Some Recent Results on Renormalization-Group Properties of Quantum Field Theories

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XXXIII Int'l Workshop on High Energy Physics, Logunov IHEP/Kurchatov NRC, 8-12 Nov. 2021

Outline

- Renormalization-group flows from UV to IR in asymptotically free gauge theories; types of IR behavior; role of an exact or approximate IR fixed point

- Higher-loop calculations of UV to IR evolution, including IR zero of β and anomalous dimension $\gamma_{\bar{\psi}\psi,IR}$ of fermion bilinear up to 5-loop level

- Scheme-independent expansions and calculations of $\gamma_{\bar{\psi}\psi,IR}$ and β'_{IR} using 5-loop beta function in asymptotically free gauge theories

- Comparison with lattice measurements for various groups G, fermion representations R, and number, N_f of fermions

- Studies of RG flows in other theories, e.g., Gross-Neveu (2D), ϕ^3 (6D), U(1) (4D), and $\lambda|\vec{\phi}|^4$ (4D),

- Conclusions

We have been interested in RG flows in QFTs for many yrs., since our solution of the O(N) nonlinear σ model and its β fn., in the large-N limit, in $d = 2+\epsilon$: Bardeen, Lee, RS, PRD 14, 985 (1976) (also studied by Brézin, Zinn-Justin, PRB 14, 3110 (1976)).

RG Flow from UV to IR; Types of IR Behavior and Role of IR Fixed Point

Consider an asymptotically free, vectorial gauge theory with gauge group G and N_f massless fermions in a representation R of G.

Asymptotic freedom $\Rightarrow$ theory is weakly coupled, properties are perturbatively calculable for large Euclidean momentum scale μ in deep ultraviolet (UV).

The nature of the renormalization-group (RG) flow from large μ in the UV to small μ in the infrared (IR) is of fundamental field-theoretic interest.

If a fermion had mass m_0, it would be integrated out in the low-energy effective field theory for $\mu < m_0$, and hence would not affect the IR limit of interest here, so there is no loss of generality in using massless fermions.

For some fermion contents, the β function may have an IR zero, which an IR fixed point (IRFP) of RG.

Notation: we denote running gauge coupling at scale μ as $g = g(\mu)$, and let $\alpha(\mu) = g(\mu)^2/(4\pi)$ and $a(\mu) = g(\mu)^2/(16\pi^2)$.

The dependence of $\alpha(\mu)$ on μ is described by the β function

$$\beta \equiv \frac{d\alpha}{dt} = -2\alpha \sum_{\ell=1}^{\infty} b_\ell \, a^\ell = -2\alpha \sum_{\ell=1}^{\infty} \bar{b}_\ell \, \alpha^\ell \ ,$$

where $dt = d \ln \mu$, $\ell =$ loop order of the coeff. b_ℓ, and $\bar{b}_\ell = b_\ell/(4\pi)^\ell$.

Coefficients b_1 and b_2 in β are independent of regularization/renormalization scheme, while b_ℓ for $\ell \geq 3$ are scheme-dependent (restrict to mass-independent schemes here). Calculations - b_1: Gross and Wilczek; Politzer, 1973 ['t Hooft, unpub.]; b_2: Caswell; Jones, 1974.

Asymptotic freedom means $b_1 > 0$, so $\beta < 0$ for small $\alpha(\mu)$, in neighborhood of UV fixed point (UVFP) at $\alpha = 0$.

As the scale μ decreases from large values, $\alpha(\mu)$ increases. Denote α_{cr} as minimum value for formation of bilinear fermion condensates and resultant spontaneous chiral symmetry breaking (SχSB).

Two generic possibilities for β and resultant UV to IR flow:

- β has no IR zero, so as μ decreases, $\alpha(\mu)$ increases, can exceed the perturbatively calculable regime.
- β has a IR zero, α_{IR}, so as μ decreases, $\alpha(\mu)$ approaches α_{IR}. In this class of theories, there are two further generic possibilities: $\alpha_{IR} < \alpha_{cr}$ or $\alpha_{IR} > \alpha_{cr}$.

If $\alpha_{IR} < \alpha_{cr}$, the zero of β at α_{IR} is an exact IR fixed point (IRFP) of the RG; as $\mu \to 0$ and $\alpha \to \alpha_{IR}$, $\beta \to \beta(\alpha_{IR}) = 0$, and the theory becomes exactly scale-invariant with nontrivial anomalous dimensions (Caswell, Banks-Zaks).

If β has no IR zero, or an IR zero at $\alpha_{IR} > \alpha_{cr}$, then as μ decreases through a scale Λ, $\alpha(\mu)$ exceeds α_{cr} and SχSB occurs, so fermions gain dynamical masses $\sim \Lambda$.

If SχSB occurs, then in low-energy effective field theory applicable for $\mu < \Lambda$, one integrates these fermions out, and β fn. becomes that of a pure gauge theory, with no IR zero. Hence, if β has a zero at $\alpha_{IR} > \alpha_{cr}$, this is only an approx. IRFP of RG.

If α_{IR} is only slightly greater than α_{cr}, then quasiconformal behavior, possible light pseudo Nambu-Goldstone boson (dilaton), which might be relevant in composite Higgs models.

Denote the n-loop β fn. as $\beta_{n\ell}$ and the IR zero of $\beta_{n\ell}$ as $\alpha_{IR,n\ell}$. For a given gauge group G and fermion rep. R, the asymptotic freedom (AF) condition restricts N_f to be less than an upper (u) value $N_u = 11C_A/(4T_f)$.

The (scheme-independent) 2-loop β has an IR zero in the interval I given by upper and lower (ℓ) ends $I: \quad N_\ell < N_f < N_u$, where

$$N_\ell = \frac{17C_A^2}{2T_f(5C_A + 3C_f)}$$

Here, the Casimir invariants $C_2(R)$ and $T(R)$ are defined as $\sum_{a=1}^{o(G)} \sum_{j=1}^{\dim(R)} (T_a^{(R)})_{ij} (T_a^{(R)})_{jk} = C_2(R)\delta_{ik}$ and $\mathrm{Tr}_R[T_a^{(R)} T_b^{(R)}] = T(R)\delta_{ab}$, where R is the representation, T_a are the generators of the Lie algebra of G, and $T^{(R)}$ is the matrix of T_a in the rep. R; also, $C_2(adj) \equiv C_A$, and for fermions transforming according to the representation R, we denote $C_2(R) \equiv C_f$ and $T(R) \equiv T_f$; e.g., for $G = \mathrm{SU}(N_c)$, $C_A = N_c$, and for $R = fund.\ (F)$, $T_f = 1/2$ and $C_f = (N_c^2 - 1)/(2N_c)$.

In expressions for N_u, N_ℓ, etc., formally continue N_f to non-integral real values, with integral physical values understood implicitly.

The interval I for $R = F$ is $5.55 < N_f < 11$ for SU(2) and $8.05 < N_f < 16.5$ for SU(3).

At 2-loop level, $\alpha_{IR,2\ell} = -4\pi b_1/b_2$ and in general, $\alpha_{IR} \searrow 0$ as $N_f \nearrow N_u$.

Define $N_f = N_{f,cr}$ at $\alpha_{IR} = \alpha_{cr}$. For $N_{f,cr} < N_f < N_u$, IR theory is in a (deconfined) chirally symmetric non-Abelian Coulomb phase (NACP) while for $N_f < N_{f,cr}$, there is SχSB, confinement.

Lattice studies of gauge theories with various gauge groups G and fermions in various representations R have been carried out; progress toward determining $N_{f,cr}$ for various G and R with applications for possible models with composite Higgs..

At an IRFP in the NACP, scale-invariance, inferred conformal invariance. It is of fundamental interest to determine the properties of the theory at this IRFP.

Denote the dimension of an operator $\mathcal{O}$ as $D_{\mathcal{O}}$; because of interactions, this differs from the free-field dimension, $D_{\mathcal{O},free}$; $D_{\mathcal{O}} = D_{\mathcal{O},\ free} - \gamma_{\mathcal{O}}$, where $\gamma_{\mathcal{O}}$ is the anomalous dimension of $\mathcal{O}$.

An example is $\bar\psi\psi = \sum_{j=1}^{N_f} \bar\psi_j \psi_j$ with anom. dim. $\gamma_{\bar\psi\psi}$. Another quantity of interest is $\beta' = d\beta/d\alpha$; the values of these at an IRFP are scheme-independent. We denote these as $\gamma_{\bar\psi\psi,IR}$ and β'_{IR}.

Higher-Loop Analysis of UV $\to$ IR Evolution of Gauge Theories

For a given G and R, as N_f decreases below N_u, $\alpha_{IR,2\ell}$ increases. This motivates calculation of the IR zero in β and anom. dim. $\gamma_{\bar\psi\psi}$ to higher-loop order. Calculations for general G and R to 4-loop order in Ryttov and RS, PRD 83, 056011 (2011) [1011.4542] and Pica and Sannino, PRD 83, 035013 (2011) [1011.5917]. 5-loop calculations in Ryttov-RS, PRD 94, 105015 (2016) [1607.06866].

Structural properties including β'_{IR} studied in RS, PRD 87, 105005 (2013) [1301.3209], PRD 87, 116007 (2013) [1302.5434].

Series expansion in $a = \alpha/(4\pi)$ for $\gamma_{\bar\psi\psi}$: $\gamma_{\bar\psi\psi} = \sum_{\ell=1}^{\infty} c_\ell \, a^\ell$, where c_ℓ is ℓ-loop coefficient. To calculate the n-loop result for the anom. dim., $\gamma_{\bar\psi\psi,n\ell}$, we first calculate $\alpha_{IR,n\ell}$, then set $\alpha = \alpha_{IR,n\ell}$ in above eq.

The 1-loop coeff. $c_1 = 6C_f$ is scheme-independent, while the c_ℓ with $\ell \ge 2$ are scheme-dependent; calcs. of b_ℓ and c_ℓ up to $\ell = 4$ loop level by van Ritbergen, Vermaseren, Larin (1997); Chetyrkin (1997); Vermaseren and Larin (1997) in $\overline{\rm MS}$ scheme. .

At the 2-loop level, we obtained (using abbreviation $\gamma_{IR,n\ell} \equiv \gamma_{\bar\psi\psi,IR,n\ell}$)

$$\gamma_{IR,2\ell} = \frac{C_f(11C_A - 4T_fN_f)[455C_A^2 + 99C_AC_f + (180C_f - 248C_A)T_fN_f + 80(T_fN_f)^2]}{12[-17C_A^2 + 2(5C_A + 3C_f)T_fN_f]^2}$$

with more complicated expressions at the 3-loop and 4-loop levels.

Approximate solution of Schwinger-Dyson eq. for fermion propagator in ladder approx. suggests SχSB at $\gamma_{\bar\psi\psi} \simeq 1$ (Yamawaki et al.; Appelquist et al., 1986).

In the chirally broken phase, just as the IR zero of β is only an approx. IRFP, so also, the $\gamma_{\bar\psi\psi,IR}$ is only approx., describing the running of $\bar\psi\psi$ and the dynamically generated running fermion mass, $\Sigma(k)$, near the zero of β.

Illustrative numerical values of $\gamma_{\bar\psi\psi,IR,n\ell} \equiv \gamma_{IR,n\ell}$ for SU(2) and SU(3) at the $n = 2,\ 3,\ 4$ loop level and fermion rep. $R = F$ with $N_f \in I$:

N_c	N_f	$\gamma_{IR,2\ell}$	$\gamma_{IR,3\ell}$	$\gamma_{IR,4\ell}$
2	7	> 2	0.457	0.0325
2	8	0.752	0.272	0.204
2	9	0.275	0.161	0.157
2	10	0.0910	0.0738	0.0748
3	10	> 2	0.647	0.156
3	11	1.61	0.439	0.250
3	12	0.773	0.312	0.253
3	13	0.404	0.220	0.210
3	14	0.212	0.146	0.147
3	15	0.0997	0.0826	0.0836
3	16	0.0272	0.0258	0.0259

Thus, the 3-loop and 4-loop results are closer to each other for a larger range of N_f than the 2-loop and 3-loop results. We also performed these higher-loop calculations for other fermion reps. R.

Calculation of α_{IR} and $\gamma_{\bar\psi\psi,IR}$ to 5-Loop Level

In Ryttov and RS, PRD 94, 105015 (2016) [1607.06866] we extended our calculations of α_{IR} and $\gamma_{\bar\psi\psi,IR}$ to the 5-loop level, using b_5 and c_5 from Baikov, Chetyrkin, and Kühn, 1606.08659 [PRL 118, 082002 (2017)] for SU(3), $R = F$, again in $\overline{\rm MS}$ scheme.

With a factor α^2 extracted, $\beta_{5\ell} \propto \bar b_1 + \bar b_2\alpha + \bar b_3\alpha^2 + \bar b_4\alpha^3 + \bar b_5\alpha^4$. One determines $\alpha_{IR,5\ell}$ as the smallest real positive root of this polynomial.

Recall that for SU(3), the integral interval of interest, I, is $9 \le N_f \le 16$. For $13 \le N_f \le 16$, we found that $\alpha_{IR,5\ell}$ is close to, and slightly larger than $\alpha_{IR,4\ell}$; e.g., for $N_f = 14$, $\alpha_{IR,4\ell} = 0.224$, while $\alpha_{IR,5\ell} = 0.233$. For $N_f \le 12$, we make use of an analysis using Padé approximants to obtain the value of $\alpha_{IR,5\ell}$.

A $[p,q]$ Padé approximant (PA) to a finite series expansion of n'th degree is the rational function

$$[p,q]_{\beta_{r,n\ell}} = \frac{1+\sum_{j=1}^{p} n_j\alpha^j}{1+\sum_{k=1}^{q} d_k\,\alpha^k}$$

with $p + q = n - 1$, where the n_j and d_j are α-independent coefficients. For example, for SU(3), $R = F$, and $N_f = 12$, we get $\alpha_{IR,5\ell} = 0.41$ from the [3,1] PA, slightly smaller than $\alpha_{IR,4\ell} = 0.47$.

We use direct calculation of $\alpha_{IR,5\ell}$ from $\beta_{5\ell}$ for $14 \le N_f \le 16$ and Padé methods for lower N_f, to get $\alpha_{IR,5\ell}$ and then evaluate $\gamma_{5\ell}$ at $\alpha = \alpha_{IR,5\ell}$. Some results for $G = \mathrm{SU}(3)$ and $R = F$ are shown in the table (with $\gamma_{IR,n\ell} \equiv \gamma_{\bar\psi\psi,IR,n\ell}$):

N_f	$\gamma_{IR,2\ell}$	$\gamma_{IR,3\ell}$	$\gamma_{IR,4\ell}$	$\gamma_{IR,5\ell}$
11	1.61	0.439	0.250	0.294
12	0.773	0.312	0.253	0.255
13	0.404	0.220	0.210	0.239
14	0.212	0.146	0.147	0.154
15	0.0997	0.0826	0.0836	0.0843
16	0.0272	0.0258	0.0259	0.0259

For small N_f near the lower end of the NACP, the coupling is too strong for these perturbative methods to be reliable.

Study of Scheme Dependence in Calculation of IR Fixed Point

Since coeffs. b_n in $\beta_{n\ell}$, and hence also $\alpha_{IR,n\ell}$, are scheme-dependent for $n \geq 3$, it is important to assess the effects of this scheme dependence. We have done this in a series of papers: Ryttov and RS, PRD 86, 065032 (2012) [1206.2366]; PRD 86, 085005 (2012) [1206.6895]; RS, PRD 88, 036003 (2013) [1305.6524]; RS, PRD 90, 045011 (2014) [1405.6244]; Choi and RS, PRD 90 125029 (2014) [1411.6645]; Choi and RS, PRD 94, 065038 (2016) [1607.03500]; see also T. Ryttov, PRD 89, 016013 (2014) [1309.3867]; PRD 89, 056001 (2014) [1311.0848]; PRD 90, 056007 (2014) [1408.5841].

A scheme transformation (ST) is a map between α and α' or equivalently, a and a', where $a = \alpha/(4\pi)$ of the form

$$a = a' f(a') \equiv F(a')$$

with $f(0) = 1$ since ST has no effect in limit of zero coupling.

$$f(a') = 1 + \sum_{s=1}^{s_{max}} k_s (a')^s = 1 + \sum_{s=1}^{s_{max}} \bar{k}_s (\alpha')^s$$

where $\bar{k}_s = k_s/(4\pi)^s$, and s_{max} may be finite or infinite.

The Jacobian $J = da/da' = d\alpha/d\alpha' = 1 + \sum_{s=1}^{s_{max}} (s+1) k_s (a')^s$, satisfying $J = 1$ at $a = a' = 0$.

After the scheme transformation is applied, the beta function in the new scheme is given by

$$\beta_{\alpha'} \equiv \frac{d\alpha'}{dt} = \frac{d\alpha'}{d\alpha}\frac{d\alpha}{dt} = J^{-1}\,\beta_\alpha$$

with the expansion

$$\beta_{\alpha'} = -2\alpha' \sum_{\ell=1}^{\infty} b'_\ell (a')^\ell$$

We calculated the b'_ℓ as functions of the b_ℓ and k_s. At 1-loop and 2-loop, this yields the well-known results $b'_1 = b_1$ and $b'_2 = b_2$; at higher orders, we obtained

$$b'_3 = b_3 + k_1 b_2 + (k_1^2 - k_2) b_1 \ ,$$

$$b'_4 = b_4 + 2k_1 b_3 + k_1^2 b_2 + (-2k_1^3 + 4k_1 k_2 - 2k_3) b_1$$

$$b'_5 = b_5 + 3k_1 b_4 + (2k_1^2 + k_2) b_3 + (-k_1^3 + 3k_1 k_2 - k_3) b_2 \\ +(4k_1^4 - 11k_1^2 k_2 + 6k_1 k_3 + 4k_2^2 - 3k_4) b_1$$

etc. for higher-order b'_ℓ.

We specified a set of conditions that a physically acceptable scheme transformation (of a perturbatively reliable calculation) must satisfy:

- C_1: the ST must map a (real positive) α to a real positive α'
- C_2: the ST should not map a moderate value of α, where perturbation theory is applicable, to a value of α' so large that perturbation theory is inapplicable.
- C_3: J should not vanish (or diverge) or else the ST would be singular.
- C_4: Existence of an IR zero of β is a scheme-independent property, so the ST should satisfy the condition that β_α has an IR zero if and only if $\beta_{\alpha'}$ has an IR zero.

These conditions can always be satisfied by an ST near the UVFP at $\alpha = \alpha' = 0$, but they are not automatic, and can be quite restrictive, at an IRFP.

For example, consider the ST (dependent on a parameter r)

$$a = \frac{\tanh(ra')}{r}$$

with inverse

$$a' = \frac{1}{2r} \ln\left(\frac{1+ra}{1-ra}\right)$$

(e.g., for $r = 4\pi$, $\alpha = \tanh \alpha'$). This is acceptable for small a, but if $a > 1/r$, i.e., $\alpha > 4\pi/r$, it maps a real α to a complex α' and hence is physically unacceptable. For $r = 8\pi$, e.g., this pathology can occur at the moderate value $\alpha = 0.5$.

Thus, the issue of scheme dependence at a zero of the beta function away from the origin in coupling constant space (e.g, at an IRFP in an asymptotically free theory) is more involved than near the origin (small α_s in QCD). For QCD in the perturbative region, many studies of optimization of schemes (e.g., Brodsky, Lepage, MacKenzie (1983); Brodsky, Mojaza, Wu, PRD 89, 014027 (2014) [1304.4631], etc.)

We have constructed several STs that are acceptable at an IRFP and have studied scheme dependence of the IR zero of $\beta_{n\ell}$ using these. For example, we have used a $\sinh$ transformation (depending on a parameter r):

$$a = \frac{\sinh(ra')}{r}$$

(where we can take $r > 0$ without loss of generality) with inverse

$$a' = \frac{1}{r} \ln\left[ra + \sqrt{1 + (ra)^2}\,\right]$$

Illustrative results with this $\sinh$ scheme transformation: Denote the IR zero of $\beta_{\alpha'}$ at the n-loop level in the transformed scheme as $\alpha'_{IR,n\ell,r}$:

For SU(3), $R = F$, $N_f = 12$, $\alpha_{IR,2\ell} = 0.754$, and:

$$\alpha_{IR,3\ell,\overline{\rm MS}} = 0.435, \quad \alpha'_{IR,3\ell,r=3} = 0.434, \quad \alpha'_{IR,3\ell,r=6} = 0.433,$$

$$\alpha_{IR,4\ell,\overline{\rm MS}} = 0.470, \quad \alpha'_{IR,4\ell,r=3} = 0.470, \quad \alpha'_{IR,4\ell,r=6} = 0.467,$$

This and other scheme transformations that we have studied suggest that the scheme dependence of our calculations of anomalous dimensions is not overly large.

This is similar to perturbative calculations of physical quantities in QCD (leading-order, LO; NLO, NNLO, etc.), which are also scheme-dependent, but very useful in predicting cross sections at Tevatron, LHC. For these QCD applications, calculations to higher order have shown reduced scheme dependence.

Since the coefficients b_ℓ at loop order $\ell \geq 3$ in the beta function are scheme-dependent, one might expect that it would be possible, at least in the vicinity of zero coupling, to construct a scheme transformation that would set $b'_\ell = 0$ for some range of $\ell \geq 3$, and, indeed a ST that would do this for all $\ell \geq 3$, so that $\beta_{\alpha'}$ would consist only of the 1-loop and 2-loop terms ('t Hooft scheme).

We have constructed explicit scheme transformations that can do this in the vicinity of zero coupling constant in Ryttov and RS, PRD 86, 065032 (2012) [arXiv:1206.2366]; PRD 86, 085005 (2012) [arXiv:1206.6895]; RS, PRD 88, 036003 (2013) [arXiv:1305.6524]; RS, PRD 90, 045011 (2014) [arXiv:1405.6244]. However, we have also shown that it is more difficult to try to do this at a zero of β away from the origin (IR zero for an asymp. free theory; UV zero for an IR-free theory).

We proceed to our scheme-independent calculations.

Scheme-Independent Series Expansions

The anomalous dimensions $\gamma_{\bar{\psi}\psi,IR}$ and $\gamma_{F^2,IR}$ are physical and hence are obviously scheme-independent. However, this property is not maintained in finite series expansions of these anom. dims. in powers of $\alpha_{IR,n\ell}$, since both $\alpha_{IR,n\ell}$ for $\ell \geq 3$ and higher-loop coefficients are scheme-dependent.

A basic property of the IRFP is that $\alpha_{IR,n\ell} \to 0$ as $N_f \nearrow N_u$, i.e., as $\Delta_f \to 0$, where $\Delta_f = N_u - N_f = [11C_A/(4T_f)] - N_f$. Hence, one can re-express a series expansion in powers of $\alpha_{IR,n\ell}$ as a series expansion in the manifestly scheme-independent variable Δ_f, e.g.,

$$\gamma_{\bar{\psi}\psi,IR} = \sum_{j=1}^{\infty} \kappa_j \Delta_f^j$$

Since $\gamma_{\bar{\psi}\psi,IR}$ and Δ_f are scheme-independent, and this property holds for variable Δ_f, it follows that each coefficient κ_f is scheme-independent.

Thus, this expansion has the appealing property that it is scheme-independent to each finite order, in contrast to the conventional series expansion in powers of α, which is scheme-dependent beyond the lowest order. With T. Ryttov, we have done this in a series of papers.

We denote the truncation of the above series to maximal order (power) p as $\gamma_{\bar\psi\psi,IR,\Delta_f^p}$.

Define a denominator factor $D = 7C_A + 11C_f$. The first two κ_j are

$$\kappa_1 = \frac{8C_fT_f}{C_AD} \ ,$$

$$\kappa_2 = \frac{4C_fT_f^2(5C_A + 88C_f)(7C_A + 4C_f)}{3C_A^2D^3} \ ,$$

and similarly for κ_3 (Ryttov, PRL 117, 071601 (2016) [1604.00687]). In Ryttov and RS, PRD 94, 105014 (2016) [1608.00068] we calculated κ_4 and hence $\gamma_{\bar\psi\psi,IR}$ to $O(\Delta_f^4)$ for SU(3) and $R = F$.

In Ryttov and RS, PRD 95, 085012 (2017) [1701.06083] and Ryttov and RS, PRD 95, 105004 (2017) [1703.08558] we have calculated κ_4 and hence $\gamma_{\bar\psi\psi,IR}$ to $O(\Delta_f^4)$ for arbitrary gauge group G and R and have studied particular reps. in detail. We use the 5-loop beta fn. coeff. b_5 from Herzog, Ruijl, Ueda, Vermaseren, Vogt, JHEP 02(2017)090 [1701.01404] in $\overline{\mathrm{MS}}$ scheme. We find:

$$
\begin{aligned}
\kappa_4 = \frac{T_f^2}{3^5 C_A^5 D^7}\Bigg[& C_A C_f T_f^2 \Big(19515671 C_A^6 - 131455044 C_A^5 C_f + 1289299872 C_A^4 C_f^2 + 2660221312 C_A^3 C_f^3 \\
& + 1058481072 C_A^2 C_f^4 + 6953709312 C_A C_f^5 + 1275715584 C_f^6\Big) \\
& + 2^{10} C_f T_f^2 D \Big(5789 C_A^2 - 4168 C_A C_f - 6820 C_f^2\Big) \frac{d_A^{abcd} d_A^{abcd}}{d_A} \\
& - 2^{10} C_A C_f T_f D \Big(41671 C_A^2 - 125477 C_A C_f - 53240 C_f^2\Big) \frac{d_R^{abcd} d_A^{abcd}}{d_A} \\
& - 2^8 \cdot 11^2 C_A^2 C_f D (2569 C_A^2 + 18604 C_A C_f - 7964 C_f^2\Big) \frac{d_R^{abcd} d_R^{abcd}}{d_A} \\
& - 2^{14} \cdot 3 C_A T_f^2 D^3 \frac{d_R^{abcd} d_A^{abcd}}{d_R} + 2^{13} \cdot 33 C_A^2 T_f D^3 \frac{d_R^{abcd} d_R^{abcd}}{d_R} \\
& + 2^8 D \Big[- 3 C_A C_f T_f^2 D \Big(4991 C_A^4 - 17606 C_A^3 C_f + 33240 C_A^2 C_f^2 - 30672 C_A C_f^3 + 9504 C_f^4\Big) \\
& - 2^4 C_f T_f^2 \frac{d_A^{abcd} d_A^{abcd}}{d_A} \Big(17206 C_A^2 - 60511 C_A C_f - 45012 C_f^2\Big) \\
& + 40 C_A C_f T_f \frac{d_R^{abcd} d_A^{abcd}}{d_A} \Big(35168 C_A^2 - 154253 C_A C_f - 88572 C_f^2\Big) \\
& - 88 C_A^2 C_f \frac{d_R^{abcd} d_R^{abcd}}{d_A} \Big(973 C_A^2 - 93412 C_A C_f - 56628 C_f^2\Big) \\
& + 1440 C_A T_f^2 D^2 \frac{d_R^{abcd} d_A^{abcd}}{d_R} - 7920 C_A^2 T_f D^2 \frac{d_R^{abcd} d_R^{abcd}}{d_R} \Big] \zeta_3
\end{aligned}
$$

$$+\frac{4505600 C_A C_f D^2}{d_A}\Big[-4T_f^2 d_A^{abcd} d_A^{abcd} + 2T_f d_R^{abcd} d_A^{abcd}(10C_A + 3C_f) + 11C_A d_R^{abcd} d_R^{abcd}(C_A - 3C_f)\Big]\zeta_5\Big]$$

where ($a,\ b,\ c,\ d$ are group indices)

$$d_R^{abcd} = \frac{1}{3!}\,\mathrm{Tr}_R\Big[T^a_{(R)}\Big(T^b_{(R)}T^c_{(R)}T^d_{(R)} + cycl.\Big)\Big]$$

$d_A^{abcd} = d_R^{abcd}$ for $R = adj$, $d_R = \dim(R)$, and $\zeta_s = \sum_{n=1}^{\infty}\frac{1}{n^s}$ is the Riemann zeta function.

This calculation of κ_j requires, as inputs, the values of the b_ℓ for $1 \le \ell \le j+1$ and the c_ℓ for $1 \le \ell \le j$. As is evident, it requires that the IRFP be exact and hence applies in the non-Abelian Coulomb phase (NACP).

If the phase change from the (conformal) NACP to the quasi-conformal regime with SχSB for N_f slightly below $N_{f,cr}$ is continuous, our scheme-independent calc. of $\gamma_{\bar\psi\psi,IR}$ may give approx. info. on $\gamma_{\bar\psi\psi,IR,eff.}$ in this quasiconformal regime.

For $G = \mathrm{SU}(N_c)$ and $R = F$, our results for general G and R reduce to

$$\kappa_{1,F} = \frac{4(N_c^2-1)}{N_c(25N_c^2-11)}, \qquad \kappa_{2,F} = \frac{4(N_c^2-1)(9N_c^2-2)(49N_c^2-44)}{3N_c^2(25N_c^2-11)^3}$$

$$\begin{aligned}
\kappa_{3,F} \ &= \ \frac{8(N_c^2-1)}{3^3N_c^3(25N_c^2-11)^5}\Bigg[\Big(274243N_c^8 - 455426N_c^6 - 114080N_c^4 + 47344N_c^2 + 35574\Big) \\
&- \ 4224N_c^2(4N_c^2-11)(25N_c^2-11)\zeta_3\Bigg]
\end{aligned}$$

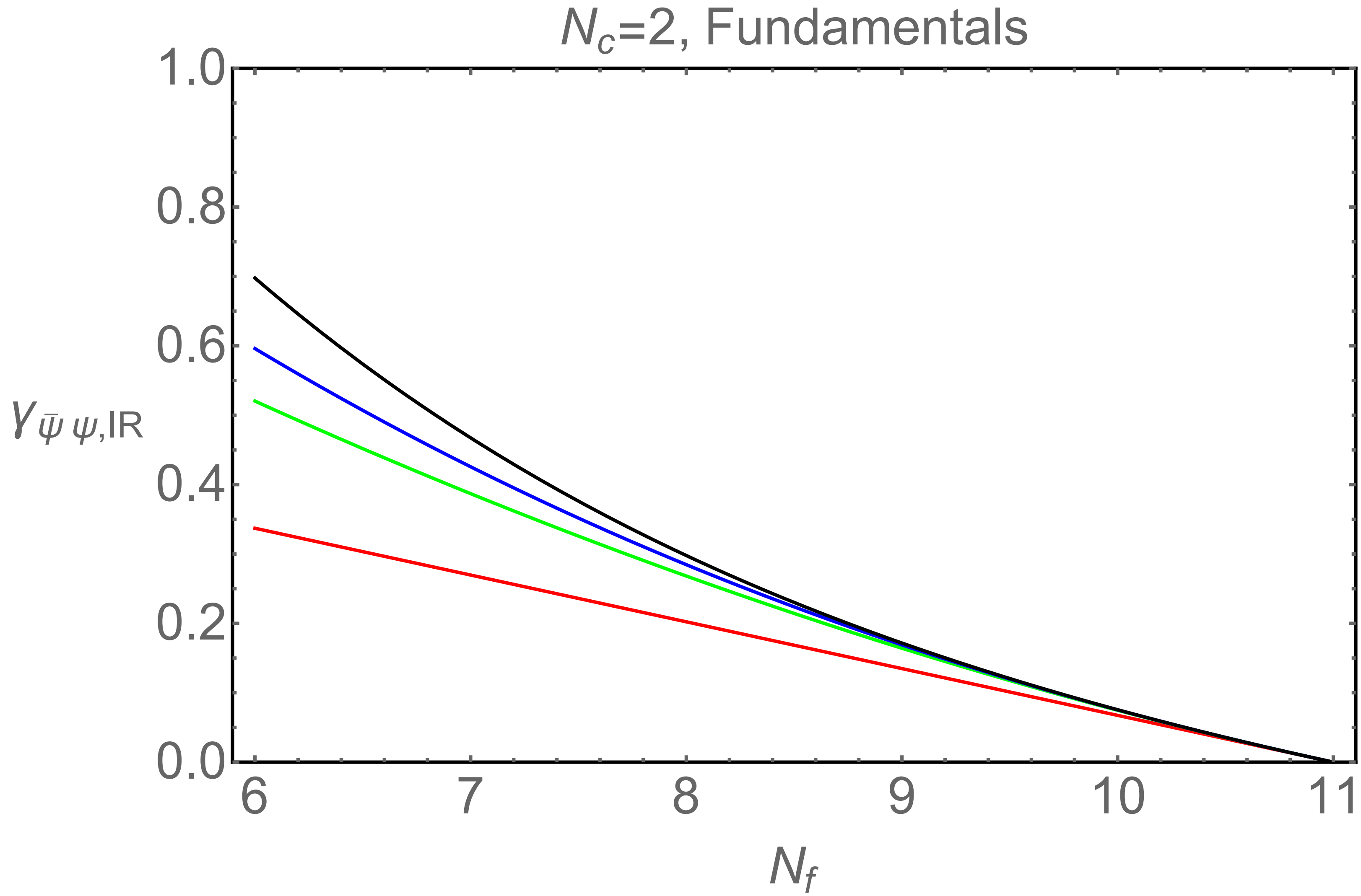
$$\begin{aligned}
\kappa_{4,F} \ &= \ \frac{4(N_c^2-1)}{3^4N_c^4(25N_c^2-11)^7}\Bigg[\Big(263345440N_c^{12} - 673169750N_c^{10} + 256923326N_c^8 \\
&- \ 290027700N_c^6 + 557945201N_c^4 - 208345544N_c^2 + 6644352\Big) \\
&+ \ 384(25N_c^2-11)\Big(4400N_c^{10} - 123201N_c^8 + 480349N_c^6 \\
&- \ 486126N_c^4 + 84051N_c^2 + 1089\Big)\zeta_3 \\
&+ \ 211200N_c^2(25N_c^2-11)^2(N_c^6 + 3N_c^4 - 16N_c^2 + 22)\zeta_5\Bigg]
\end{aligned}$$

Numerical results for SU(2) and SU(3):

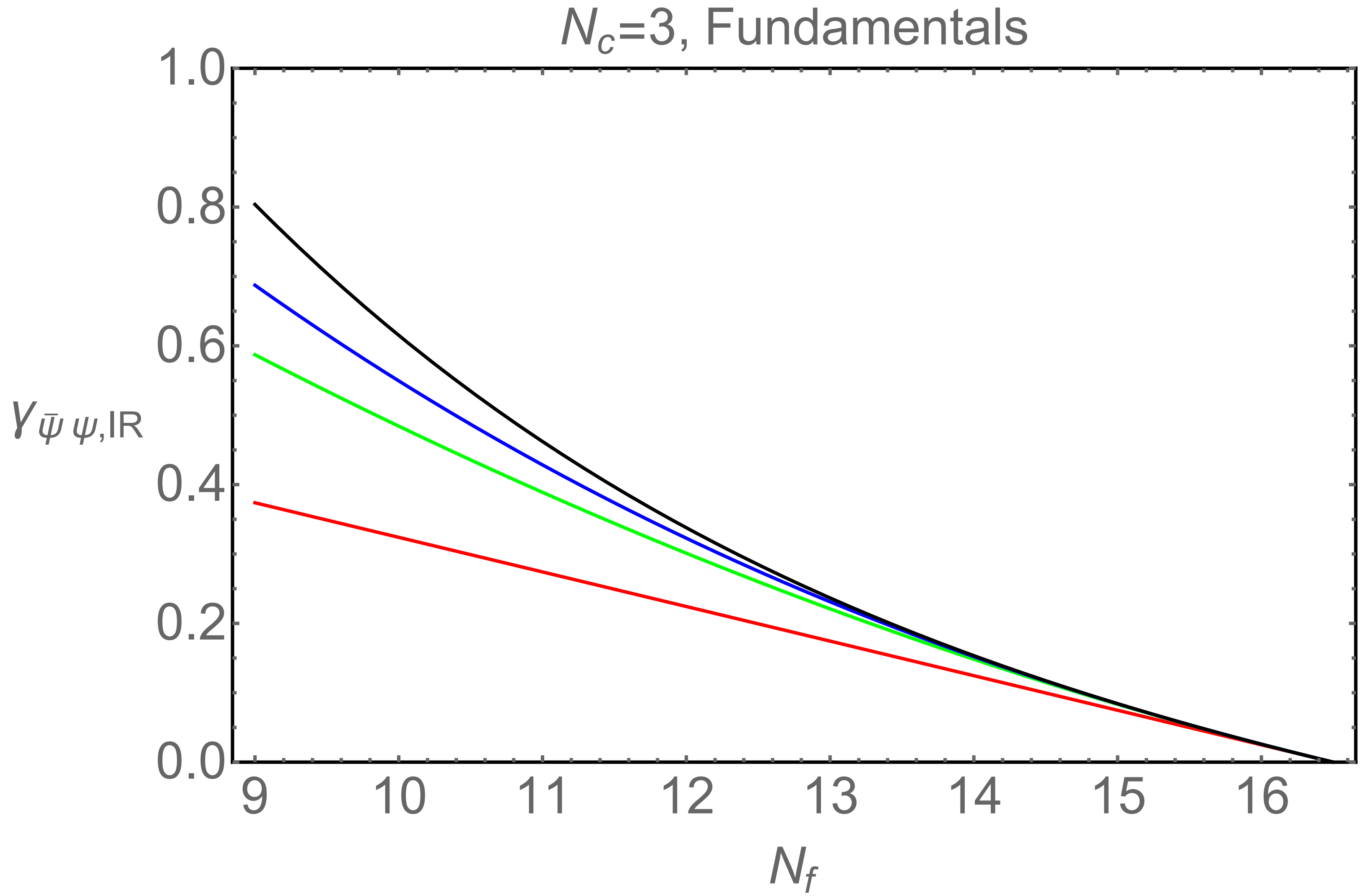
$$\mathbf{SU(2):}\ \gamma_{\bar{\psi}\psi,IR,F,\Delta_f^4} = \Delta_f\Big[0.067416 + (0.73308\times 10^{-2})\Delta_f + (0.60531\times 10^{-3})\Delta_f^2 + (1.62662\times 10^{-4})\Delta_f^3\Big]$$

$$\mathbf{SU(3):}\ \gamma_{\bar{\psi}\psi,IR,F,\Delta_f^4} = \Delta_f\Big[0.049844 + (0.37928\times 10^{-2})\Delta_f + (0.23747\times 10^{-3})\Delta_f^2 + (0.36789\times 10^{-4})\Delta_f^3\Big]$$

Plots of $\gamma_{\bar{\psi}\psi,IR,\Delta_f^p}$ with $1 \le p \le 4$ for SU(2) and SU(3) and fermion rep. $R = F$, as functions of $N_f \in I$. Curves: $\gamma_{\bar{\psi}\psi,IR,F,\Delta_f}$ (red), $\gamma_{\bar{\psi}\psi,IR,F,\Delta_f^2}$ (green), $\gamma_{\bar{\psi}\psi,IR,F,\Delta_f^3}$ (blue), $\gamma_{\bar{\psi}\psi,IR,F,\Delta_f^4}$ (black).



N_c=2, Fundamentals
$\gamma_{\bar{\psi}\psi,\mathrm{IR}}$
N_f
1.0
0.8
0.6
0.4
0.2
0.0
6
7
8
9
10
11



N_c=3, Fundamentals
1.0
0.8
0.6
0.4
0.2
0.0
$\gamma_{\bar{\psi}\psi,\mathrm{IR}}$
9
10
11
12
13
14
15
16
N_f

N_c	N_f	γ_{IR,F,Δ_f}	γ_{IR,F,Δ_f^2}	γ_{IR,F,Δ_f^3}	γ_{IR,F,Δ_f^4}
2	6	0.337	0.520	0.596	0.698
2	7	0.270	0.387	0.426	0.467
2	8	0.202	0.268	0.285	0.298
2	9	0.135	0.164	0.169	0.172
2	10	0.0674	0.07475	0.07535	0.0755
3	8	0.424	0.698	0.844	1.036
3	9	0.374	0.587	0.687	0.804
3	10	0.324	0.484	0.549	0.615
3	11	0.274	0.389	0.428	0.462
3	12	0.224	0.301	0.323	0.338
3	13	0.174	0.221	0.231	0.237
3	14	0.125	0.148	0.152	0.153
3	15	0.0748	0.0833	0.0841	0.0843
3	16	0.0249	0.0259	0.0259	0.0259

Values of $\gamma_{\bar\psi\psi,IR,\Delta_f^p} = \gamma_{IR,\Delta_f^p}$ with $1 \le p \le 4$ for SU(2), SU(3), and $R = F$.

We have also calculated and analyzed the special cases of our general results for $G = \mathrm{SU}(N_c)$ and other fermion representations R, including adjoint (adj), and symmetric and antisymmetric rank-2 tensor reps. S_2 and A_2.

Positivity properties: κ_1 and κ_2 are manifestly positive for all G and R. For $G = \mathrm{SU}(N_c)$ and all above fermion reps., we find κ_3 and κ_4 are also positive.

We derive two monotonicity properties for SU(N_c) and these R:

- For a fixed p with $1 \le p \le 4$, the anom. dim. $\gamma_{\bar\psi\psi,IR,\Delta_f^p}$ is a monotonically increasing function of Δ_f, i.e., increases monotonically with decreasing N_f, for $N_f \in I$.
- For a fixed $N_f \in I$, $\gamma_{\bar\psi\psi,IR,\Delta_f^p}$ is a monotonically increasing function of p in the range $1 \le p \le 4$.

These monotonicity properties are evident in the figures. In addition to the manifestly positive κ_1 and κ_2, a conjecture is that, for these fermion representations R, $\kappa_j > 0$ for all $j \ge 3$. We find this is true in $\mathcal{N} = 1$ SQCD: see Ryttov-RS, PRD 96, 105018 (2017) [arXiv:1706.06422] using results from Novikov, Shifman, Vainshtein, Zakharov, Seiberg; see also Kataev, Stepanyanz.

From our comparison of γ_{IR,Δ_f^p} with $\gamma_{IR,n\ell}$ with $n \simeq p$, we find that these agree very well with each other for N_f in the upper end of the non-Abelian Coulomb phase, NACP, i.e., with N_f not too far below N_u.

In our papers we have discussed the accuracy of these finite order-p calculations and resultant γ_{IR,Δ_f^p} values. A rough estimate can be obtained from the figures. Where the curves for the γ_{IR,Δ_f^p} with different p are close to each other, higher-order terms are expected to be small. As N_f decreases, these curves deviate progressively more from each other, and higher-order terms are more important.

For example, for SU(2) and $R = F$, the curves for γ_{IR,Δ_f^p} with $p = 2,\ 3,\ 4$ are very close to each other for $N_f \geq 8$ and deviate from each other for $N_f = 7$ and 6.

For SU(3) and $R = F$, the curves for γ_{IR,Δ_f^p} with $p = 2,\ 3,\ 4$ are very close to each other for $N_f \geq 13$ and moderately close for $N_f = 12$, deviating more from each other as N_f decreases further to 9.

This suggests that the Δ_f expansion may be reasonably reliable for a substantial portion of the non-Abelian Coulomb phase, including, in particular, the upper part.

In Ryttov and RS, Phys. Rev. D 96, 105015 (2017) [1709.05358], we have extended our analysis by evaluating our general-(G, R) results for gauge groups $G = \mathrm{SO}(N_c)$ and Sp(N_c) for a variety of fermion representations R.

Further insight can be obtained from calculation and analysis of Padé approximants (PAs) for $\gamma_{\bar{\psi}\psi,IR}$. We have done this in Ryttov and RS, PRD 97, 025004 (2018) [1710.06944] and get results in agreement with our previous calculations.

We have generalized our analysis to the case of theories with fermions in multiple representations in Ryttov-RS, PRD 97, 016020 (2018) [1710.00096]; PRD 98, 096003 (2018) [arXiv:1809.02242]; Girmohanta, Ryttov, RS, PRD 99, 116022 (2019) [arXiv:1903.09672]. For simplicity, we only consider theories with fermions in a single representation here.

It is of interest to compare our higher-loop calculations of $\gamma_{\bar{\psi}\psi,IR}$ with lattice measurements. (Note that for several G and R, there is not yet a consensus as to the value of $N_{f,cr}$, i.e., the lower end of conformal regime.)

A heavily studied case is $G = \mathrm{SU}(3)$, $R = F$, and $N_f = 12$ - Appelquist et al. (LSD Collab.), Hasenfratz et al., Lombardo et al., and LatKMI find this is IR-conformal, while the Fodor, Kuti, Nogradi et al. find it is not IR-conformal.

For this SU(3) theory with $N_f = 12$, our series calculation in powers of the IR coupling gives $\gamma_{IR,3\ell} = 0.312 \qquad \gamma_{IR,4\ell} = 0.253 \qquad \gamma_{IR,5\ell} = 0.255$. Some lattice results for this SU(3), $R = F$, $N_f = 12$ case are the following, with $\gamma_{\bar\psi\psi,IR} \equiv \gamma$ (see papers for details of uncertainty estimates):

$0.2 \lesssim \gamma \lesssim 0.4$ - Kuti et al., [1205.1878]; Fodor et al., PRD 94, 091501 (2016) [1607.06121]; PoS (Lattice2019)019 121, [1912.07653] (with SχSB, so $\gamma_{eff.}$).

$\gamma = 0.27(3)$ - Hasenfratz et al., PoS(Lattice 2012)034 [1207.7162]; $\gamma \simeq 0.25$; Hasenfratz et al., PoS(Lattice 2013)075 [1310.1124]; Hasenfratz and Schaich, JHEP 1802 (2018) 132 [1610.10004]; $\gamma = 0.26(2)$ - Carroso, Hasenfratz, Neil, PRL 121, 201601 (2018) [1806.01385].

$\gamma = 0.235(46)$ - Lombardo, Miura, Nunes, Pallante, JHEP 12(2014)183 [1410.0298].

So our 4-loop and 5-loop calculations are in good agreement with these lattice measurements. Our scheme-independent values are $\gamma_{IR,\Delta_f^3} = 0.32$, $\gamma_{IR,\Delta_f^4} = 0.34$. See our papers for comparisons with lattice measurements for other G, R, and N_f.

We have also calculated series expansions of the anom. dim. of the $\bar\psi\sigma_{\mu\nu}\psi$ at the IRFP in Ryttov-RS, PRD 94, 125005 (2016) [arXiv:1610.00387] and of the anom. dim. of baryon operators in Gracey, Ryttov, and RS, PRD 97, 116018 (2018) [arXiv:1805.02729].

Another interesting scheme-independent quantity is $\frac{d\beta}{d\alpha}\Big|_{\alpha_{IR}} = \beta'_{IR} = -\gamma_{F^2,IR}$ at the IRFP, where $\gamma_{F^2,IR}$ gives the anom. dim. of $\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})$. We calculated series expansions for this in powers of α_{IR} in RS, PRD 87, 105005 (2013); RS, PRD 87,116007 (2013) and scheme-independent series expansions up to $O(\Delta_f^4)$ in Ryttov and RS, PRD 94, 125005 (2016) [1610.00387] and to $O(\Delta_f^5)$ in Ryttov and RS, PRD 95, 085012 (2017) [1701.06083], PRD 95, 105004 (2017) [1703.08558]; PRD 96, 105015 (2017) [1709.05358]; PRD 97, 025004 (2018) [arXiv:1710.06944].

We write

$$\beta'_{IR} = \sum_{j=2}^{\infty} d_j \, \Delta_f^j$$

We find

$$d_2 = \frac{2^5 T_f^2}{3^2 C_A D}, \qquad d_3 = \frac{2^7 T_f^3 (5C_A + 3C_f)}{3^3 C_A^2 D^2}$$

$$
\begin{aligned}
d_4 &= -\frac{2^3 T_f^2}{3^6 C_A^4 D^5}\Bigg[-3C_A T_f^2\Big(137445C_A^4 + 103600C_A^3 C_f + 72616C_A^2 C_f^2 + 951808C_A C_f^3 - 63888C_f^4\Big)\\
&- 5120T_f^2 D\frac{d_A^{abcd}d_A^{abcd}}{d_A} + 90112C_A T_f D\frac{d_R^{abcd}d_A^{abcd}}{d_A} - 340736C_A^2 D\frac{d_R^{abcd}d_R^{abcd}}{d_A}\\
&+ 8448D\Bigg[C_A^2 T_f^2\Big(21C_A^2 + 12C_A C_f - 33C_f^2\Big) + 16T_f^2\frac{d_A^{abcd}d_A^{abcd}}{d_A} - 104C_A T_f\frac{d_R^{abcd}d_A^{abcd}}{d_A} + 88C_A^2\frac{d_R^{abcd}d_R^{abcd}}{d_A}\Bigg]\zeta_3\Bigg].
\end{aligned}
$$

$$
\begin{aligned}
d_5 &= \frac{2^4 T_f^3}{3^7 C_A^5 D^7}\Bigg[-C_A T_f^2\Big(39450145C_A^6 + 235108272C_A^5 C_f + 1043817726C_A^4 C_f^2 + 765293216C_A^3 C_f^3\\
&- 737283360C_A^2 C_f^4 + 730646400C_A C_f^5 - 356750592C_f^6\Big) - 2^9 T_f^2 D\frac{d_A^{abcd}d_A^{abcd}}{d_A}(6139C_A^2 + 2192C_A C_f - 3300C_f^2)\\
&+ 2^9 C_A T_f D\frac{d_R^{abcd}d_A^{abcd}}{d_A}(43127C_A^2 - 28325C_A C_f - 2904C_f^2) + 15488C_A^2 D\frac{d_R^{abcd}d_R^{abcd}}{d_A}(2975C_A^2 + 8308C_A C_f - 12804C_f^2)\\
&+ 2^7 D\Bigg[3C_A T_f^2 D\Big(6272C_A^4 - 49823C_A^3 C_f + 40656C_A^2 C_f^2 + 13200C_A C_f^3 + 2112C_f^4\Big)\\
&+ 2^4 T_f^2\frac{d_A^{abcd}d_A^{abcd}}{d_A}(19516C_A^2 - 18535C_A C_f - 21780C_f^2) - 2^3 C_A T_f\frac{d_R^{abcd}d_A^{abcd}}{d_A}(182938C_A^2 - 297649C_A C_f - 197472C_f^2)\\
&- 88C_A^2\frac{d_R^{abcd}d_R^{abcd}}{d_A}(245C_A^2 + 62524C_A C_f + 42108C_f^2)\Bigg]\zeta_3\\
&+ 2^{10}\cdot 55C_A D^2\Bigg[9C_A T_f^2 D(C_A + 2C_f)(C_A - C_f) + 160T_f^2\frac{d_A^{abcd}d_A^{abcd}}{d_A}\\
&- 80T_f(10C_A + 3C_f)\frac{d_R^{abcd}d_A^{abcd}}{d_A} - 440C_A(C_A - 3C_f)\frac{d_R^{abcd}d_R^{abcd}}{d_A}\Bigg]\zeta_5\Bigg].
\end{aligned}
$$

We have analyzed the results for $G = \mathrm{SU}(N_c)$ and fermion reps. $R = F$, adj, S_2, and A_2. Note that the d_j are not all positive, in contrast to our calculated κ_j. For $R = F$ we find:

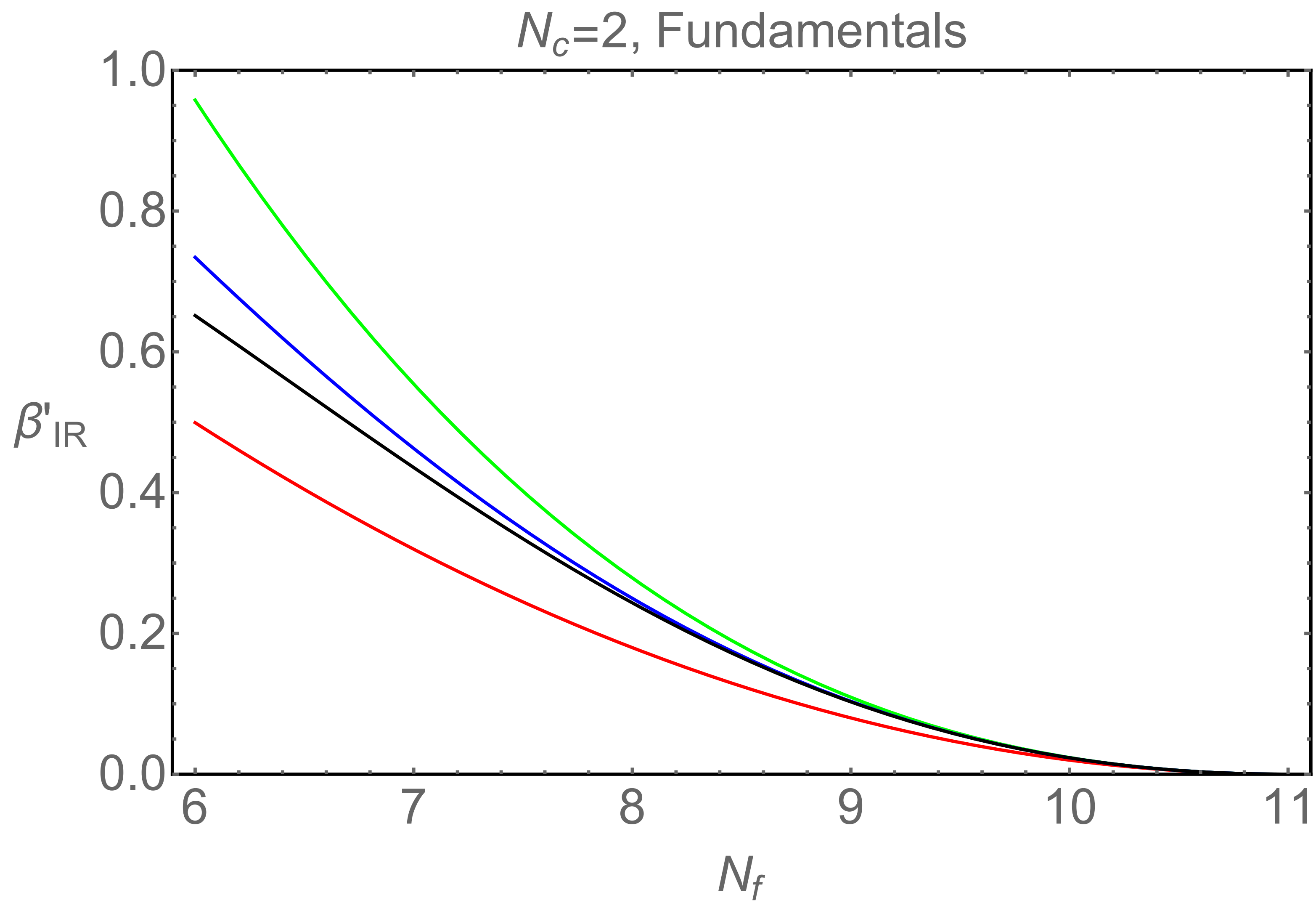
$$\mathrm{SU}(2): \quad \beta'_{IR,F,\Delta_f^5} = \Delta_f^2 \Big[(1.99750 \times 10^{-2} + (3.66583 \times 10^{-3})\Delta_f - (3.57303 \times 10^{-4})\Delta_f^2 - (2.64908 \times 10^{-5})\Delta_f^3 \Big]$$

$$\mathrm{SU}(3): \quad \beta'_{IR,F,\Delta_f^5} = \Delta_f^2 \Big[(0.83074 \times 10^{-2}) + (0.98343 \times 10^{-3}\Delta_f - (0.46342 \times 10^{-4})\Delta_f^2 - (0.56435 \times 10^{-5})\Delta_f^3 \Big]$$

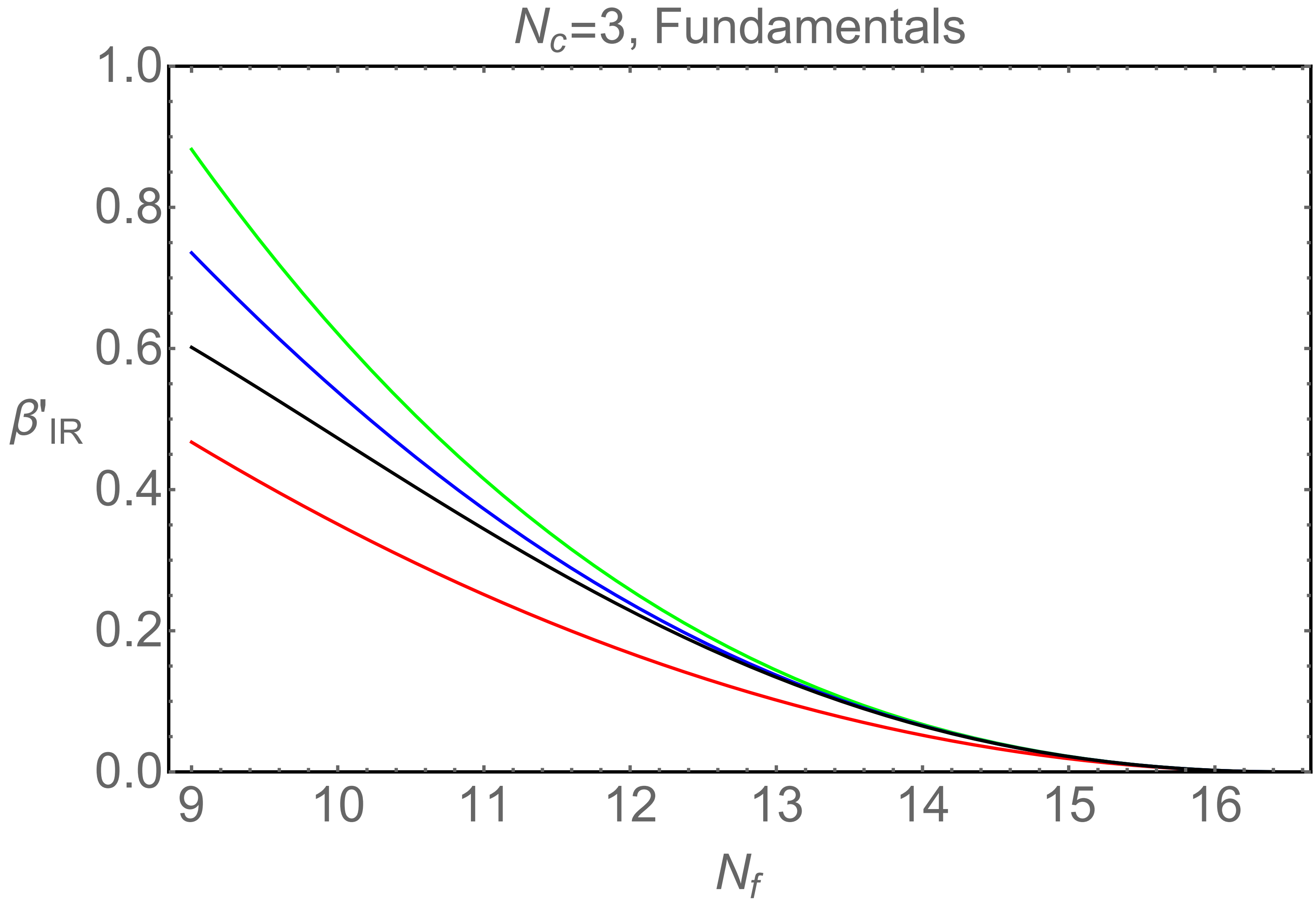
Table and plots for SU(2), SU(3), $R = F$: curves are for β'_{IR,F,Δ_f^2} (red); β'_{IR,F,Δ_f^3} (green), β'_{IR,F,Δ_f^4} (blue); and β'_{IR,F,Δ_f^5} (black).

N_c	N_f	$\beta'_{IR,F,2\ell}$	$\beta'_{IR,F,3\ell,\overline{\rm MS}}$	$\beta'_{IR,F,4\ell,\overline{\rm MS}}$	β'_{IR,F,Δ_f^2}	β'_{IR,F,Δ_f^3}	β'_{IR,F,Δ_f^4}	β'_{IR,F,Δ_f^5}
2	6	6.061	1.620	0.975	0.499	0.957	0.734	0.6515
2	7	1.202	0.728	0.677	0.320	0.554	0.463	0.436
2	8	0.400	0.318	0.300	0.180	0.279	0.250	0.243
2	9	0.126	0.115	0.110	0.0799	0.109	0.1035	0.103
2	10	0.0245	0.0239	0.0235	0.0200	0.0236	0.0233	0.0233
3	9	4.167	1.475	1.464	0.467	0.882	0.7355	0.602
3	10	1.523	0.872	0.853	0.351	0.621	0.538	0.473
3	11	0.720	0.517	0.498	0.251	0.415	0.3725	0.344
3	12	0.360	0.2955	0.282	0.168	0.258	0.239	0.228
3	13	0.174	0.1556	0.149	0.102	0.144	0.137	0.134
3	14	0.0737	0.0699	0.0678	0.0519	0.0673	0.0655	0.0649
3	15	0.0227	0.0223	0.0220	0.0187	0.0220	0.0218	0.0217
3	16	2.21e-3	2.20e-3	2.20e-3	2.08e-3	2.20e-3	2.20e-3	2.20e-3

Lattice calculation by Hasenfratz and Schaich, JHEP 02(2018) 132 obtains $\beta'_{IR} = 0.26(2)$ for SU(3) with $N_f = 12$, in agreement with our calculations.



N_c=2, Fundamentals
1.0
0.8
0.6
0.4
0.2
0.0
β'_{IR}
6
7
8
9
10
11
N_f


N_c=3, Fundamentals
1.0
0.8
0.6
0.4
0.2
0.0
β'_{IR}
9
10
11
12
13
14
15
16
N_f

RG Flows in Other Theories

We have also performed higher-loop calculations of RG flows and investigated possible zeros of beta functions for other theories, both asymptotically free and non-asymptotically free:

- 2D Gross-Neveu model
- 6D ϕ^3 theories
- 4D U(1) gauge theory with (charged) fermions
- 4D $\lambda|\vec{\phi}|^4$ theory

We briefly discuss these next.

(In separate work, we have studied RG evolution of asymptotically free chiral gauge theories; e.g., see Appelquist and RS, PLB 548, 204 (2002); PRL 90, 201801 (2003); PRD 88, 105012 (2013); Quigg and RS, PRD 79, 096002 (2009); Shi and RS, PRD 91, 045004 (2015); PRD 92, 105032 (2015); PRD 92, 125009 (2015); PRD 94, 065001 2016); Ryttov and RS, PRD 97, 016020 (2018).)

Study of RG Flows for the Finite-N Gross-Neveu Model

The Gross-Neveu (GN) model is a QFT in $d = 2$ dimensions with an N-component massless fermion, ψ_j, $j = 1, ..., N$ and a four-fermion interaction. The Lagrangian is

$$\mathcal{L} = i\bar{\psi}\partial\!\!\!/\psi + \frac{G}{2}(\bar{\psi}\psi)^2$$

This model has been of interest because it exhibits some properties similar to QCD, namely asymptotic freedom and formation of massive bound states of fermions. The model was solved exactly in the $N \to \infty$ limit with NG fixed by Gross and Neveu (1974). In this limit, the beta function has no IR zero.

This leaves open the question of whether the beta function has an IR zero for finite N. We investigated this, using the beta function up to the 4-loop level, in Choi, Ryttov, and RS, PRD 95, 025012 (2017) [arXiv:1612.05580].

As part of our study, we calculate and analyze Padé approximants and evaluate effects of scheme dependence.

From our study, we find that in the range of coupling where the perturbative calculation of the beta function is reliable, it does not exhibit robust evidence for an IR zero.

Study of RG Flows for ϕ^3 Theories in $d = 6$ Dimensions

ϕ^3 theories in $d = 6$ dimensions are asymptotically free, and it is of interest to investigate whether they exhibit IRFPs. We have done this, using beta functions calculated up to the 4-loop order, in Gracey, Ryttov, and RS, PRD 89, 045019 (2014) [1311.5268]; Ryttov and RS, PRD 102, 056016 (2020) [2008.06772].

As before, without loss of generality, we take the matter field to be massless, since a ϕ field with nonzero mass m_ϕ would be integrated out of the low-energy effective theory for momentum scales $\mu < m_\phi$.

Lagrangian for this ϕ^3_6 theory with real 1-component ϕ field:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{g}{3!}\phi^3 \ .$$

Lagrangian for the ϕ^3_6 theory with ϕ field transforming according to the fundamental representation of a global SU(N) symmetry:

$$\mathcal{L}_2 = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) - \frac{g}{3!} d_{ijk}(\phi^i \phi^j \phi^k + h.c.)$$

where d_{ijk} is the symmetric rank-3 tensor for SU(N).

For both of these ϕ^3 theories in $d = 6$, we find evidence against an IRFP.

Study of RG Flows for Non-Asymptotically-Free Theories

If the β function of a theory is positive near zero coupling, then this theory is IR-free; as the reference scale μ decreases, the coupling decreases toward 0. As μ increases from the IR, the coupling increases, and a basic question is whether the beta function has a UV zero (in the perturbatively calculable range), which would be a UV fixed point of the RG.

At $d = 2$, the O(N) nonlinear σ model is asymptotically free. For $d = 2 + \epsilon$, this model is IR-free and provides an example of a UVFP in an IR-free theory. From a solution of this model in the $N \longrightarrow \infty$ limit (involving a sum of an infinite number of Feynman diagrams that dominate in this limit), one finds, for small ϵ,

$$\beta(\xi) = \epsilon\xi\Big(1 - \frac{\xi}{\xi_c}\Big)$$

where ξ is the effective coupling and $\xi_c = 2\pi\epsilon$: Bardeen, B. W. Lee, RS, PRD 14, 985 (1976); and Brézin, Zinn-Justin, PRB 14, 3110 (1976).

Hence, assuming that ξ is small for small μ, it follows that as μ increases, ξ approaches the UV fixed point at ξ_c as $\mu \longrightarrow \infty$.

U(1) Gauge Theory

In the early history of QED, it was noted that the beta function is positive, so the theory is IR-free. The 1-loop and 2-loop coefficients are both positive, so there is no UV zero in β at the maximal scheme-independent order.

We have studied this further up to 5-loop level in RS, PRD 89, 045019 (2014) [1311.5268], using b_4 from Gorishny, Kataev, Larin, Surguladze (1991) and b_5 from Kataev, and Larin, JETP Lett. 96, 61 (2012); Baikov, Chetyrkin, Kühn, Rittinger, NPB 867, 182 (2013) in $\overline{\mathrm{MS}}$ scheme.

We find evidence against a UVFP in the U(1) gauge thy. As before, we have studied effects of scheme transf; see also Kataev and Molokoedov, PRD 92, 054008 (2015).

This does not imply that a Landau pole occurs, since the coupling α becomes too large for perturbative calculations to be reliable as one approaches this pole.

In the Standard Model, the U$(1)_{em}$ gauge symmetry results from the electroweak symmetry breaking of $\mathrm{SU}(2)_L \otimes \mathrm{U}(1)_Y$, so the question is then the UV behavior of the hypercharge U$(1)_Y$. In grand unified theories, the U$(1)_Y$ theory is embedded in the non-abelian GUT gauge group, so the question of asymptotic behavior of a U(1) gauge interaction does not arise.

RG Flows in the O(N) $\lambda|\vec{\phi}|^4$ Theory

The 4D O(N)-symmetric $\lambda|\vec{\phi}|^4$ theory with an N-component scalar field $\vec{\phi}$ is another interesting IR-free theory. We consider

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \vec{\phi}) \cdot (\partial^\mu \vec{\phi}) - \frac{\lambda}{4!}|\vec{\phi}|^4$$

where $\vec{\phi} = (\phi_1, ..., \phi_N)$. In RS, PRD 90, 065023 (2014) [1408.3141]; PRD 94, 125026 (2016) [1610.03733]; PRD 96, 056010 (2017) [arXiv:1707.06248] we have investigated RG flows and searched for a possible UVFP using the beta function up to 6-loop order (5-loop term from Kleinert, Neu, Schulte-Frohlinde, Chetyrkin, Larin, 1991, 1993); 6-loop term from Kompaniets and Panzer, PRD 96, 036016 (2017) [1606.09210] in $\overline{\mathrm{MS}}$ scheme).

With $dt = d\ln\mu$ as before, we write

$$\beta_\lambda = \frac{d\lambda}{dt} = \lambda \sum_{\ell=1}^{\infty} b_\ell \, a_\lambda^\ell \ ,$$

where $a_\lambda = \lambda/(4\pi)^2$.

Ideally, for a UVFP here, we would find that the beta function calculated to progressively higher loop orders would exhibit a zero for each order, and the values $a_{\lambda,UV,n\ell}$ at which this zero occurs at the n-loop order would be similar for successive loop orders.

As is evident in the table, we do not find this. The notation u means "unphysical"; the zero in β_λ closest to origin is negative or consists of a complex-conjugate pair.

As before, we use a combination of analysis of the power series expansion in a_λ, Padé resummation methods, and scheme transformations to check this.

So we do not find convincing evidence of a UVFP in the $\lambda|\vec{\phi}|^4$ theory.

N	$a_{\lambda,UV,2\ell}$	$a_{\lambda,UV,3\ell}$	$a_{\lambda,UV,4\ell}$	$a_{\lambda,UV,5\ell}$	$a_{\lambda,UV,6\ell}$
1	0.5294	u	0.2333	u	0.1604
2	0.5000	u	0.2217	u	0.1529
3	0.4783	u	0.2123	u	0.1467
4	0.4615	u	0.2044	u	0.1414
5	0.4483	u	0.1978	u	0.1368
6	0.4375	u	0.1920	u	0.1328
7	0.4286	u	0.1869	u	0.1292
8	0.42105	u	0.1823	u	0.1259
9	0.4146	u	0.1783	u	0.1229
10	0.4091	u	0.1746	u	0.1202
30	0.3654	u	0.1362	u	0.09033
100	0.3439	u	0.1012	u	0.05965
300	0.3370	u	0.07944	u	0.03783
500	0.3355	u	0.07341	0.08045	0.03074
800	0.3347	u	0.07137	0.02871	0.02866
890	0.3346	u	0.07164	0.02559	0.03829
900	0.3346	u	0.07170	0.02530	u
1000	0.3344	u	0.07241	0.02276	u
2000	0.3339	u	0.1054	0.01231	u
3000	0.3337	u	0.5475	0.008850	u
4000	0.3336	u	u	0.007042	u
10^4	0.3334	u	u	0.003460	u

Conclusions

- Understanding the UV to IR evolution of an asymptotically free gauge theory and the nature of the IR behavior is of fundamental field-theoretic interest.

- Our higher-loop calculations give information on this UV to IR flow and on determination of IR properties; we now have results up to the 5-loop level.

- We have investigated effects of scheme-dependence of IR zero in the beta function in higher-loop calculations.

- We have calculated scheme-independent series expansions for $\gamma_{\bar{\psi}\psi,IR}$ to $O(\Delta_f^4)$ and β'_{IR} to $O(\Delta_f^5)$ using 5-loop inputs.

- In addition to these $O(\Delta_f^4)$ and $O(\Delta_f^5)$ calculations, extrapolations and Padé approximants give further insight.

- These continuum calculations provide useful comparisons with lattice measurements.

- We have also studied RG flows and investigated possible RG fixed points in other interesting theories.

Appendices

Scheme Transformations to Remove $\ell \geq 3$ Loop Terms in the Beta Function

We describe the construction of a scheme transformation (ST), denoted S_{R,m,k_1}, that removes the terms in the beta function from loop order 3 up to $m+1$, inclusive, for small coupling. In the limit $m \to \infty$, this transforms to the 't Hooft scheme.

To construct this ST, first, we take advantage of the property that in b'_ℓ, the ST coefficient $k_{\ell-1}$ appears only linearly. For example, $b'_3 = b_3 + k_1 b_2 + (k_1^2 - k_2) b_1$, etc. for higher-$\ell$ b'_ℓ. So solve eq. $b'_3 = 0$ for k_2, obtaining

$$k_2 = \frac{b_3}{b_1} + \frac{b_2}{b_1} k_1 + k_1^2$$

This determines $S_{R,2,k_1}$.

To get $S_{R,3,k_1}$, substitute this k_2 into expression for b'_4 and solve eq. $b'_4 = 0$, obtaining

$$k_3 = \frac{b_4}{2b_1} + \frac{3b_3}{b_1} k_1 + \frac{5b_2}{2b_1} k_1^2 + k_1^3$$

This determines $S_{R,3,k_1}$.

We continue this procedure iteratively to calculate S_{R,m,k_1} for higher m. In general, the equation $b'_\ell = 0$ is a linear equation for $k_{\ell-1}$, so one is guaranteed a unique solution.

So the ST S_{R,m,k_1} has nonzero k_s, $s = 1, ..., m$ and in the transformed beta function, sets $b'_\ell = 0$ for $\ell = 3, ..., m + 1$.

The coefficients k_s for this ST depend on the b_n in the original beta function for $n = 1, ..., m + 1$, and on the parameter k_1.

In addition to the successful application near the origin, $\alpha = 0$, we have shown that this ST S_{R,m,k_1} can be applied over part, but not all, of the interval I where the 2-loop beta function has an IR zero.

Method for Scheme-Independent Series Calculations

The method of calculation is as follows. To calculate κ_j, one begins by writing $a_{IR} = \alpha_{IR}/(4\pi)$ as a series expansion in Δ_f:

$$a_{IR} = \sum_{j=1}^{\infty} a_j \, \Delta_f^j$$

For β, extracting a prefactor, define a reduced

$$\beta_r = \frac{\beta}{(-8\pi a^2)} = \sum_{\ell=1}^{\infty} b_\ell \, a^{\ell-1}$$

so the condition for the IR zero is $\beta_r = 0$ at $a = a_{IR}$. Next, one expands the coefficients b_ℓ in Taylor series around $N_f = N_u$, i.e., $\Delta_f = 0$, and substitutes the resulting expansions for b_ℓ and the Δ_f expansion for a_{IR} in the eq. $\beta_r = 0$ at $a = a_{IR}$. The result can be written as

$$\beta_r \Big|_{\alpha=\alpha_{IR}} = 0 = \sum_{n=1}^{\infty} k_n \, \Delta_f^n$$

Since the sum is zero for all Δ_f, each $k_n = 0$. This yields a set of linear equations that one can solve for the a_n in terms of the Taylor series coefficients of b_ℓ in the expansion about $\Delta_f = 0$.

Next, one inserts these expressions for the a_n in the Δ_f expansion for a_{IR}. Then, one carries out similar Taylor series expansions of the c_ℓ around $\Delta_f = 0$ and substitutes these, with the a_{IR}, in the expansion $\gamma_{\bar\psi\psi,IR} = \sum_\ell c_\ell \, a_{IR}^\ell$. This yields the κ_j.

LNN Limit

For $G = \mathrm{SU}(N_c)$ and $R = F$, one may consider the LNN (large N_c, large N_f 't Hooft-Veneziano) limit

$$LNN: \quad N_c \to \infty\ , \quad N_f \to \infty \quad \text{with } \ r \equiv \frac{N_f}{N_c} \text{ fixed and finite}$$
$$\text{and } \ \xi(\mu) \equiv \alpha(\mu) N_c \text{ finite} \tag{1}$$

Let $x = \xi/(4\pi)$ and define a rescaled beta function that is finite in this limit:

$$\beta_\xi = \frac{d\xi}{dt} = \lim_{LNN} N_c\, \beta = -8\pi x \sum_{\ell=1}^{\infty} \hat{b}_\ell x^\ell$$

where the n-loop coeffs. that are finite in this LNN limit are

$$\hat{b}_\ell = \lim_{LNN} \frac{b_\ell}{N_c^\ell}\ .$$

Asymptotic freedom requires $r < 11/2$. The interval of r where $\beta_{\xi,2\ell}$ has an IR zero is

$$I_r: \quad r_\ell < r < r_u, \quad i.e., \quad 2.615 < r < 5.500$$

where

$$r_\ell = \lim_{LNN} \frac{N_\ell}{N_c} = \frac{34}{13} = 2.615\ , \qquad r_u = \lim_{LNN} \frac{N_u}{N_c} = \frac{11}{2}$$

We have studied the approach to the LNN limit and find that this is quite rapid, with leading correction terms in physical quantities suppressed by $1/N_c^2$: RS, PRD 87, 105005, 116007 (2013).

Define the scaled scheme-independent expansion parameter for the LNN limit

$$\Delta_r \equiv \frac{\Delta_f}{N_c} = r_u - r = \frac{11}{2} - r \ .$$

As r decreases from r_u to r_ℓ in the interval I_r, Δ_r increases from 0 to a maximal value

$$(\Delta_r)_{max} = r_u - r_\ell = \frac{75}{26} = 2.8846$$

For the scheme-independent expansion, define rescaled coefficients that are finite in this LNN limit:

$$\hat{\kappa}_{j,F} \equiv \lim_{LNN} N_c^j \, \kappa_{j,F}$$

The anomalous dimension $\gamma_{\bar\psi\psi,IR}$ is finite in the LNN limit and is given by

$$R = F: \quad \lim_{LNN} \gamma_{\bar\psi\psi,IR} = \sum_{j=1}^{\infty} \kappa_{j,F} \Delta_f^j = \sum_{j=1}^{\infty} \hat{\kappa}_{j,F} \Delta_r^j \ .$$

From our general results, we obtain

$$\hat{\kappa}_{1,F} = \frac{2^2}{5^2} = 0.1600 \ , \qquad \hat{\kappa}_{2,F} = \frac{588}{5^6} = 0.037632 \ ,$$

$$\hat{\kappa}_{3,F} = \frac{2193944}{3^3 \cdot 5^{10}} = 0.83207 \times 10^{-2} \ ,$$

$$\hat{\kappa}_{4,F} = \frac{210676352}{3^4 \cdot 5^{13}} + \frac{90112}{3^3 \cdot 5^{10}}\zeta_3 + \frac{11264}{3^3 \cdot 5^8}\zeta_5 = 0.36489 \times 10^{-2}$$

Hence, numerically, to order $O(\Delta_r^4)$,

$$\gamma_{IR,LNN,\Delta_r^4} = 0.160000\Delta_r + 0.037632\Delta_r^2 + 0.0083207\Delta_r^3 + 0.003649\Delta_r^4$$

r	$\gamma_{IR,F,2\ell}$	$\gamma_{IR,F,3\ell}$	$\gamma_{IR,F,4\ell}$	γ_{IR,F,Δ_r}	γ_{IR,F,Δ_r^2}	γ_{IR,F,Δ_r^3}	γ_{IR,F,Δ_r^4}
2.8	> 2	1.708	0.190	0.432	0.706	0.870	1.064
3.0	> 2	1.165	0.225	0.400	0.635	0.765	0.908
3.2	> 2	0.854	0.264	0.368	0.567	0.668	0.770
3.4	> 2	0.656	0.293	0.336	0.502	0.579	0.650
3.6	1.853	0.520	0.308	0.304	0.440	0.497	0.5445
3.8	1.178	0.420	0.306	0.272	0.381	0.422	0.452
4.0	0.785	0.341	0.288	0.240	0.325	0.353	0.371
4.2	0.537	0.277	0.257	0.208	0.272	0.290	0.300
4.4	0.371	0.222	0.217	0.176	0.2215	0.233	0.238
4.6	0.254	0.1735	0.1745	0.144	0.1745	0.1805	0.183
4.8	0.170	0.129	0.131	0.112	0.130	0.133	0.134
5.0	0.106	0.0889	0.0900	0.0800	0.0894	0.09045	0.0907
5.2	0.0562	0.0512	0.0516	0.0480	0.0514	0.0516	0.0516
5.4	0.0168	0.0164	0.0164	0.0160	0.0164	0.0164	0.0164

Table of values of n-loop $(n\ell)$ $\gamma_{IR,F,n\ell}$ and scheme-independent γ_{IR,F,Δ_f^j} values for $r \in I_r$.

Appendix: Proof that $\gamma_{\mathcal{O},IR}$ is scheme-independent, where $\mathcal{O}$ is some (gauge-invariant) operator, such as $\bar{\psi}\psi$, etc.

Denote the (vertex) renormalization constant for $\mathcal{O}$ as $Z_{\mathcal{O}}$. The anomalous dimension of $\mathcal{O}$ is (with $dt = d\ln\mu$)

$$\gamma_{\mathcal{O}} = \frac{d\ln Z_{\mathcal{O}}}{dt}$$

Denote the coupling and renorm. const. in the transformed scheme as α' and $Z'_{\mathcal{O}}(\alpha')$, given by $Z'_{\mathcal{O}}(\alpha') = Z_{\mathcal{O}}(\alpha)G(\alpha)$, and the anom. dim. as $\gamma'_{\mathcal{O}}(\alpha')$. Then

$$\begin{aligned} \gamma'_{\mathcal{O}}(\alpha') &= \frac{d\ln Z'_{\mathcal{O}}}{dt} = \frac{d}{dt}\Big[\ln(Z_{\mathcal{O}}(\alpha)) + \ln(G(\alpha))\Big] = \gamma_{\mathcal{O}}(\alpha) + \frac{d\alpha}{dt}\frac{d\ln(G(\alpha))}{d\alpha} \\ &= \gamma_{\mathcal{O}}(\alpha) + \beta(\alpha)\frac{d\ln(G(\alpha))}{d\alpha} \end{aligned}$$

Hence, at $\alpha = \alpha_{IR}$ where $\beta(\alpha_{IR}) = 0$, it follows that $\gamma'_{\mathcal{O}}(\alpha'_{IR}) = \gamma_{\mathcal{O}}(\alpha_{IR})$, i.e., $\gamma_{\mathcal{O}}(\alpha_{IR}) = \gamma_{\mathcal{O},IR}$ is scheme-independent.

Proof that $\left.\frac{d\beta}{d\alpha}\right|_{\alpha=\alpha_{IR}}$ is scheme-independent: denote the scheme transformation (ST) function as $F(\alpha')$, so $\alpha = F(\alpha')$, satisfying $F(\alpha') \to \alpha'$ as $\alpha' \to 0$, since a ST has no effect in the free-field limit. We have

$$\beta(\alpha) = \frac{d\alpha}{dt}, \qquad \beta'(\alpha') = \frac{d\alpha'}{dt}, \qquad\qquad \frac{d\alpha}{d\alpha'} = \frac{dF(\alpha')}{d\alpha'} = \left[\frac{d\alpha'}{d\alpha}\right]^{-1}$$

Now

$$\beta(\alpha) = \frac{d\alpha}{dt} = \frac{d}{dt}\left[F(\alpha')\right] = \frac{d\alpha'}{dt}\frac{dF(\alpha')}{d\alpha'} = \beta'(\alpha')\frac{dF(\alpha')}{d\alpha'}$$

so

$$\frac{d\beta(\alpha)}{d\alpha} = \frac{d}{d\alpha}\left[\beta'(\alpha')\frac{dF(\alpha')}{d\alpha'}\right] = \frac{d\alpha'}{d\alpha}\frac{d}{d\alpha'}\left[\beta'(\alpha')\frac{dF(\alpha')}{d\alpha'}\right]$$

$$= \left[\frac{dF(\alpha')}{d\alpha'}\right]^{-1}\left[\frac{d\beta'(\alpha')}{d\alpha'}\frac{dF(\alpha')}{d\alpha'} + \beta'(\alpha')\frac{d^2F(\alpha')}{d\alpha'^2}\right]$$

Hence, at α'_{IR} where $\beta'(\alpha'_{IR}) = 0$ and equivalently, $\beta(\alpha_{IR}) = 0$,

$$\frac{d\beta(\alpha)}{d\alpha} = \left[\frac{dF(\alpha')}{d\alpha'}\right]^{-1}\frac{d\beta'(\alpha')}{d\alpha'}\frac{dF(\alpha')}{d\alpha'} = \frac{d\beta'(\alpha')}{d\alpha'}$$

which shows that $\left.\frac{d\beta}{d\alpha}\right|_{\alpha=\alpha_{IR}}$ is scheme-independent.

Relation between β'_{IR} and $\gamma_{F^2,IR}$:

We have $\mathcal{L}_{F^2} = \frac{1}{4g^2} F^a_{\mu\nu,r} F^{a,\mu\nu}_r$, where $F^a_{\mu\nu,r} = gF^a_{\mu\nu}$. Denote $F^a_{\mu\nu,r}F^{a,\mu\nu}_r \equiv F^2$, with anomalous dimension γ_{F^2}. The energy-momentum tensor T^ν_μ is given in terms of $F^a_{\mu\rho}F^{a,\rho\nu}$, and its trace is

$$T \equiv T^\rho_\rho = \frac{\beta}{16\pi\alpha^2} F^2$$

The scaling dimension $\Delta_{\mathcal{O}}$ of an operator $\mathcal{O}$ is given by $\frac{d}{dt}\mathcal{O} = -\Delta_{\mathcal{O}}\,\mathcal{O}$. The energy-momentum tensor is conserved, so it has canonical dimension 4; so same is true for trace: $\frac{d}{dt}T = -4T$. Hence,

$$\begin{aligned}
\frac{d}{dt}T = -4T = \frac{d}{dt}\Big[\frac{\beta}{16\pi\alpha^2}F^2\Big] &= \frac{1}{16\pi}\Big[\frac{1}{\alpha^2}(\frac{d\beta}{d\alpha}\frac{d\alpha}{dt})F^2 - \frac{2\beta}{\alpha^3}\frac{d\alpha}{dt}F^2 + \frac{\beta}{\alpha^2}\frac{dF^2}{dt}\Big] \\
&= \frac{1}{16\pi}\Big[\frac{\beta'\,\beta}{\alpha^2}F^2 - \frac{2\beta^2}{\alpha^3}F^2 + \frac{\beta}{\alpha^2}(-\Delta_{F^2}F^2)\Big] \\
&= \frac{\beta}{16\pi\alpha^2}F^2\left[\beta' - \frac{2\beta}{\alpha} - \Delta_{F^2}\right]
\end{aligned}$$

Hence, $-4 = \beta' - \frac{2\beta}{\alpha} - \Delta_{F^2}$, so $\Delta_{F^2} = 4 + \beta' - \frac{2\beta}{\alpha} \equiv 4 - \gamma_{F^2}$ and therefore $\gamma_{F^2} = -\beta' + \frac{2\beta}{\alpha}$. At a zero of β (here, the IRFP), this reduces to

$$\gamma_{F^2,IR} = -\beta'_{IR}$$