Rigorous reconstruction of gluon propagator in the presence of complex singularities

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Introduction

- Analytic structure of a propagator: **states and spectrum**

  Physical case: **Källén-Lehmann spectral representation**

$$D(k^2) = \int_0^\infty d\sigma^2 \frac{\rho(\sigma^2)}{\sigma^2 - k^2},$$

$$\theta(k_0)\rho(k^2) := (2\pi)^d \sum_n |\langle 0|\phi(0)|P_n\rangle|^2 \delta^D(P_n - k),$$

singularities on complex k^2-plane

$\longleftrightarrow$ **states** non-orthogonal to $\phi(0)\,|0\rangle$

- **Analytic structures of the QCD propagators** would be useful for understanding fundamental aspects of QCD, e.g., **confinement**.

Based on the progress on the Landau-gauge gluon, ghost, and quark propagators, there has been an increasing interest in their analytic structures.

Review: recent approaches to analytic structure

Recent analyses suggest **complex singularity** of the gluon propagator invalidating the Källén-Lehmann spectral representation.

1. Modeling gluon propagator to fit lattice results
   - (refined-)Gribov-Zwanziger model [Dudal et. al. 2008]...
   - Massive-like gluon model [Siringo 2016] [YH and Kondo, 2018, 2020]
   - Padé approximation [Falcão, Oliveira, and Silva 2020]
   - (A variant of) Schlessinger-point method [Binosi and Tripolt 2019]
2. Dyson-Schwinger equation on the complex momentum plane

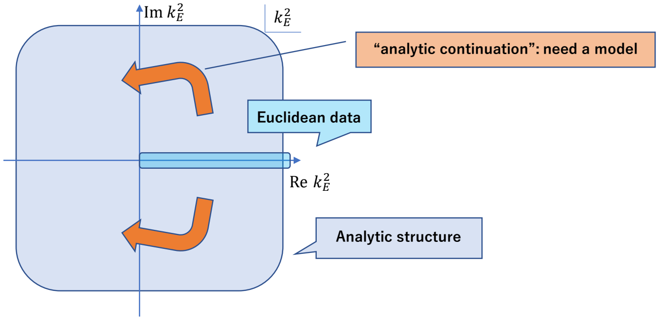
   [Strauss, Fischer, and Kellermann 2012] [Binosi and Tripolt 2019] [Huber and Fischer 2020]

However, the interpretation of complex singularities has been less studied...(short-lived gluon? non-locality? [Stingl 1986] etc.)

Plan

How to investigate analytic structures

- Aim: investigating analytic structures of the propagators from Euclidean data through "analytic continuation".
- The "analytic continuation" from finite data is in principle an ill-posed problem: **use models consistent with the Euclidean data with some theoretical backgrounds**.

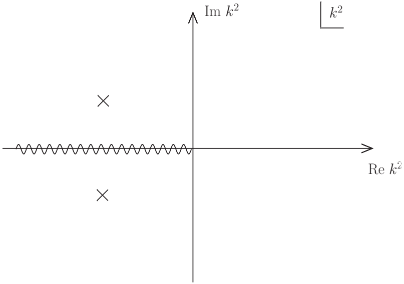


What is "complex singularity" exactly?

Complex singularity: singularity off the real axis in the complex Euclidean momentum plane k_E^2 of an analytically continued Euclidean propagator $D(k_E^2)$.

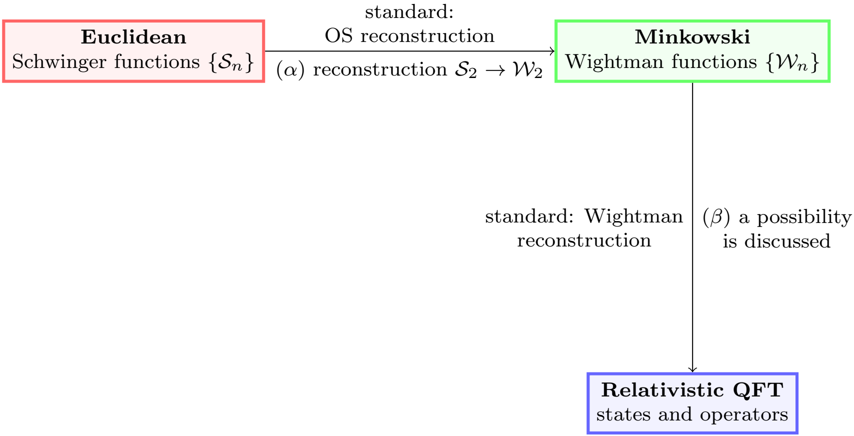
e.g.) complex poles: poles not on the real axis of analytically continued Euclidean propagator

$$D(k_E^2) = \frac{Z}{w + k_E^2} + \frac{Z^*}{w^* + k_E^2} + \int_0^\infty d\sigma^2 \frac{\rho(\sigma^2)}{\sigma^2 + k_E^2},$$



Reconstruction from Euclidean field theory to QFT

[Osterwalder and Schrader 1973, 1975]



Main Questions

In the presence of complex singularities, natural questions on this procedures are,

(α) Is it possible to reconstruct a Wightman function $W(\xi^0, \vec{\xi})$ on the Minkowski spacetime from the Schwinger function? Which conditions of the Wightman/OS axioms are preserved/violated?

(β) Does there exist a quantum theory reproducing the reconstructed Wightman function $W(\xi^0, \vec{\xi})$ as a vacuum expectation value: $W(\xi) = \langle 0|\phi(\xi)\phi(0)|0\rangle$? If it exists, what states cause complex singularities?

We will answer these questions affirmatively.

Plan

General properties of complex singularities

Wightman function $W(t, \vec{x})$ is reconstructed from Schwinger function $S(\tau, \vec{x})$ by identifying $S(\tau, \vec{x}) = W(-i\tau, \vec{x})$ $(\tau > 0)$.
In the presence of complex singularities (bounded in k_E^2-plane), we rigorously prove:

List of properties

- Holomorphy of $W(\xi - i\eta)$ in the tube $\mathbb{R}^4 - iV_+$ [V_+: forward lightcone]
- Existence of the boundary value $W(t, \vec{x}) = \lim_{\tau \to +0} W(t - i\tau, \vec{x})$ as a distribution.
- $W(t, \vec{x})$ satisfies Lorentz symmetry and locality (i.e. spacelike commutativity).
- ◦ Non-temperedness of the boundary value $W(t, \vec{x})$
- ◦ Violation of the positivity of $W(t, \vec{x})$.

Sketches of proofs: (0)holomorphy

Reconstruction: $S(\tau, \vec{x}) \to W(\xi = (t, \vec{x}))$

$$S(\tau, \vec{x}) \overset{\tau \geq 0}{=} W(-i\tau, \vec{x}) \;\to\; W(\xi - i\eta) \text{ for } \xi \in \mathbb{R}^4,\; \eta \in V_+$$
$$\to\; W(\xi) = \lim_{\eta \to 0,\; \eta \in V_+} W(\xi - i\eta)$$

e.g.) Complex poles ($E_{\vec{p}} := \sqrt{\vec{p}^2 + M^2}$ with $\mathrm{Re}\, E_{\vec{p}} > 0$)

$$D(k_E^2) = \frac{Z}{k_E^2 + M^2} + \frac{Z^*}{k_E^2 + (M^*)^2}$$
$$\to S(\tau, \vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}} \left[\frac{Z e^{-E_{\vec{p}}\tau}}{2E_{\vec{p}}} + \frac{Z e^{-E^*_{\vec{p}}\tau}}{2E^*_{\vec{p}}} \right]$$
$$\to W(\xi - i\eta) = \int \frac{d^3\vec{p}}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{\xi} - i\vec{\eta})} \left[\frac{Z e^{-iE_{\vec{p}}(\xi^0 - i\eta^0)}}{2E_{\vec{p}}} + \frac{Z^* e^{-iE^*_{\vec{p}}(\xi^0 - i\eta^0)}}{2E^*_{\vec{p}}} \right]$$

converges (and is holomorphic in $\xi - i\eta$) for $\eta^0 > |\vec{\eta}|$, i.e., $\eta \in V_+$.

Sketches of proofs:
(1) boundary value and non-temperedness

Reconstruction: $S(\tau,\vec{x}) \to W(\xi = (t,\vec{x}))$

$$S(\tau,\vec{x}) \overset{\tau\geq 0}{=} W(-i\tau,\vec{x}) \;\to\; W(\xi - i\eta) \text{ in } \mathbb{R}^4 - iV_+$$
$$\to\; W(\xi) = \lim_{\eta\to 0,\ \eta\in V_+} W(\xi - i\eta)$$

e.g.) Gribov-type propagator (cont'd)

$$W(\xi - i\eta) = \int \frac{d^3\vec{p}}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{\xi}-i\vec{\eta})} \left[\frac{Z e^{-iE_{\vec{p}}(\xi^0 - i\eta^0)}}{2E_{\vec{p}}} + \frac{Z^* e^{-iE^*_{\vec{p}}(\xi^0 - i\eta^0)}}{2E^*_{\vec{p}}} \right]$$

- If smeared by a smooth compactly-supported function of ξ, $W(\xi - i\eta)$ has a limit $\eta \to 0$ $(\eta \in V_+)$: $^\exists W(\xi) = \lim_{\eta\to 0,\ \eta\in V_+} W(\xi - i\eta)$ as a distribution.
- Since $E_{\vec{p}}$ is complex, $W(\xi)$ grows exponentially for $\xi^0 \to \pm\infty$. $W(\xi)$ is **not tempered**.

Sketches of proofs: (2) violation of positivity

The positivity is violated due to the non-temperedness.

For this, we show

$$\text{positivity} \Longrightarrow \text{temperedness}$$

Rough idea:

- Positivity of 2pt.-function $\rightarrow$ the sector $\{\phi(x)\,|0\rangle\}_{x\in\mathbb{R}^4}$ has a positive metric.
- translational invariance $\rightarrow$ translation operator $U(a)$: $U(a)\phi(x)\,|0\rangle = \phi(x+a)\,|0\rangle$ is unitary.

Therefore, the Wightman function $W(a) = \langle 0|\phi(0)U(-a)\phi(0)|0\rangle$ will be bounded above $\Rightarrow$ tempered.

Sketches of proofs: (3) Lorentz symmetry

The Schwinger function has the Euclidean rotation $SO(4)$ invariance.

- $\rightarrow$ $W(\xi - i\eta)$ is invariant under infinitesimal Euclidean rotations.
- $\rightarrow$ $W(\xi - i\eta)$ is invariant under infinitesimal complex Lorentz transformations.
- $\rightarrow$ [1] $W(\xi - i\eta)$ is invariant under the proper complex Lorentz symmetry $L_+(\mathbb{C})$ (within its domain of definition), where $L_+(\mathbb{C}) := \{\Lambda \in \mathbb{C}^{4\times 4}\ ;\ \Lambda^T G \Lambda = G, \det \Lambda = 1\}$ with the metric $G = \mathrm{diag}(1, -1, -1, -1)$.
- $\rightarrow$ The Wightman function $W(\xi)$ is invariant under the restricted Lorentz transformation[2].

[1] An argument similar to Bargmann-Hall-Wightman theorem is used here.

[2] In the case of a scalar field, the invariance under Lorentz boosts can be explicitly checked by a contour deformation.

Sketches of proofs: (4) locality

The spacelike commutativity $[W_{ij}(\xi) = (-1)^\sigma W_{ji}(-\xi)]$[3] follows from

- permutation symmetry of Schwinger function $S_{ij}(x-y) = (-1)^\sigma S_{ji}(y-x)$,
- single-valued continuation $W_{ij}(\xi - i\eta)$ in the 'extended tube' $L_+(\mathbb{C})[\mathbb{R}^4 - iV_+]$ including spacelike points ('Jost points').
- $W_{ij}(z) = (-1)^J W_{ij}(-z)$ from $-1 \in SO(4) \subset L_+(\mathbb{C})$.[4]

$$
\begin{array}{ccc}
W_{ij}(\xi - i\eta) = (-1)^{\sigma+J} W_{ji}(\xi - i\eta) & = & (-1)^\sigma W_{ji}(-\xi + i\eta) \\
\Big\downarrow \begin{array}{l} \eta \to 0 \\ (\eta \in V_+) \end{array} & & \Big\downarrow \begin{array}{l} \eta \to 0 \\ \text{holomorphy at spacelike } \xi \end{array} \\
W_{ij}(\xi) & = & (-1)^\sigma W_{ji}(-\xi)
\end{array}
$$

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[3] $(-1)^\sigma$: statistical factor

[4] $(-1)^J$: the number of dotted indices in the correlator

Summary: answer to the question (α)

Main question (α)

Is it possible to reconstruct a Wightman function $W(\xi^0, \vec{\xi})$ on the Minkowski spacetime from the Schwinger function?
Which conditions of the Wightman/OS axioms are preserved/violated?

In this section, we have seen that it is possible to reconstruct the Wightman function as a distribution.

The violated/preserved conditions of the Wightman/OS axioms are summarized in the next slide.

Summary of Wightman/OS axioms

Minkowski: Wightman axioms for Wightman functions

[W0] Temperedness	violated ✗
[W1] Poincaré Symmetry	preserved ✓
[W2] Spectral Condition	violated ✗
[W3] Spacelike Commutativity	preserved ✓
[W4] Positivity	violated ✗
[W5] Cluster property	irrelevant

Euclidean: Osterwalder-Schrader axioms for Schwinger functions

[OS0] Temperedness	assumed ✓
[OS1] Euclidean Symmetry	assumed ✓
[OS2] Reflection Positivity	violated ✗
[OS3] Permutation Symmetry	assumed ✓
[OS4] Cluster property	irrelevant
[OS0'] Laplace transform condition	violated (but irrelevant)

Interpretation in an indefinite metric state space (0)

Main question (β)

Does there exist a quantum theory reproducing the reconstructed Wightman function $W(\xi)$ as a vacuum expectation value: $W(\xi) = \langle 0|\phi(\xi)\phi(0)|0\rangle$?
If exists, what states cause complex singularities?

In this section, we argue that complex singularities can be realized in indefinite-metric QFTs and correspond to pairs of zero-norm eigenstates of complex energies.

Interpretation in an indefinite metric state space (1)

An important observation is that complex-energy spectra can appear in an indefinite metric state space.

- Complex conjugate eigenvalues of a hermitian Hamiltonian can be realized by zero-norm pairs:

$$(|E\rangle, |E^*\rangle) \begin{cases} H|E\rangle = E|E\rangle, \quad H|E^*\rangle = E^*|E^*\rangle \\ \langle E|E\rangle = \langle E^*|E^*\rangle = 0, \quad \langle E|E^*\rangle \neq 0 \end{cases}$$

- This pair contributes to the Wightman function,

$$\begin{aligned} \langle 0|\phi(t)\phi(0)|0\rangle \supset & (\langle E^*|E\rangle)^{-1} e^{-iEt} \langle 0|\phi(0)|E\rangle \langle E^*|\phi(0)|0\rangle \\ & + (\langle E|E^*\rangle)^{-1} e^{-iE^*t} \langle 0|\phi(0)|E^*\rangle \langle E|\phi(0)|0\rangle . \end{aligned}$$

- By preparing such states for all momentum $\vec{p}$, we can reproduce the Wightman function reconstructed from complex poles.

Example: covariant operator formulation of Lee-Wick model (1)

To show this more concretely, we consider an example, the covariant operator formulation of Lee-Wick model.
The essential ingredients are as follows. [Lee and Wick 1969, Nakanishi 1972]

- Annihilation operators $\alpha(\vec{p}), \beta(\vec{p})$:
  $[\alpha(\vec{p}), \beta^\dagger(\vec{q})] = [\beta(\vec{p}), \alpha^\dagger(\vec{q})] = (2\pi)^3\delta(\vec{p}-\vec{q})$, and other commutators vanish.
- The scalar field operator is

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left[\alpha(\vec{p})e^{i\vec{p}\cdot\vec{x}-iE_{\vec{p}}t} + \beta^\dagger(\vec{p})e^{-i\vec{p}\cdot\vec{x}+iE_{\vec{p}}t}\right],$$

  where $E_{\vec{p}} := \sqrt{M^2+\vec{p}^2}$ and $M^2 \in \mathbb{C}$.
- Vacuum state $|0\rangle$: $\alpha(\vec{p})|0\rangle = \beta(\vec{p})|0\rangle = 0$.
- Hamiltonian

$$H = \int \frac{d^3p}{(2\pi)^3}\left[E_{\vec{p}}\beta^\dagger(\vec{p})\alpha(\vec{p}) + E^*_{\vec{p}}\alpha^\dagger(\vec{p})\beta(\vec{p})\right].$$

Example: covariant operator formulation of Lee-Wick model (2)

The states $|\vec{p},\alpha\rangle := \alpha^\dagger(\vec{p})\,|0\rangle$ and $|\vec{p},\beta\rangle := \beta^\dagger(\vec{p})\,|0\rangle$ form the pair of complex-energy zero-norm states for every $\vec{p}\in\mathbb{R}^3$:

$$H\,|\vec{p},\alpha\rangle = E^*_{\vec{p}}\,|\vec{p},\alpha\rangle\,,\quad H\,|\vec{p},\beta\rangle = E_{\vec{p}}\,|\vec{p},\beta\rangle\,,$$
$$\langle\vec{p},\alpha|\vec{q},\alpha\rangle = \langle\vec{p},\beta|\vec{q},\beta\rangle = 0,\quad \langle\vec{p},\alpha|\vec{q},\beta\rangle = (2\pi)^3\delta(\vec{p}-\vec{q}).$$

From these pairs, the Wightman functions are

$$\langle 0|\phi(x)\phi(0)|0\rangle = \int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}e^{i\vec{p}\cdot\vec{x}-iE_{\vec{p}}t},$$
$$\langle 0|\phi(x)\phi^\dagger(0)|0\rangle = \langle 0|\phi^\dagger(x)\phi(0)|0\rangle = 0,$$
$$\langle 0|\phi^\dagger(x)\phi^\dagger(0)|0\rangle = \int\frac{d^3p}{(2\pi)^3}\frac{1}{2E^*_{\vec{p}}}e^{i\vec{p}\cdot\vec{x}-iE^*_{\vec{p}}t},$$

which are indeed Wightman functions reconstructed from complex poles $\frac{1}{k^2+M^2}$ and $\frac{1}{k^2+(M^*)^2}$.

Remarks on the interpretation

Thus, complex singularities can be realized in an indefinite-metric QFT and understood as pairs of zero-norm states.

- Complex-energy states violate the spectral condition and make asymptotic states ill-defined.
- Complex singularity of the gluon propagator suggests

$$A_\mu^A(0)\left|0\right\rangle = \left|E\right\rangle + \left|E^*\right\rangle + \cdots$$

- Such zero-norm states should be **confined**. In the Kugo-Ojima scenario, $\left|E\right\rangle$ and $\left|E^*\right\rangle$ should be in BRST quartets.
- Both states $\left|E\right\rangle$ and $\left|E^*\right\rangle$ should contain BRST-parent states $\rightarrow$ complex singularities in ghost-gluon bound states?

Summary

Many studies suggest that the Landau-gauge gluon propagator has complex singularities. Therefore, we have considered the reconstruction procedure in the presence of complex singularities

- The 2pt. Wightman function can be reconstructed as a distribution.
- Complex singularities lead to **non-temperedness** of the Wightman function $\Rightarrow$ **violation of the positivity and spectral condition**
- Complex singularities are **consistent with Lorentz symmetry and locality**.
- Complex singularities in a propagator can be realized in an indefinite-metric QFT and understood as **pairs of zero-norm confined states**.