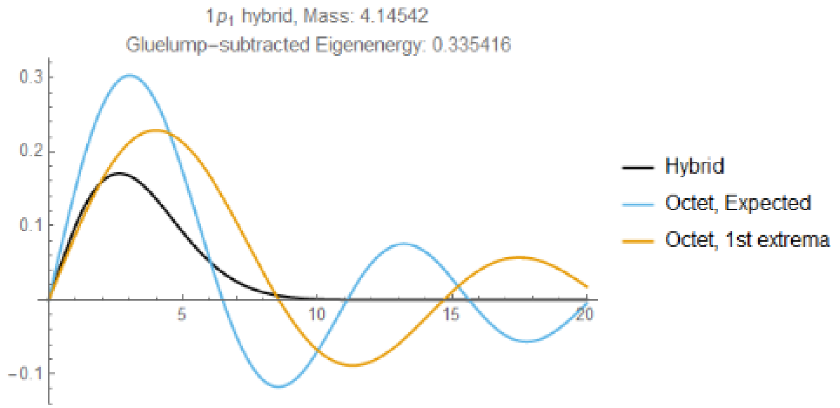
H_2-multiplet, $l = 1, J^{PC} = [1^{++}, (0,1,2)^{+-}]$
$H_2(4145)$:



Radial integral of $f^i_{mq}(E)$ vs E (GeV)



Radial hybrid wave function vs r (GeV^{-1})

- The actual peak is slightly off (at a lower E) from the expected peak at $E = E_m - \Lambda$.
- The peak is broad, with width ~ 1 GeV. The assumption that $f^i_{mq}(E)$ is nonzero only when $E_m \approx E + \Lambda$ is not true.

Quarkonium spectrum

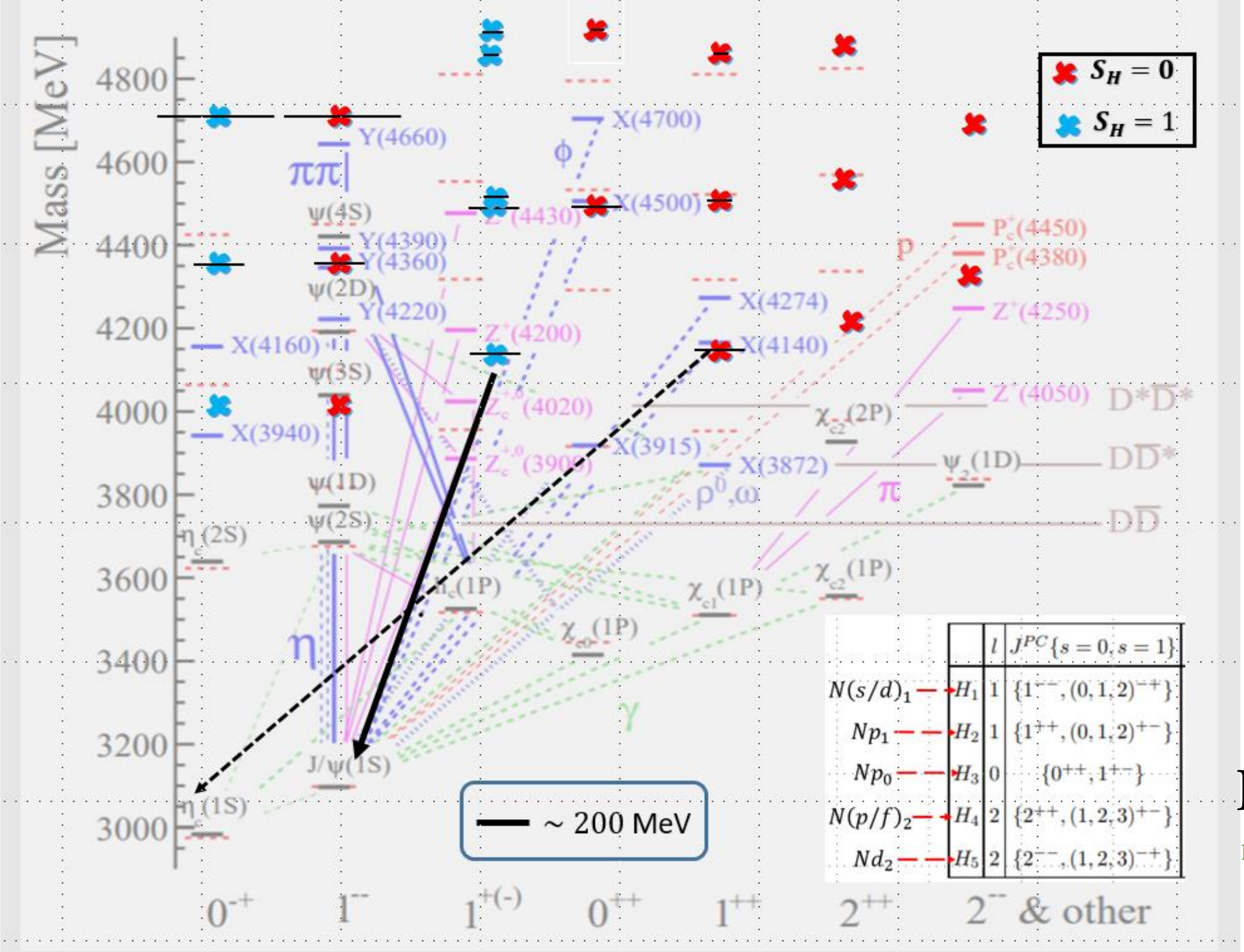
N. Brambilla, W.K. Lai, AM, A. Vairo

(in progress)

nL	$M_{c\bar{c}}$	E_{exp}	$M_{b\bar{b}}$	E_{exp}
$1s$	2991	3068	9460	9445
$2s$	3489	3674	9847	10017
$3s$	3894	4039	10133	10355
$4s$	4251	4421	10379	10579
$5s$	4577	?	10602	10876
$1p$	3311	3525	9762	9900
$2p$	3736	3927	10046	10260
$3p$	4106	?	10295	?
$4p$	4441	?	10521	?
$5p$	4751	?	10731	?

$$m_c^{RS} = 1.477(40)\ \text{GeV}$$

$$m_b^{RS} = 4.863(55)\ \text{GeV}$$



		l	$J^{PC}\{s=0, s=1\}$
$N(s/d)_1$	H_1	1	$\{1^{--}, (0,1,2)^{-+}\}$
Np_1	H_2	1	$\{1^{++}, (0,1,2)^{+-}\}$
Np_0	H_3	0	$\{0^{++}, 1^{+-}\}$
$N(p/f)_2$	H_4	2	$\{2^{++}, (1,2,3)^{+-}\}$
Nd_2	H_5	2	$\{2^{--}, (1,2,3)^{-+}\}$

Ex. of Charm Hybrids: Spin conserving Decays

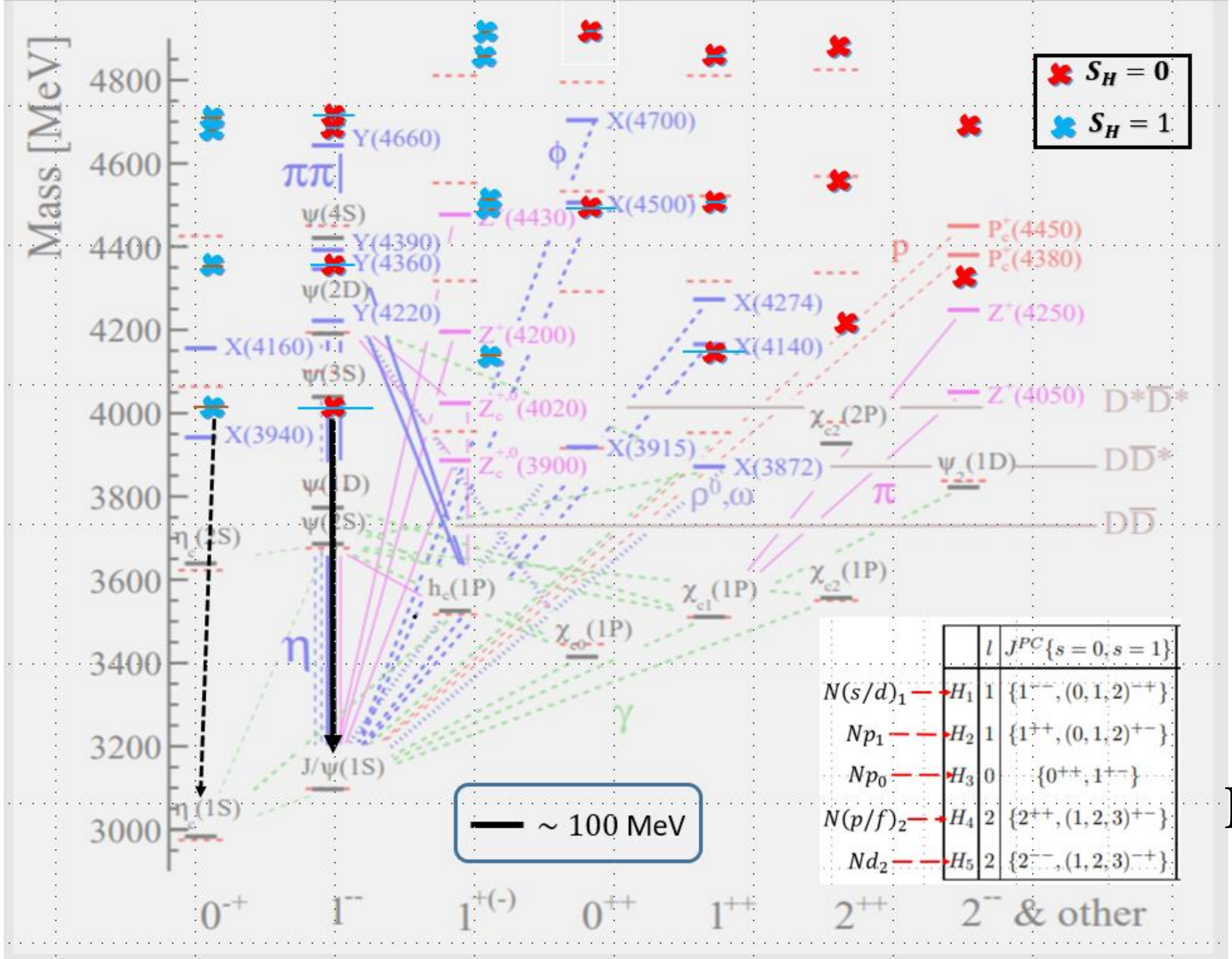
Background Fig from
S.L. Olsen, T. Skwarnicki, D. Zieminska
Rev. Mod. Phys. 90, 015003 (2018)

Ex. for illustration

$1p_1 \rightarrow 1s$

Preliminary Results

N. Brambilla, W.K. Lai, AM, A. Vairo (in progress)



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Ex. of Charm Hybrids: Spin flipping Decays

Background Fig from

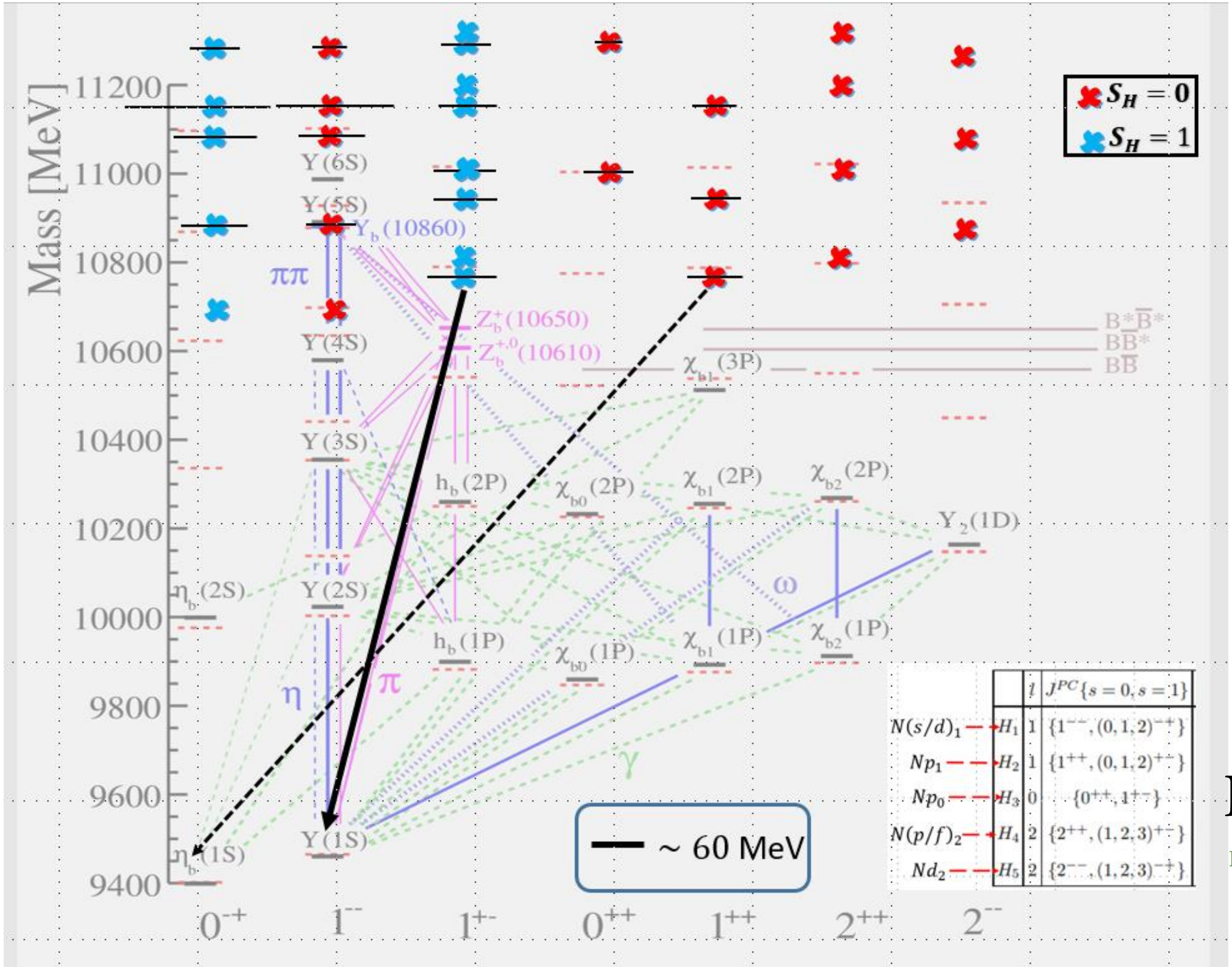
S.L. Olsen, T. Skwarnicki, D. Zieminska

Rev. Mod. Phys. 90, 015003 (2018)

Ex. for illustration

$$1(s/d)_1 \to 1s$$

Preliminary Results



Ex. of Bottom Hybrids: Spin conserving Decays

Background Fig from

S.L. Olsen, T. Skwarnicki, D. Zieminska

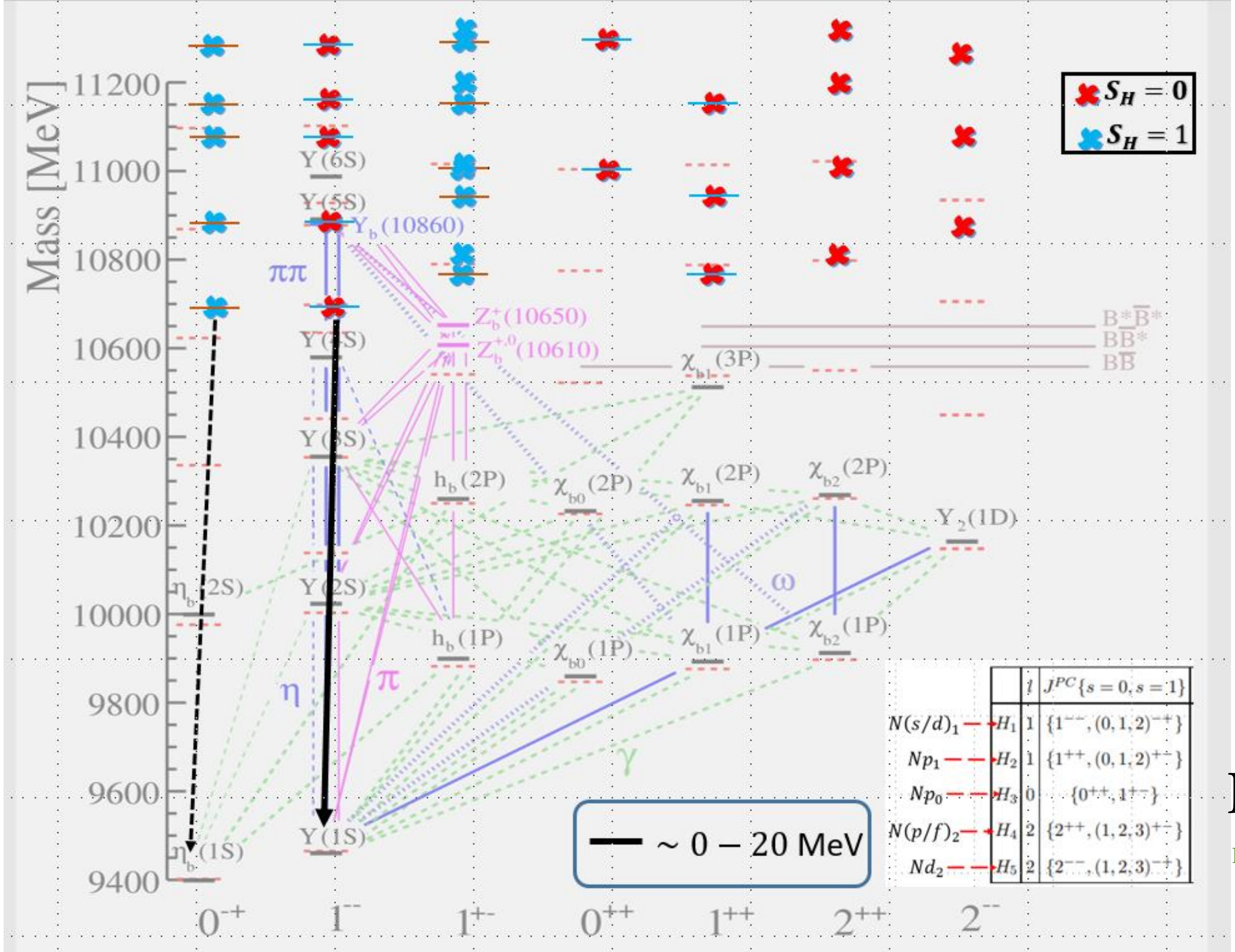
Rev. Mod. Phys. 90, 015003 (2018)

Ex. for illustration

$$1p_1 \to 1s$$

Preliminary Results

N. Brambilla, W.K. Lai, AM, A. Vairo (in progress)



Ex. of Bottom Hybrids: Spin flipping Decays

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