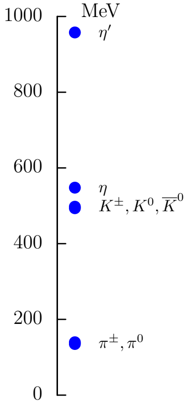
The η/η' system and Large-N_c ChPT: A Lattice QCD study

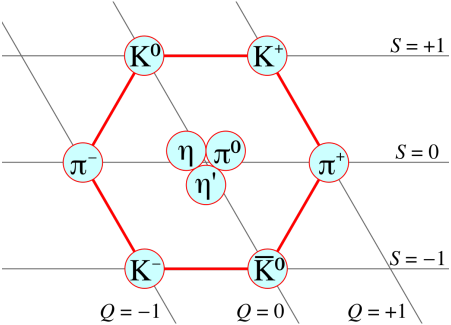
Gunnar Bali
Universität Regensburg

with Jakob Simeth, Sara Collins, Vladimir Braun, Andreas Schäfer

XXXIII International Workshop on HEP at IHEP November 10, 2021

Pseudoscalar meson nonet





Pseudoscalar meson nonet 2

If $\exists$ SU(3) flavour symmetry (u,d,s) then for $\bar{q}q$ we have $\bar{3} \otimes 3 = 8 \oplus 1$.

$$\text{octet: } \pi^0, \pi^\pm, K^\pm, K^0, \overline{K}^0, \eta\,, \qquad\qquad \text{singlet: } \eta'.$$

$$\eta = \eta_8 \sim \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})\,, \qquad \eta' = \eta_0 \sim \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})\,.$$

Classical global symmetries of $\mathscr{L}_{QCD}$ for $m_u = m_d = m_s$ broken for $m_q \to 0$:

$$\text{SU}_A(3) \times \text{SU}_V(3) \times \text{U}_A(1) \times \text{U}_V(1) \longrightarrow \text{SU}_V(3) \times \text{U}_V(1)$$

$\text{SU}_A(3)$ chiral symmetry spontaneously broken at $T < T_c$,
8 Nambu-Goldstone bosons: $\pi^0, \pi^\pm, K^\pm, K^0, \overline{K}^0, \eta_8$.

$\text{U}_A(1)$ symmetry broken due to quantum corrections (axial anomaly).

η_0 is heavier than octet mesons.

Physical ($m_s > m_u \approx m_d > 0$) η and η' are no flavour eigenstates.

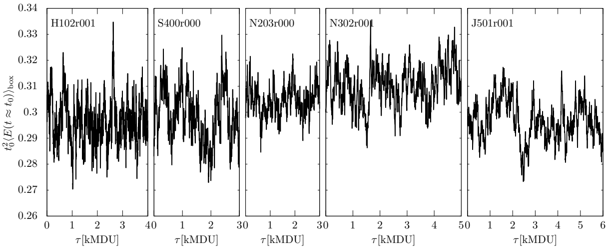
$\rightsquigarrow$ state mixing picture between η_8 and η_0 based on an effective Lagrangian.

$N_f = 2 + 1$ CLS ensembles

Coordinated Lattice Simulations (CLS): Berlin, CERN, Mainz, UA Madrid, Milano Bicocca, Münster, Odense, Regensburg, Rome I and II, Wuppertal, DESY-Zeuthen.

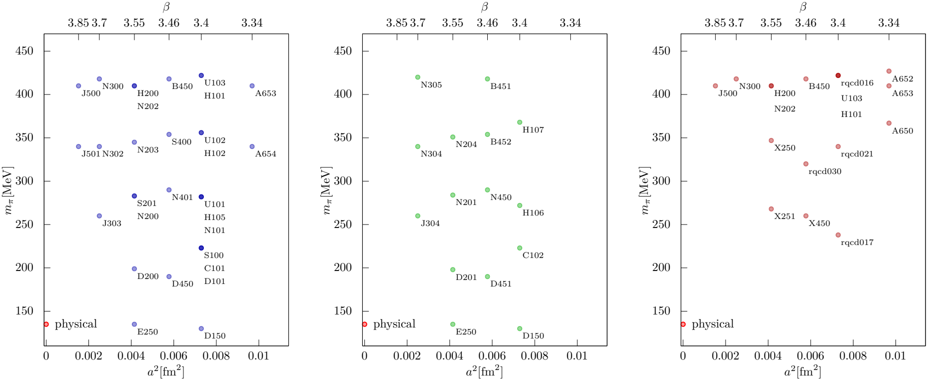
- ⋆ Non-perturbatively improved clover fermion action and tree-level Lüscher-Weisz gauge action.
- ⋆ Six (four) lattice spacings: $a = 0.1 - 0.04$ fm.
- ⋆ $LM_\pi \gtrsim 4$ and multiple spatial volumes.
- ⋆ Mostly open boundary conditions in time.

Wilson flow action density, $t_0^2 E(t \approx t_0)$, $M_\pi \approx 340$ MeV, averaged over ≈ 1 fm slice.



CLS ensembles: M_π vs a^2

- ⋆ Three trajectories, physical point ensembles.
- ⋆ Typically 6000–10000 MDUs.

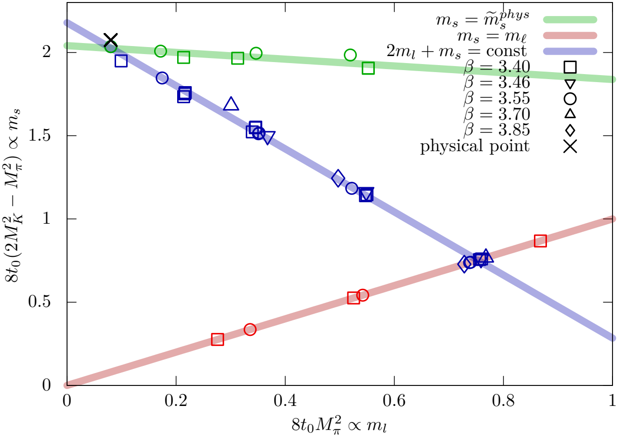


$2m_\ell + m_s = \text{const.}$

$m_s \approx \text{const.}$

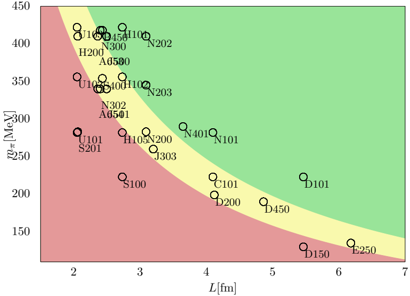
$m_\ell = m_s$

CLS ensembles: m_ℓ-m_s plane

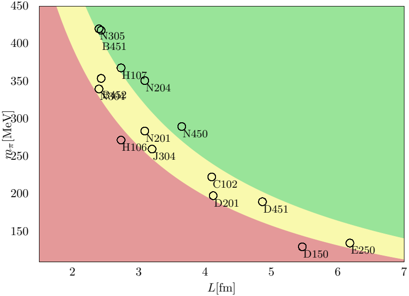


CLS ensembles: spatial volume

$2m_\ell + m_s = \mathrm{const.}$



$m_s \approx \mathrm{const.}$



$\mathrm{LM}_\pi < 4$ **$4 \leq \mathrm{LM}_\pi < 5$** **$\mathrm{LM}_\pi \geq 5$**

The topological charge density

$$\omega(x) = -\frac{1}{16\pi^2}\,\mathrm{tr}\left[F_{\mu\nu}(x)\widetilde{F}_{\mu\nu}(x)\right] = -\frac{1}{32\pi^2}F^a_{\mu\nu}(x)\widetilde{F}^a_{\mu\nu}(x).$$

Singlet axial Ward identity (AWI) in the massless limit:

$$\partial_\mu \widehat{A}^0_\mu = \sqrt{2N_f}\,\widehat{\omega}.$$

Remark: we use $\mathscr{L} = \frac{1}{4g^2}F^a_{\mu\nu}F^a_{\mu\nu} + \cdots$. In pQCD $\mathscr{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \cdots$. Then $\omega \sim g^2 F\widetilde{F}$ (instead of $\omega \sim F\widetilde{F}$) and $F\widetilde{F}$ runs with the inverse β-function. It is then explicit that the anomaly vanishes with $g^2 \to 0$.

ω (after gradient flow) is protected on the lattice by topology and cannot acquire an anomalous dimension:

$$Q = \int \mathrm{d}^4x\,\omega(x) \in \mathbb{Z}.$$

But $\partial_\mu A^0_\mu$ has the same dimension and symmetries as ω. Hence in $\overline{\mathrm{MS}}$ scheme:

$$\widehat{\omega}(\mu) = Z_\omega \omega + Z_{\omega A}(\mu)\partial_\mu A^0_\mu.$$

Mixing with $a^{-1}P^0$ is removed if ω is defined via the gradient flow (or from the overlap operator). It is likely that $Z_\omega = 1$ when ω is defined through the gradient flow/cooling, e.g., [Del Debbio & Pica, hep-lat/0309145].

The topological susceptibility

$$\hat{\tau} = \sum_x \langle \widehat{\omega}(0)\widehat{\omega}(x)\rangle = \frac{1}{V}\sum_{x,y}\langle \widehat{\omega}(x)\widehat{\omega}(y)\rangle = \frac{\langle \widehat{Q}^2\rangle}{V} = Z_\omega^2 \frac{\langle Q^2\rangle}{V}$$

We determine this for the gradient flow time $t_0 = t_0^*$, $\sqrt{8t_0^*} \approx 0.413\,\text{fm}$.

For ensembles with open boundary conditions in time, a distance $\sim 1.9\,\text{fm}$ is kept from the boundaries.

We see no evidence of mass-dependent lattice spacing effects but cut-off effects are substantial.

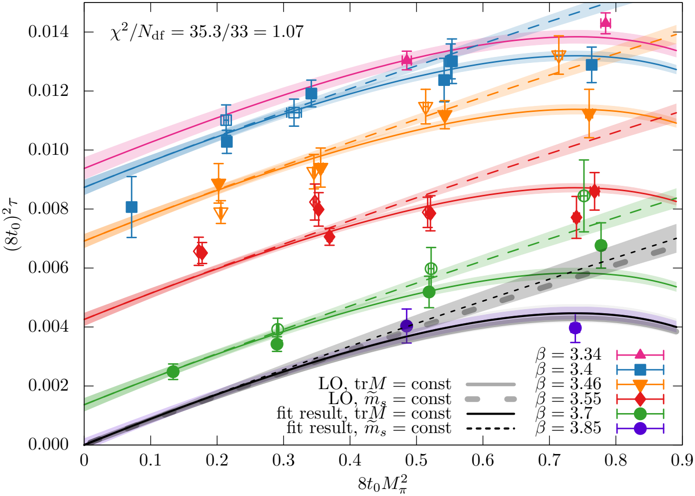
Leading order ChPT expectation plus lattice effects:

$$(8t_0)^2\tau = \frac{(8t_0)^2 F^2}{2Z_\omega^2}\left(\frac{1}{2M_K^2 - M_\pi^2} + \frac{2}{M_\pi^2}\right)^{-1} + l_\tau^{(2)}\frac{a^2}{t_0^*} + l_\tau^{(3)}\frac{a^3}{(t_0^*)^{3/2}} + l_\tau^{(4)}\frac{a^4}{(t_0^*)^2}.$$

Fit to 37 CLS ensembles with free F/Z_ω:

$$l_\tau^{(2)} = -0.072(10), \quad l_\tau^{(3)} = 0.355(34), \quad l_\tau^{(4)} = -0.324(30).$$

The topological susceptibility 2



LO uses F determined from η, η' decay constants and $Z_\omega = 1$.

Problem: $Z_{\omega A}$ is unknown (except for its scale dependence)!

We will revisit this issue.

Definitions: fermionic bilinears and their renormalization

We define ($\psi = (u, d, s)^\intercal$, $N_f = 3$)

$$P^a = \bar{\psi} t^a \gamma_5 \psi, \quad A^a_\mu = \bar{\psi} t^a \gamma_\mu \gamma_5 \psi, \quad t^0 = \frac{1}{\sqrt{2N_f}} \mathbb{1}, \quad t^8 = \frac{\lambda^8}{2}.$$

Then ($A^q_\mu = \bar{q}\gamma_\mu\gamma_5 q$, $m_\ell = m_u = m_d$, $A^\ell_\mu = (A^u_\mu + A^d_\mu)/\sqrt{2}$)

$$A^8_\mu = \frac{1}{12}\left(A^u_\mu + A^d_\mu - 2A^s_\mu\right) = \frac{1}{\sqrt{6}} A^\ell_\mu - \frac{1}{\sqrt{3}} A^d_\mu,$$

$$A^0_\mu = \frac{1}{6}\left(A^u_\mu + A^d_\mu + A^s_\mu\right) = \frac{1}{\sqrt{3}} A^\ell_\mu + \frac{1}{\sqrt{6}} A^d_\mu.$$

Renormalization (ignoring $O(a)$ improvement terms):

$$\widehat{A}^8_\mu = Z_A(g^2) A^8_\mu, \qquad \widehat{P}^8(\mu) = Z_P(g^2, \mu a) P^8,$$

$$\widehat{A}^0_\mu(\mu) = Z^s_A(g^2, \mu a) A^0_\mu, \qquad \widehat{P}^0(\mu) = Z^s_P(g^2, \mu a) P^0.$$

$\widehat{A}^0_\mu$: 1-loop anomalous dimension vanishes, i.e $\widehat{A}^0_\mu(\infty)/\widehat{A}^0_\mu(\mu) =$ finite
$\Rightarrow$ natural to set $\mu = \infty$ in this case.

$r_P = Z^s_P / Z_P = 1 + O(g^6)$ only depends on g^2, not on $\ln(\mu a)$.

Large-N_c ChPT

Axial Ward identities:

$$\partial_\mu \widehat{A}^a_\mu = \left(\overline{\psi}\gamma_5 \widehat{\{\mathcal{M}, t^a\}}\psi\right) + \sqrt{2N_f}\,\delta^{a0}\widehat{\omega}.$$

Mass matrix: $\mathcal{M} = \mathrm{diag}(m_\ell, m_\ell, m_s)$.

Witten-Veneziano relation: at large N_c limit at fixed N_f (t'Hooft limit)

$$\frac{F^2}{2N_f} M_0^2 = \tau_0 \overset{N_f=3}{\approx} \frac{F_\pi^2}{2N_f}\left(M_\eta^2 + M_{\eta'}^2 - 2M_K^2\right),$$

where τ_0 is the topological susceptibility of the quenched theory and $F_\pi \approx 92\,\mathrm{MeV}$.

Since $F \propto \sqrt{N_c}$, we have $M_0^2 \propto N_c^{-1}$.

$\rightsquigarrow$ Simultaneous expansion in $1/N_c$ and the quark masses m_q (Large-N_c ChPT):

$$p = \mathcal{O}(\delta^{1/2}), \quad m_q \sim M^2 = \mathcal{O}(\delta), \quad M_0^2 \propto N_c^{-1} = \mathcal{O}(\delta).$$

Large-N_c ChPT 2

Assume $m_\ell = m_u = m_d$ (isospin symmetry).
(otherwise also mixing with π^0 [CSSM/QCDSF/UKQCD: Kordov et al.,2110.11533])

Contribution of the η_8/η_0 sector (in the adjoint representation of U(3)) to the leading order Large-N_c ChPT Lagrangian ($\eta^\intercal = (\eta_8, \eta_0)$):

$$\mathscr{L} = \ldots + \frac{1}{2}\partial_\mu \eta^\intercal \partial^\mu \eta - \frac{1}{2}\eta^\intercal \mu^2 \eta, \quad \mu^2 = \begin{pmatrix} \mu_8^2 & \mu_{80}^2 \\ \mu_{80}^2 & \mu_0^2 \end{pmatrix}.$$

$$R\mu^2 R^\intercal = \begin{pmatrix} M_\eta^2 & 0 \\ 0 & M_{\eta'}^2 \end{pmatrix}, \quad R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Define

$$\overline{M}^2 := \frac{1}{3}(2M_K^2 + M_\pi^2) \approx 2B_0 \overline{m}, \quad \delta M^2 := 2(M_K^2 - M_\pi^2) \approx 2B_0 \delta m,$$

where $\overline{m}$ is the average quark mass, $\delta m = m_s - m_\ell$ and $B_0 = -\langle \bar{u}u \rangle / F^2 > 0$.

At leading order:

$$\mu_8^2 = 2B_0(m_\ell + 2m_s) = \overline{M}^2 + \frac{1}{3}\delta M^2, \quad \mu_0^2 = 2B_0(2m_\ell + m_s) + M_0^2 = \overline{M}^2 + M_0^2,$$

$$m_{80}^2 = -\frac{2\sqrt{2}}{3}B_0(m_s - m_\ell) = -\frac{\sqrt{2}}{3}\delta M^2, \quad \tan(2\theta) = -2\sqrt{2}\frac{\delta M^2}{3M_0^2 - \delta M^2}.$$

Decay constants

Decay constants in singlet/octet ($a = 0, 8$) or light/strange ($q = \ell, s$) basis:

$$\langle 0|\widehat{A}^{a\mu}|n\rangle = i\, F_n^a p^\mu, \quad \langle 0|\widehat{A}^{q\mu}|n\rangle = i\sqrt{2}\, F_n^q p^\mu, \quad n = \eta, \eta'$$

(normalized so that physical $F_\pi = F_{\pi^0}^3 \approx 92$ MeV)

Four independent decay constants:

$$\begin{pmatrix} F_\eta^8 & F_\eta^0 \\ F_{\eta'}^8 & F_{\eta'}^0 \end{pmatrix} = \begin{pmatrix} F^8 \cos\theta_8 & -F^0 \sin\theta_0 \\ F^8 \sin\theta_8 & F^0 \cos\theta_0 \end{pmatrix}$$

In the SU(3) limit ($m_u = m_d = m_s$): $\theta_8 = \theta_0 = 0$.

At LO $F^0 = F^8 = F$, $\theta_0 = \theta_8 = \theta$:

$$F_\eta^8 = F_{\eta'}^0 = F\cos\theta, \quad -F_\eta^0 = F_{\eta'}^8 = F\sin\theta.$$

Since $F_\eta^8 \neq F_{\eta'}^0$ and $F_\eta^0 \neq -F_{\eta'}^8$, at least NLO is needed!

Other popular choice: "flavour basis"

$$\begin{pmatrix} F_\eta^\ell & F_\eta^s \\ F_{\eta'}^\ell & F_{\eta'}^s \end{pmatrix} = \begin{pmatrix} F^\ell \cos\phi_\ell & -F^s \sin\phi_s \\ F^\ell \sin\phi_\ell & F^s \cos\phi_s \end{pmatrix} = \begin{pmatrix} F_\eta^8 & F_\eta^0 \\ F_{\eta'}^8 & F_{\eta'}^0 \end{pmatrix} \frac{1}{\sqrt{3}} \begin{pmatrix} -1 & -\sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}.$$

Decay constants and FKS model

Scale-independent: F^8, θ_8, θ_0. Scale-dependent: F^0, F^ℓ, F^s, ϕ_ℓ, ϕ_s.

NLO Large-N_c ChPT gives

$$(F^8)^2 = \frac{4F_K^2 - F_\pi^2}{3}, \quad (F^0)^2 = \frac{2F_K^2 + F_\pi^2}{3}\left(1 + \Lambda_1(\mu)\right),$$
$$(F^\ell)^2 = F_\pi^2 + \frac{2}{3}\Lambda_1(2F_K^2 + F_\pi^2), \quad (F^s)^2 = 2F_K^2 - F_\pi^2 - \frac{1}{3}\Lambda_1(2F_K^2 + F_\pi^2).$$

Flavour-basis AWIs are popular because flavour-diagonal:

$$\partial_\mu \widehat{A}^s_\mu = 2\widehat{m}_s\widehat{P}^s + 2\,\widehat{\omega}, \quad \partial_\mu \widehat{A}^\ell_\mu = 2\widehat{m}_\ell\widehat{P}^\ell + 2\sqrt{2}\,\widehat{\omega}.$$

In the Feldmann-Kroll-Stech model, hypothetical η_s and η_ℓ states are introduced and a separation between non-OZI and OZI-violating contributions is assumed within $\partial_\mu\langle\Omega|A^q_\mu|n\rangle = \sqrt{2}M_n^2F_n^q = \ldots$.

This is equivalent to setting the NLO LEC $\Lambda_1(\mu) = 0$ and neglecting any scale dependence. Then $\phi = \phi_\ell = \phi_s$ and

$$\sin^2\phi = \frac{\left(M^2_{\eta'} - (2M_K^2 - M_\pi^2)\right)\left(M_\eta^2 - M_\pi\right)}{2(M^2_{\eta'} - M^2_\eta)(M_K^2 - M_\pi^2)}, \quad F^\ell = F_\pi, \quad F^s = \sqrt{2F_K^2 - F_\pi^2}.$$

Obviously, this cannot be correct (scale-dependence). Too good to be true?

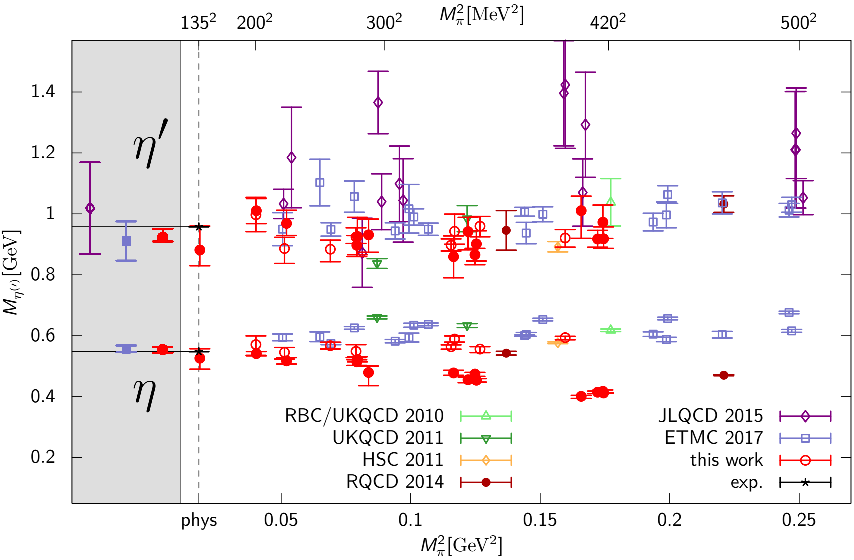
NLO Large-N_c ChPT

$$
\begin{aligned}
(\mu_8^{\rm NLO})^2 =& (\mu_8^{\rm LO})^2 + \frac{8}{3F^2}\,(2L_8 - L_5)\,\delta M^4, \\
(\mu_0^{\rm NLO})^2 =& (\mu_0^{\rm LO})^2 + \frac{4}{3F^2}\,(2L_8 - L_5)\,\delta M^4 - \frac{8}{F^2} L_5 \overline{M}^2 M_0^2 - \tilde{\Lambda}\overline{M}^2 - \Lambda_1 M_0^2 \\
(\mu_{80}^{\rm NLO})^2 =& (\mu_{80}^{\rm LO})^2 - \frac{4\sqrt{2}}{3F^2}\,(2L_8 - L_5)\,\delta M^4 + \frac{4\sqrt{2}}{3F^2} L_5 M_0^2 \delta M^2 + \frac{\sqrt{2}}{6}\tilde{\Lambda}\delta M^2.
\end{aligned}
$$

where $\tilde{\Lambda} = \Lambda_1(\mu) - 2\Lambda_2(\mu)$ is scale-independent and $M_0 = M_0(\mu)$. No chiral logs/ChPT renormalization scale at this order!

$$
\begin{aligned}
F_\eta^8 =& \quad F\left[\cos\theta + \frac{4L_5}{3F^2}\left(3\cos\theta\overline{M}^2 + (\sqrt{2}\sin\theta + \cos\theta)\delta M^2\right)\right], \\
F_{\eta'}^8 =& \quad F\left[\sin\theta + \frac{4L_5}{3F^2}\left(3\sin\theta\overline{M}^2 + (\sin\theta - \sqrt{2}\cos\theta)\delta M^2\right)\right], \\
F_\eta^0 =& -F\left[\sin\theta\left(1 + \frac{\Lambda_1}{2}\right) + \frac{4L_5}{3F^2}\left(3\sin\theta\overline{M}^2 + \sqrt{2}\cos\theta\delta M^2\right)\right], \\
F_{\eta'}^0 =& \quad F\left[\cos\theta\left(1 + \frac{\Lambda_1}{2}\right) + \frac{4L_5}{3F^2}\left(3\cos\theta\overline{M}^2 - \sqrt{2}\sin\theta\delta M^2\right)\right].
\end{aligned}
$$

Masses: Results and comparison with previous studies



Physical point extrapolation

$$f_O(a, \overline{M}^2, \delta M^2) =$$

$$f_O^{\text{cont}}(\overline{M}^2, \delta M^2 | F, L_5, L_8, M_0^2, \Lambda_1, \Lambda_2) \qquad \text{continuum}$$

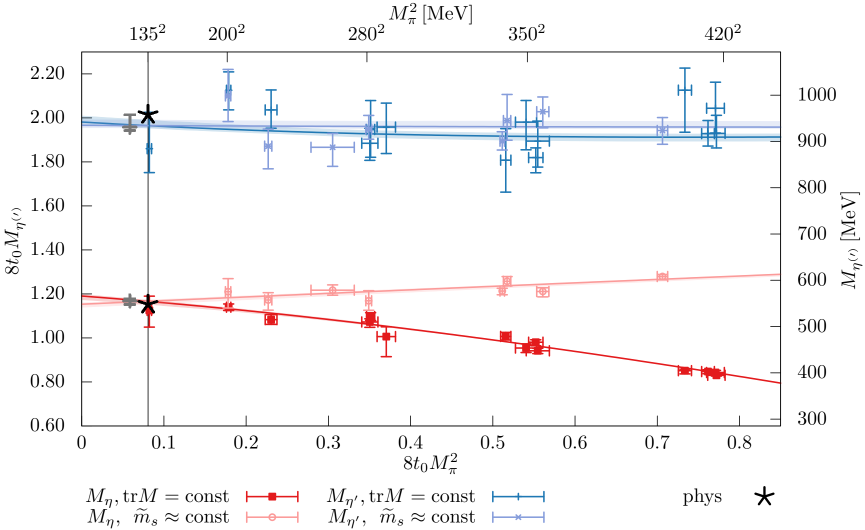
$$\times\, h_O^{(1)}(a, am_\ell, am_s | f_A^l, d_A^l, \tilde{d}_A^l, \delta c_A^l) \qquad \mathcal{O}(a) \text{ improvement}$$

$$\times\, h_O^{(2)}(a^2/t_0^*, a^2\overline{M}^2, a^2\delta M^2 | l_O, m_O, n_O) \qquad \mathcal{O}(a^2) \text{ terms}$$

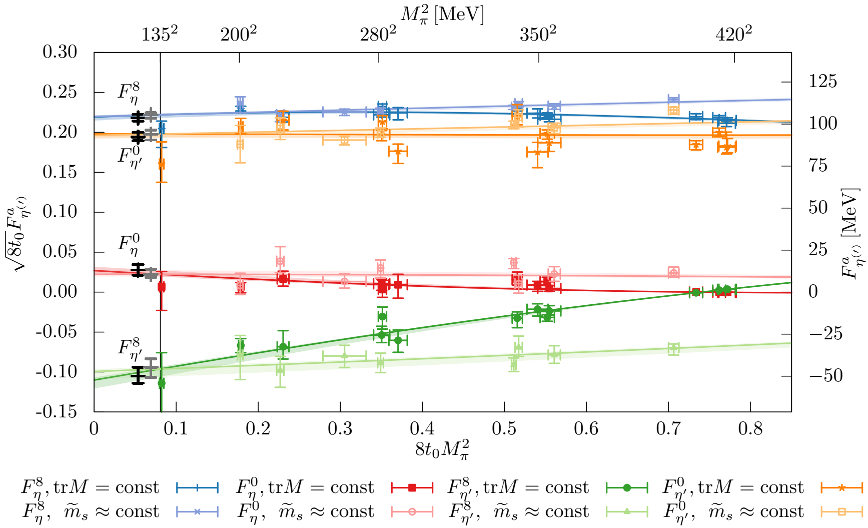
where $O \in \{M_\eta, M_{\eta'}, F_\eta^8, F_\eta^0, F_{\eta'}^8, F_{\eta'}^0\}$ and $h^{(1)}_{M_\eta} = h^{(1)}_{M_{\eta'}} = 1$

- Fix numerically irrelevant lattice spacing terms to zero
- $\mathcal{O}(a)$ improvement for decay constants: d_A (singlet) and f_A (octet) seem to be particularly important
- Combined, fully correlated fit gives $\chi^2/N_{\text{df}} \approx 179/122 \approx 1.47$

Physical point results: masses

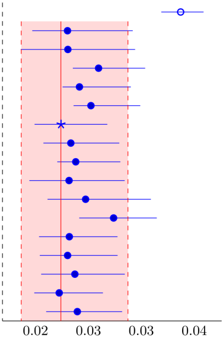


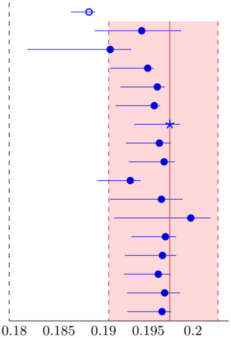
Physical point results: decay constants



Systematics

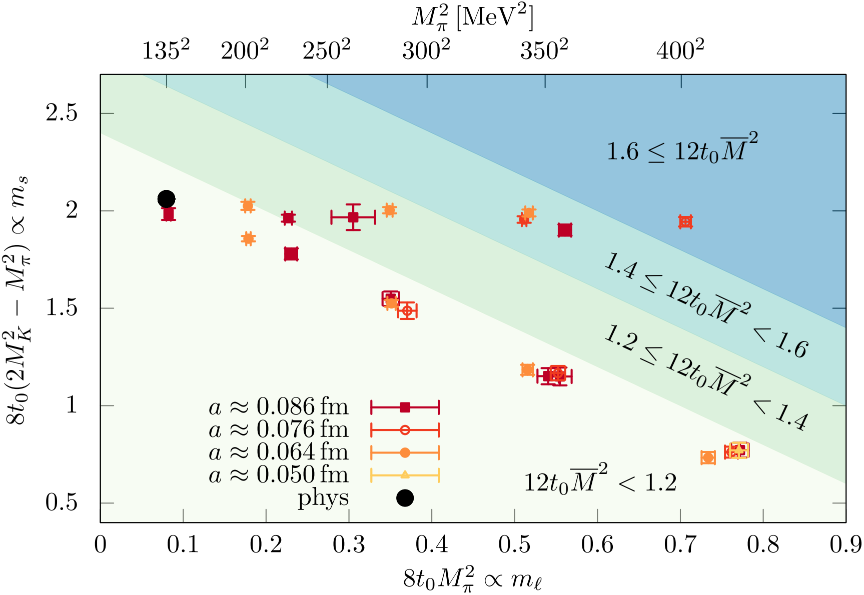
- Volume: only large volumes: $L_s^3 > (2.2\,\mathrm{fm})^3 \gg R_\eta^3 \approx R_\pi^3$ [Bernstein,1511.03242] and typically $L_s M_\pi > 4$.
- Lattice spacing: vary parametrization of discretization effects.
- NLO Large-N_c ChPT: impose cutoffs on the average (non-singlet) pseudoscalar mass: $\overline{M}^2 \leq \overline{M}^2_{\mathrm{max}}$, $12t_0\overline{M}^2_{\mathrm{max}} \in \{1.2, 1.4, 1.6\}$.
- Renormalization: matching to PT done at $\mu \in \{a^{-1}/2, a^{-1}, 2a^{-1}\}$.





id	$\sqrt{8t_0^{\mathrm{ph}}}F_\eta^0$	$\sqrt{8t_0^{\mathrm{ph}}}F_{\eta'}^0$
1	$0.0337\,(^{22}_{18})$	$0.1884\,(^{7}_{20})$
2	$0.0230\,(^{62}_{33})$	$0.1943\,(^{44}_{53})$
3	$0.0231\,(^{64}_{45})$	$0.1907\,(^{24}_{93})$
4	$0.0260\,(^{44}_{24})$	$0.1949\,(^{7}_{42})$
5	$0.0242\,(^{49}_{16})$	$0.1960\,(^{9}_{42})$
6	$0.0252\,(^{47}_{16})$	$0.1957\,(^{6}_{44})$
7	$0.0224\,(^{44}_{25})$	$0.1974\,(^{12}_{40})$
8	$0.0233\,(^{46}_{26})$	$0.1962\,(^{13}_{37})$
9	$0.0238\,(^{42}_{17})$	$0.1968\,(^{12}_{39})$
10	$0.0232\,(^{53}_{38})$	$0.1930\,(^{12}_{37})$
11	$0.0247\,(^{62}_{36})$	$0.1965\,(^{24}_{58})$
12	$0.0274\,(^{41}_{32})$	$0.1998\,(^{22}_{86})$
13	$0.0232\,(^{45}_{29})$	$0.1969\,(^{12}_{38})$
14	$0.0230\,(^{47}_{26})$	$0.1966\,(^{15}_{42})$
15	$0.0237\,(^{47}_{32})$	$0.1961\,(^{14}_{38})$
16	$0.0222\,(^{41}_{24})$	$0.1968\,(^{17}_{42})$
17	$0.0240\,(^{42}_{29})$	$0.1965\,(^{10}_{39})$

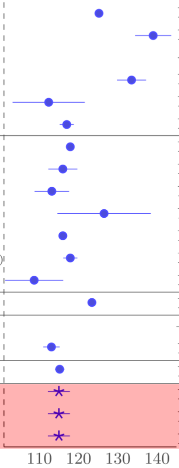
ChPT cuts

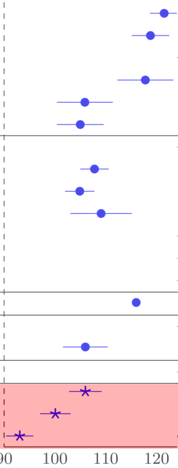


M_π^2 [MeV^2]
135^2 200^2 250^2 300^2 350^2 400^2
$1.6 \leq 12t_0\overline{M}^2$
$1.4 \leq 12t_0\overline{M}^2 < 1.6$
$1.2 \leq 12t_0\overline{M}^2 < 1.4$
$12t_0\overline{M}^2 < 1.2$
$a \approx 0.086$ fm
$a \approx 0.076$ fm
$a \approx 0.064$ fm
$a \approx 0.050$ fm
phys
$8t_0(2M_K^2 - M_\pi^2) \propto m_s$
0.5 1 1.5 2 2.5
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9
$8t_0 M_\pi^2 \propto m_\ell$

Physical point results

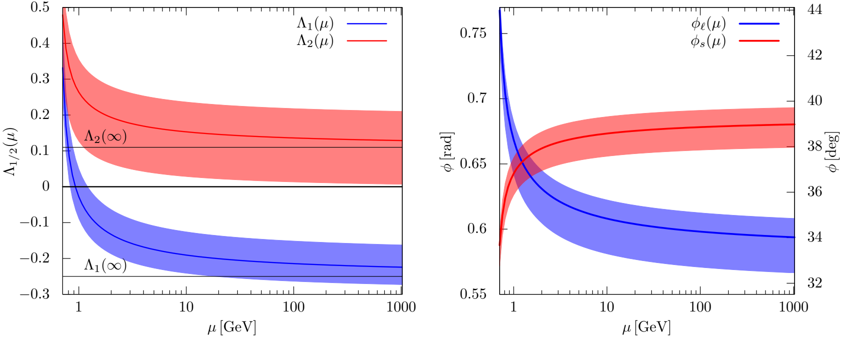
ref	F^8/MeV	F^0/MeV
Benayoun et al. [101]	125.2(9)	121.5(2.8)
Escribano and Frere [102]	139.0(4.6)	118.8(3.7)
Escribano et al. [103]	– – –	—
Chen et al. [104]	133.5(3.7)	117.8(5.5)
Escribano et al. [105]	112.4(9.2)	105.9(5.5)
Escribano et al. [98]	117.0(1.8)	105.0(4.6)
Leutwyler [6]	118	—
Feldmann [97]	116.0(3.7)	107.8(2.8)
Guo et al. [81] NLO-A	113.2(4.4)	104.9(2.9)
Guo et al. [81] NNLO-B	126(12)	109.1(6.0)
Bickert et al. [42] NLO-I	116.0(9)	—
[42] NNLO w/o Ci ($\mu_{\text{EFT}} = 1$GeV)	117.9(1.8)	—
[42] NNLO w/ Ci ($\mu_{\text{EFT}} = 1$GeV)	109(7)	—
Ding et al. [106]	123.4	116.0
ETMC [19]	—	—
Gu et al. [107] NNLO-A9p(F_π)	113.1(2.1)	106.0(4.4)
eq. (7.16)	115.2(1.2)	—
this work ($\mu = 1$GeV)	115.0(2.8)	106.0(3.2)
this work ($\mu = 2$ GeV)	115.0(2.8)	100.1(3.0)
this work ($\mu = \infty$)	115.0(2.8)	93.1(2.7)





$$F^8 = \sqrt{(F^8_\eta)^2 + (F^8_{\eta'})^2} \qquad F^0 = \sqrt{(F^0_\eta)^2 + (F^0_{\eta'})^2}$$

QCD scale dependence and the FKS model



$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} \frac{F_0(\mu)}{\sqrt{1+\Lambda_1(\mu)}} = 0, \qquad \frac{\sqrt{2}}{3} F_\pi^2 \Lambda_1 = F^\ell F^s \sin(\phi_\ell - \phi_s)$$

at NLO. $\Lambda_1 = 0$ (FKS model) gives scale-independent F_0 and $\phi_s = \phi_\ell$.

Works at low scales.

Gluonic matrix elements: reminder

$$\partial_\mu \widehat{A}_\mu^a = \left(\widehat{\overline{\psi}\gamma_5\{M, t^a\}\psi}\right) + \sqrt{2N_f}\,\delta^{a0}\widehat{\omega}.$$

Mass matrix: $M = \mathrm{diag}(m_\ell, m_\ell, m_s)$. AWIs in the octet/singlet basis ($\delta\widehat{m} = \widehat{m}_s - \widehat{m}_\ell$):

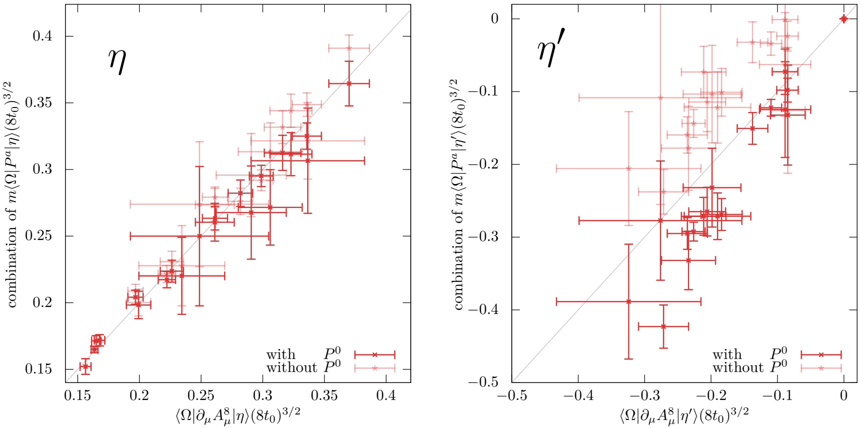
$$\partial_\mu \widehat{A}_\mu^8 = \frac{2}{3}\left(\widehat{m}_\ell + 2\widehat{m}_s\right)\widehat{P}^8 - \frac{2\sqrt{2}}{3}\delta\widehat{m}\,\widehat{P}^0,$$

$$\partial_\mu \widehat{A}_\mu^0 = \frac{2}{3}\left(2\widehat{m}_\ell + \widehat{m}_s\right)\widehat{P}^0 - \frac{2\sqrt{2}}{3}\delta\widehat{m}\,\widehat{P}^8 + \sqrt{6}\,\widehat{\omega}.$$

Renormalization of the topological charge density ($Z_\omega \approx 1$):

$$\widehat{\omega}(\mu) = Z_\omega \omega + Z_{\omega A}(\mu)\partial_\mu A_\mu^0.$$

Test of the octet AWI for $\eta^{(\prime)}$ mesons

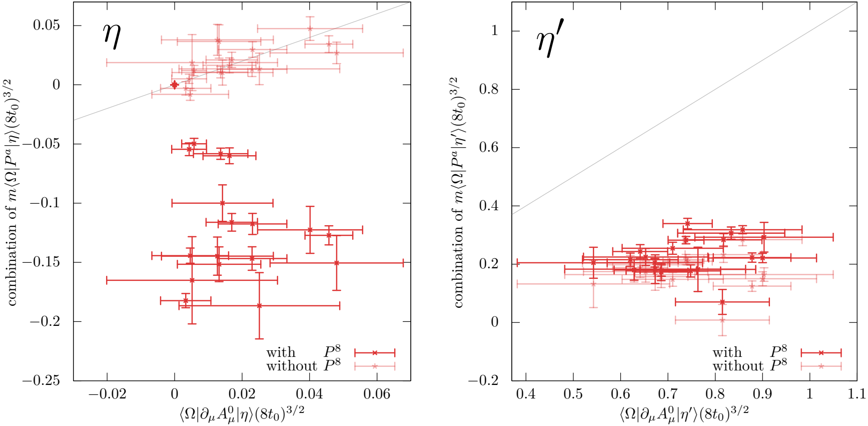


$$\partial_\mu\langle\Omega|A^8_\mu|n\rangle = \frac{2}{3}\left(\widetilde{m}_\ell + 2\widetilde{m}_s\right)\langle\Omega|P^8|n\rangle - \frac{2\sqrt{2}}{3}\delta\widetilde{m}\, r_P\langle\Omega|P^0|n\rangle + \mathcal{O}(a).$$

We assume $r_P = 1$. $\widetilde{m}$: bare AWI mass (renormalizes with Z_A/Z_P).

Deviations due to incomplete $\mathcal{O}(a)$ improvement and $\mathcal{O}(a^2)$ terms are expected.

Confirmation that ∃ gluonic contribution to singlet AWI



$$\partial_\mu\langle\Omega|A^0_\mu|n\rangle = \frac{2}{3}\left(2\widetilde{m}_\ell+\widetilde{m}_s\right)\langle\Omega|P^0|n\rangle - \frac{2\sqrt{2}}{3}\delta\widetilde{m}\langle\Omega|P^8|n\rangle$$

(missing:) $$+\sqrt{\frac{3}{2}}\langle\Omega|2\widehat{\omega}|n\rangle + \mathcal{O}(a)$$

Gluonic matrix elements from fermions

We can obtain renormalized gluonic matrix elements through the singlet AWI from the singlet decay constants F_n^0 ($n \in \{\eta, \eta'\}$) and pseudoscalar matrix elements H_n^0 and H_n^8

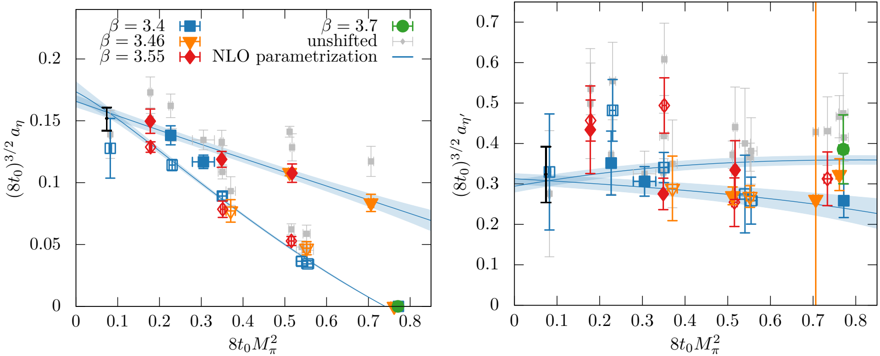
$$Z_A^s(\mu)\partial_\mu \langle \Omega | A_\mu^0 | n \rangle = M_n^2 F_n^0(\mu), \quad H_n^a(\mu) = Z_P^{(s)}(\mu) \langle \Omega | P^a | n \rangle,$$

$$\begin{aligned} a_n(\mu) &:= \langle \Omega | 2\widehat{\omega} | n \rangle \\ &= \sqrt{\frac{2}{3}} M_n^2 F_n^0(\mu) + \frac{4}{3\sqrt{3}} \delta \widehat{m}\, H_n^8 - \frac{2}{3}\sqrt{\frac{2}{3}} (2\widehat{m}_\ell + \widehat{m}_s) H_n^0. \end{aligned}$$

Note that $\widehat{m} H_n^8 = Z_A \widetilde{m} \langle \Omega | P^8 | n \rangle$, $\widehat{m} H_n^0 = Z_A r_P \widetilde{m} \langle \Omega | P^0 | n \rangle$, $r_P = 1 + \mathcal{O}(g^6)$.

We will later check if the above results from fermionic matrix elements are consistent with the gluonic definition.

Gluonic matrix elements from the singlet AWI



Parametrization is NLO U(3) Large-N_c ChPT. 6 LECs (with priors from analysis of decay constants) plus 3 parameters to account for $\mathcal{O}(a)$ effects. $\chi^2/N_{\rm df} \approx 34/31$. At $\mu = \infty$:

$$(8t_0^{\rm ph})^{3/2}a_\eta = 0.1564\,(^{37}_{63}) \quad \text{and} \quad (8t_0^{\rm ph})^{3/2}a_{\eta'} = 0.308\,(^{16}_{17})\,.$$

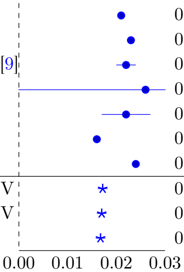
The NLO Large-N_c ChPT prediction from the decay constants alone reads:

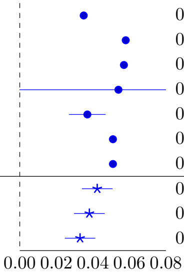
$$(8t_0^{\rm ph})^{3/2}a_\eta = 0.1609\,(^{17}_{27}) \quad \text{and} \quad (8t_0^{\rm ph})^{3/2}a_{\eta'} = 0.383\,(^{11}_{17})\,.$$

We take the difference as our systematic error (black error bar).

Comparison with the literature

ref	a_η/GeV^3	$a_{\eta'}/\mathrm{GeV}^3$
Novikov et al. [116]	0.021	0.035
Feldmann [97]	0.023	0.058
Beneke and Neubert [9]	0.022(2)	0.057(2)
Cheng et al. [118]	0.026(28)	0.054(57)
Singh [117]	0.0220(50)	0.037(10)
Qin et al. [119]	0.016	0.051
Ding et al. [106]	0.024	0.051
this work at $\mu = 1\mathrm{GeV}$	0.0172(10)	0.0424(84)
this work at $\mu = 2\mathrm{GeV}$	0.0170(10)	0.0381(84)
this work at $\mu = \infty$	0.0168(10)	0.0330(83)





Systematics from parametrization, renormalization and scale setting included.

If anomaly dominates [Novikov et al., NPB165(80)55]:

$$R(J/\psi) = \frac{\Gamma[J/\psi \to \eta'\gamma]}{\Gamma[J/\psi \to \eta\gamma]} \approx \frac{a_{\eta'}^2}{a_\eta^2} \left(\frac{k_{\eta'}}{k_\eta}\right)^3$$

with k_n: momentum of the meson in the rest frame of J/ψ. From this:

$$R(J/\psi, \mu = 2\,\mathrm{GeV}) = 5.03\,({}^{19}_{45})_{\mathrm{stat}}\,(1.94)_{\mathrm{syst}}\,, \quad \text{PDG:} \quad R(J/\psi) = 4.74(13).$$

Renormalizing the gluonic definition

We compute $\langle \Omega | 2\omega(0) | n \rangle$, $n \in \{\eta, \eta'\}$. Then the renormalized matrix element is given as

$$a_n(\mu) = Z_\omega \langle \Omega | 2\omega | n \rangle + 2 \frac{Z_{\omega A}}{Z_A^s} M_n^2 F_n^0(\mu).$$

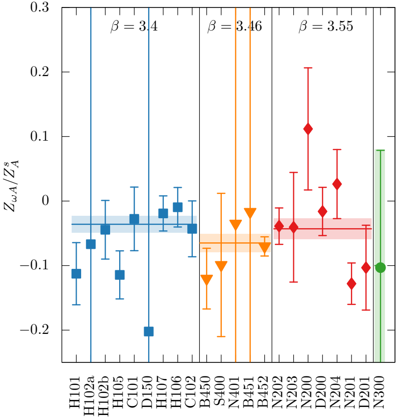
Problem: $Z_{\omega A}$ is unknown. However, only one $Z_{\omega A}$ per lattice spacing.
$\rightarrow$ Isolate the scale independent combination $Z_{\omega A}/Z_A^s$ ($Z_\omega = 1$):

$$\frac{Z_{\omega A}}{Z_A^s} = \frac{a_n - \langle \Omega | 2\omega | n \rangle}{2 M_n^2 F_n^0}.$$

Shown for $n = \eta'$.

Denominator near zero for the η.

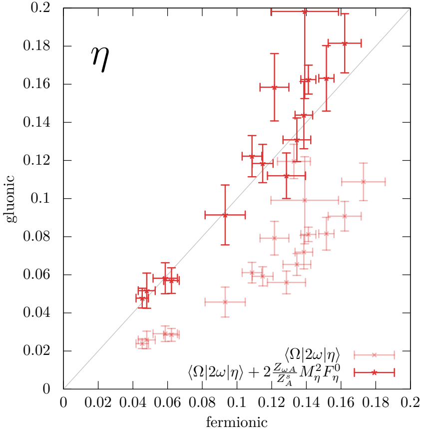
Consistent with constant values, slowly varying with g^2.

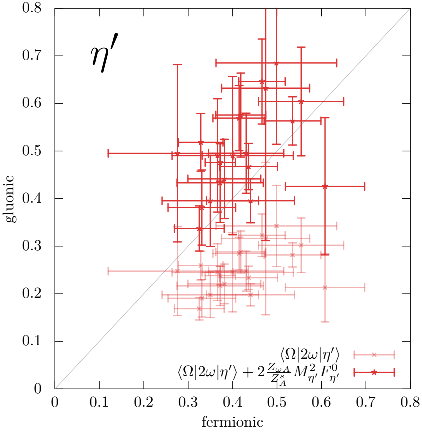


Comparison between fermionic and gluonic determinations

$Z_{\omega A} \neq 0$ is necessary!

Light symbols: without $Z_{\omega A}$, dark symbols: with $Z_{\omega A}$.





Summary

- The topological susceptibility seems well described by Large-N_c ChPT, albeit with large cut-off effects (also observed for other fermion actions: [MILC, arXiv:1003.5695], [Chowdury et al., arXiv:1110.6013], [ETMC, arXiv:1312.5161], [ALPHA, arXiv:1406.5363], [Bonati et al., arXiv:1512.06746], [BMWc, arXiv:1606.07494].)
- First direct lattice determination of the η and η' decay constants and gluonic matrix elements.
- First verification of the singlet AWI with Wilson fermions.
- Scale dependence of $F_0(\mu)$, $a_\eta(\mu)$, $a_{\eta'}(\mu)$ included for the first time.
- We have derived the NLO Large-N_c ChPT expressions for $a_\eta/a_{\eta'}$.
- NLO Large-N_c ChPT describes all data (two meson masses, four decay constants, two gluonic matric elements) reasonably well with just six LECs, but there are some tensions. NNLO?
- The Feldmann-Kroll-Stech model works OK where $\Lambda_0(\mu)$ is small ($\mu \in [0.8, 1.5]$ GeV).