Hadron matrix elements, lattice QCD and the Feynman–Hellmann approach

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– QCDSF-UKQCD–CSSM Collaboration –

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Contents

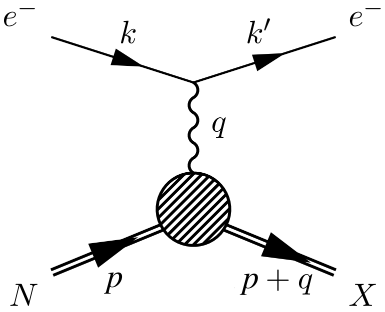
- Deep Inelastic Scattering
  - Hadronic Tensor
  - Compton Amplitude
  - Physical/unphysical regions
- Feynman-Hellmann theorem (perturbation in parameter λ)
  - Mathematical derivation via Dyson series
  - $O(\lambda^2)$ term and relation to Compton Amplitude
- Lattice
  - $O(\lambda^2)$ terms
    - Compton Amplitude structure functions
    - Physical moments
  - $O(\lambda)$ terms
    - Energy degeneracies
    - Nucleon scattering form factors
- Conclusions and future perspectives

Papers:

- 'A Lattice Study of the Glue in the Nucleon'
  arXiv:1205.6410 (PLB)
- 'A Feynman-Hellmann approach to the spin structure of hadrons'
  arXiv:1405.3019 (PRD)
- 'A novel approach to nonperturbative renormalization of singlet and nonsinglet lattice operators'
  arXiv:1410.3078 (PLB)
- 'Disconnected contributions to the spin of the nucleon'
  arXiv:1508.06856 (PRD)
- 'Electromagnetic form factors at large momenta from lattice QCD'
  arXiv:1702.01513 (PRD)
- 'Nucleon structure functions from lattice operator product expansion'
  arXiv:1703.01153 (PRL)
- 'Lattice QCD evaluation of the Compton amplitude employing the Feynman-Hellmann theorem'
  arXiv:2007.01523 (PRD)
- 'Generalised parton distributions from the off-forward Compton amplitude in lattice QCD'
  arXiv:2110.11532

+ Various Lattice conferences, including Lattice 2021

DIS



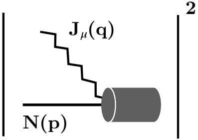
Deep ($Q^2 \gg M_N^2$)
Inelastic ($M_X^2 > M_N^2$)
Scattering (DIS)

- k, k': incoming, outgoing lepton momenta
- p: 4-momentum of the incoming nucleon of mass M_N
- $M_X^2 = (p+q)^2$: invariant mass of the recoiling system X
- $Q^2 = -q^2$: photon virtuality, momentum transfered to nucleon
- $x = \frac{Q^2}{2p\cdot q}$: Bjorken scaling variable $[x > 0]$
  $[M_X^2 > M_N^2 \;\Rightarrow\; 0 < x < 1]$
- $\omega = x^{-1}$: inverse Bjorken variable

DIS and the Hadronic Tensor

$$d\sigma \sim L_J^{\mu\nu}\, W_{\mu\nu}^J \qquad J \sim \gamma, Z \text{ (neutral) or } W \text{ (charged)}$$

- $L_J^{\mu\nu}$: Leptonic tensor
- $W_{\mu\nu}^J$: Hadronic tensor

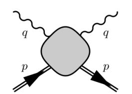


$$W_{\mu\nu} \equiv \frac{1}{4\pi}\int d^4z\, e^{iq\cdot z} \rho_{ss'}\, {}_{\mathrm{rel}}\langle p, s'|[J_\mu(z), J_\nu(0)]|p, s\rangle_{\mathrm{rel}}$$
$$= \left(-\eta_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) F_1(x, Q^2) + \left(p_\mu - \frac{p\cdot q}{q^2} q_\mu\right)\left(p_\nu - \frac{p\cdot q}{q^2} q_\nu\right)\frac{F_2(x, Q^2)}{p\cdot q}$$

F_i are structure functions

$\rho_{ss'} = \frac{1}{2}\delta_{ss'}$, unpolarised; ${}_{\mathrm{rel}}\langle p|p\rangle_{\mathrm{rel}} = 2E_N(\vec{p})$

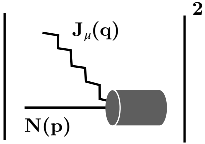
Scaling $F_i = F_i(x)$ only

Forward Compton Amplitude: 

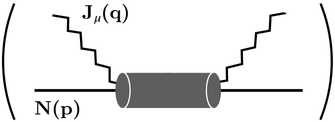
$$T_{\mu\nu}(p,q) \equiv i \int d^4z\, e^{iq\cdot z} \rho_{ss'}\, {}_{\rm rel}\langle p,s'|T(J_\mu(z)J_\nu(0))|p,s\rangle_{\rm rel}$$

$$= \left(-\eta_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right)\mathcal{F}_1(\omega,Q^2) + \left(p_\mu - \frac{p\cdot q}{q^2}q_\mu\right)\left(p_\nu - \frac{p\cdot q}{q^2}q_\nu\right)\frac{\mathcal{F}_2(\omega,Q^2)}{p\cdot q}$$

Related via the Optical theorem:



$\sim \quad 2\,\mathrm{Im}$



DIS Cross Section ~ Hadronic Tensor

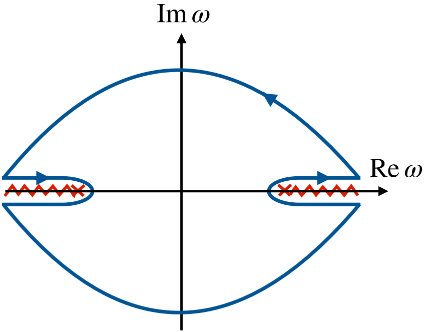
Forward Compton Amplitude
~ Compton Tensor

Nucleon Structure Functions I

- (Photon) crossing symmetry $N \to \bar{N}$:
  $T_{\mu\nu}(p,q) = T_{\nu\mu}(p,-q)$
  gives $\mathcal{F}_1(-\omega, Q^2) = \mathcal{F}_1(\omega, Q^2)$
  Schwarz reflection: $f^*(z) = f(z^*)$
  ...
- Optical theorem relates the Compton SF to DIS SF

$$\mathrm{Im}\mathcal{F}_1(\omega, Q^2) = 2\pi F_1(x, Q^2)$$

- So we can write a (subtracted) dispersion relation:



$$\begin{aligned}\mathcal{F}_1(\omega, Q^2) &= \frac{2\omega}{\pi}\int_1^\infty d\omega' \left[\frac{\mathrm{Im}\mathcal{F}_1(\omega', Q^2)}{\omega'(\omega' - \omega - i\epsilon)} - \frac{\mathrm{Im}\mathcal{F}_1(\omega', Q^2)}{\omega'(\omega' + \omega - i\epsilon)}\right] + \mathcal{F}_1(0, Q^2) \\ &= \underbrace{4\omega^2 \int_0^1 dx' \frac{x' F_1(x', Q^2)}{1 - x'^2\omega^2 - i\epsilon}}_{\overline{\mathcal{F}}_1(\omega, Q^2)} + \underbrace{\mathcal{F}_1(0, Q^2)}_{\textit{once subtracted}:\, S_1(Q^2)}\end{aligned}$$

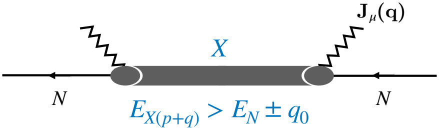
Nucleon Structure Functions II

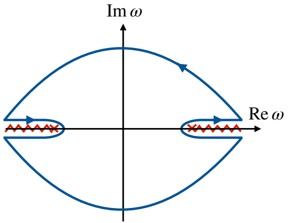
- As long as we are in the unphysical region [ie below elastic threshold]

$$|\omega| < 1 \iff M_X^2 < M_N^2$$

  - No singularity in previous integral
  - Time ordering irrelevant, can drop ϵ so Minkowski and Euclidean amplitudes are identical [Direct computation: see later]

- Physically $|\omega| < 1$ means states propagating between currents cannot go on-shell:





Consistent:

$$-\left(1 - \frac{M_N^2 - M_X^2(\vec{p} - \vec{q})}{Q^2}\right) < \omega < 1 - \frac{M_N^2 - M_X^2(\vec{p} + \vec{q})}{Q^2}$$

Nucleon Structure Functions III

In unphysical region $|\omega| < 1$, no need for $i\epsilon$, Taylor expand denominator of

$\omega = 2p \cdot q/Q^2$

$$\overline{\mathcal{F}}_1(\omega, Q^2) = 4\omega^2 \int_0^1 dx' \, \frac{x' F_1(x', Q^2)}{1 - x'^2\omega^2} = 2\sum_{n=1}^{\infty} \omega^{2n} M_{2n}^{(1)}(Q^2)$$

where Mellin moments of the nucleon structure function $F_1(x, Q^2)$ are

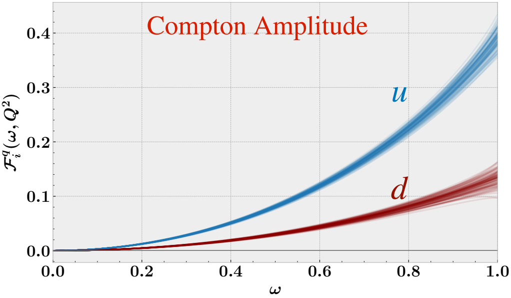
$$M_{2n}^{(1)}(Q^2) = 2\int_0^1 dx' \, x'^{2n-1} F_1(x', Q^2)$$

Furthermore consider Compton amplitude with $\mu = \nu = 3, \ p_z = q_z = 0$

$$T_{33}(p, q) = \overline{\mathcal{F}}_1(\omega, Q^2) = \sum_{n=1}^{\infty} 2\omega^{2n} M_{2n}^{(1)}(Q^2)$$

So from Compton amplitude data we can extract the Mellin moments

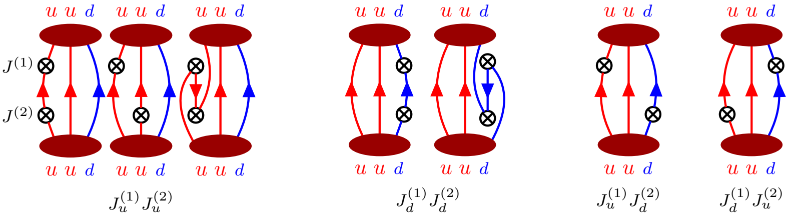
Expected shape of the Compton Amplitude



$$T_{33}(p,q) = \overline{\mathcal{F}}_1(\omega, Q^2) = 2\sum_{n=1}^{\infty} \omega^{2n} M_{2n}^{(1)}(Q^2)$$

- ω in unphysical region
- Cross section positivity $\Rightarrow F_1 > 0 \Rightarrow M_2^{(1)} \geq M_4^{(1)} \geq \ldots M_{2n}^{(1)} \geq \ldots > 0$

Lattice: Just need to compute (Euclidean) Compton Amplitude – $T_{\mu\nu}$ . . . !?



- Picture from: Fukaya, Hashimoto, Keneko, Ohki, arXiv:2010.01253
- $J_u^{(i)} = \bar{u}\Gamma^{(i)} u$, $J_d^{(i)} = \bar{d}\Gamma^{(i)} d$
- Computationally complicated: Loads of diagrams
- [+ disconnected diagrams]

Alternative: Feynman-Hellmann approach

Feynman–Hellmann — some Mathematical Details

Consider the 2-point nucleon correlation function

$$C_{fo\,\lambda}(t;\vec{p},\vec{q}) = {}_\lambda\langle 0| \underbrace{\hat{\tilde{B}}_{N_f}(0;\vec{p})}_{\text{Sink: mom Op}} \hat{S}(\vec{q})^t \underbrace{\hat{\bar{B}}_{N_o}(0,\vec{0})}_{\text{Source: spatial}} |0\rangle_\lambda$$

where $\hat{S}$ is the $\vec{q}$-dependent transfer matrix

$$\hat{S}(\vec{q}) = e^{-\hat{H}(\vec{q})}$$

and in the presence of a perturbation

$$\hat{H}(\vec{q}) = \hat{H}_0 - \sum_\alpha \lambda_\alpha \hat{\tilde{\mathcal{O}}}_\alpha(\vec{q})$$

where

$$\hat{\tilde{\mathcal{O}}}_\alpha(\vec{q}) = \int_{\vec{x}} \left(\hat{O}_\alpha(\vec{x}) e^{i\vec{q}\cdot\vec{x}} + \hat{O}^\dagger_\alpha(\vec{x}) e^{-i\vec{q}\cdot\vec{x}} \right)$$

[Can generalise $\lambda_\alpha \hat{O}_\alpha(\vec{x}) \to |\lambda_\alpha| \underbrace{e^{i\phi_\alpha} \hat{O}_\alpha(\vec{x})}$]

Now insert two complete sets of unperturbed states $|X\rangle \to \frac{|X\rangle}{\sqrt{\langle X|X\rangle}}$, $|0\rangle \to |0\rangle$

$$|N(\vec{p})\rangle\langle N(\vec{p})| + \sum\!\!\!\!\!\!\int_{E_X(\vec{p}_X)>E_N(\vec{p})} |X(\vec{p})_X)\rangle\langle X(\vec{p}_X)| = 1$$

where

- $\hat{H}_0|X(\vec{p}_X)\rangle = E_X(\vec{p}_X)|X(\vec{p}_X)\rangle$
- Lowest state $|N(\vec{p})\rangle$ well separated from other states

before and after $\hat{S}^t$ to give

$$C_{fo\,\lambda}(t;\vec{p},\vec{q}) = \sum\!\!\!\!\!\!\int_{Y(\vec{p}_Y)} {}_\lambda\langle 0|\hat{\tilde{B}}_{N_f}(\vec{p})|N(\vec{p})\rangle \underbrace{\langle N(\vec{p})|\hat{S}(\vec{q})^t|Y(\vec{p}_Y)\rangle}_{\text{need to evaluate}} \langle Y(\vec{p}_Y)|\hat{\tilde{B}}_{N_o}(\vec{0})|0\rangle_\lambda$$

Time dependent perturbation theory via the Dyson Series for

$$\exp\{-(\hat{H}_0 - \textstyle\sum_\alpha \lambda_\alpha \hat{\tilde{\mathcal{O}}}_\alpha(\vec{q}))t\}$$

Result – factorisation

$$C_{fo\,\lambda}(t;\vec{p},\vec{q}) = {}_\lambda\langle 0|\hat{\tilde{B}}_{N_f}(\vec{p})|N(\vec{p})\rangle \times {}_\lambda\langle N(\vec{p})|\hat{\bar{B}}_{N_o}(\vec{0})|0\rangle_\lambda \times e^{-E_{N\,\lambda}(\vec{p},\vec{q})t}$$

where we have defined ${}_\lambda\langle N(\vec{p})|$ as

$${}_\lambda\langle N(\vec{p})| = \langle N(\vec{p})| + \lambda_\alpha \sum\!\!\!\!\!\!\!\int_{E_Y(\vec{p}_Y)>E_N(\vec{p})} \frac{\langle N(\vec{p})|\hat{\tilde{\mathcal{O}}}_\alpha(\vec{q})|Y(\vec{p}_Y)\rangle}{E_Y(\vec{p}_Y) - E_N(\vec{p})} \langle Y(\vec{p}_Y)| \Bigg|_{E_Y>E_N}$$

The modified energy is given by

$$\begin{aligned} E_{N\,\lambda}(\vec{p},\vec{q}) \\ = \quad & E_N(\vec{p}) - \lambda_\alpha \langle N(\vec{p})|\hat{\tilde{\mathcal{O}}}_\alpha(\vec{\ })|\mathcal{N}(\vec{\ })\rangle \\ & -\lambda_\alpha\lambda_\beta \sum\!\!\!\!\!\!\!\int_{E_X(\vec{p}_X)>E_N(\vec{p})} \frac{\langle X(\vec{p}_X)|\hat{\tilde{\mathcal{O}}}_\alpha(\vec{q})|N(\vec{p})\rangle^* \langle X(\vec{p}_X)|\hat{\tilde{\mathcal{O}}}_\beta(\vec{q})|N(\vec{p})\rangle}{E_X(\vec{p}_X) - E_N(\vec{p})} \\ & +O(\lambda^3) \end{aligned}$$

The $O(\lambda^2)$ terms

Comments I

For the matrix elements we have $[\hat{O}(\vec{x}) = e^{-i\hat{\vec{p}}\cdot\vec{x}}\,\hat{O}(\vec{0})\,e^{i\hat{\vec{p}}\cdot\vec{x}}]$

$$\begin{aligned}&\langle X(\vec{p}_X)|\hat{\tilde{\mathcal{O}}}_\alpha(\vec{q})|N(\vec{p})\rangle \\ &\quad = \langle X(\vec{p}_X)|\hat{O}_\alpha(\vec{0})|N(\vec{p})\rangle\,\delta_{\vec{p}_X,\vec{p}+\vec{q}} + \langle X(\vec{p}_X)|\hat{O}^\dagger_\alpha(\vec{0})|N(\vec{p})\rangle\,\delta_{\vec{p}_X,\vec{p}-\vec{q}}\end{aligned}$$

so matrix elements step up or down in $\vec{q}$

- Also valid for $X = N(\vec{p})$ so $O(\lambda)$ term vanishes ($\vec{q} \neq \vec{0}$)
- Generalise: each λ inserts another $\hat{\tilde{\mathcal{O}}}$ into the matrix element, so need an even number of λs ie odd powers of λ vanish
  At $O(\lambda^{2n})$ need $E_X(\vec{p} \pm n\vec{q}) > E_N(\vec{p})$

which gives

$$\begin{aligned}&E_{N\,\lambda}(\vec{p},\vec{q}) \\ &\quad = E_N(\vec{p}) - \sum_{X:\,E_X(\vec{p}\pm\vec{q})>E_N(\vec{p})}\left[\frac{|\langle X(\vec{p}+\vec{q})|\lambda_\alpha\hat{O}_\alpha(\vec{0})|N(\vec{p})\rangle|^2}{E_X(\vec{p}+\vec{q}) - E_N(\vec{p})}\right. \\ &\qquad\qquad \left. + \frac{|\langle X(\vec{p}-\vec{q})|(\lambda_\alpha\hat{O}_\alpha(\vec{0}))^\dagger|N(\vec{p})\rangle|^2}{E_X(\vec{p}-\vec{q}) - E_N(\vec{p})}\right]\end{aligned}$$

Comments II

- Need $E_N(\vec{p} \pm \vec{q}) > E_N(\vec{p})$ [$X = N$ worst case] giving

$$-1 < \omega < 1 \qquad\qquad \omega = \frac{2\vec{p}\cdot\vec{q}}{\vec{q}^2}$$

  - usual definition of ω (with $q_0 = 0$)
  - ω in unphysical region – safe

What has all this to do with the Compton Amplitude?

Interpretation of results – Compton Amplitude

Compton Amplitude

$$T^{(\mathcal{M})}_{\mu\nu}(p,q) = i\int d^4x\, e^{iq\cdot x}\, \rho_{ss'}\,{}_{\rm rel}\langle N(\vec{p}),s'|\, T\, \hat{O}^\dagger_\mu(x)\hat{O}_\nu(0)\,|N(\vec{p}),s\rangle_{\rm rel}$$

Insert complete set of states

$$T^{(\mathcal{M})}_{\mu\nu}(p,q) = i \sum\!\!\!\!\!\!\int_{X(\vec{p}_X)} \int d^3x\, e^{-i\vec{q}\cdot\vec{x}}$$
$$\times\Big[\int_0^\infty dx^0 e^{iq^0x^0}\langle N(\vec{p})|\hat{O}^\dagger_\mu(x)|X(\vec{p}_X)\rangle\langle X(\vec{p}_X)|\hat{O}_\nu(0)|N(\vec{p})\rangle$$
$$+\int_{-\infty}^0 dx^0 e^{iq^0x^0}\langle N(\vec{p})|\hat{O}_\nu(0)|X(\vec{p}_X)\rangle\langle X(\vec{p}_X)|\hat{O}^\dagger_\mu(x)|N(\vec{p})\rangle\Big]$$

giving

$$T^{(\mathcal{M})}_{\mu\nu}(p,q)$$
$$= \sum_X\Big[\frac{\langle X(\vec{p}+\vec{q})|\hat{O}_\mu(\vec{0})|N(\vec{p})\rangle^*\,\langle X(\vec{p}+\vec{q})|\hat{O}_\nu(\vec{0})|N(\vec{p})\rangle}{E_X(\vec{p}+\vec{q})-E_N(\vec{p})-q^0-i\epsilon}$$
$$+\frac{\langle X(\vec{p}-\vec{q})|\hat{O}^\dagger_\nu(\vec{0})|N(\vec{p})\rangle^*\langle X(\vec{p}-\vec{q})|\hat{O}^\dagger_\mu(\vec{0})|N(\vec{p})\rangle}{E_X(\vec{p}-\vec{q})-E_N(\vec{p})+q^0-i\epsilon}\Big]$$

Comparing with previous result

- $q^0 = 0$
- Choose $\vec{p}, \vec{q}$ geometry so that $E_X(\vec{p} \pm \vec{q}) > E_N(\vec{p})$, ie $-1 < \omega < 1$ so can also drop $i\epsilon$
- Gives:

$$E_{N\,\lambda}(\vec{p}, \vec{q}) = E_N(\vec{p}) - \frac{\lambda^*_\alpha \lambda_\beta}{{}_{\rm rel}\langle N(\vec{p}) | N(\vec{p}) \rangle_{\rm rel}} T^{(\mathcal{M})}_{\alpha\beta}((E_N(\vec{p}), \vec{p}), (0, \vec{q})) + O(\lambda^4)$$

- As $T^{(\mathcal{M})}_{\alpha\beta}(p,q)^* = T^{(\mathcal{M})}_{\beta\alpha}(p,q)$ then
  real part of Compton amplitude is symmetric (unpolarised) [λ real];
  imaginary part is anti-symmetric (polarised) [λ complex]

For the DIS case considered here set $\mu = \nu = 3$; $p_z = q_z = 0$, giving $T_{33}(p,q) = \mathcal{F}_1(\omega, Q^2)$. So with $O_\alpha \to J_3$ and $\lambda_3 \to \lambda$ we have

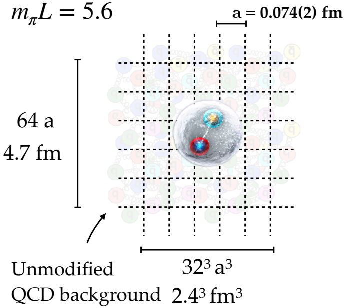
$$\begin{aligned} \Delta E_{N\,\lambda}(\vec{p}, \vec{q}) &\equiv E_{N\,\lambda}(\vec{p}, \vec{q}) - E_N(\vec{p}) \\ &= -\frac{\lambda^2}{2E_N(\vec{p})} \mathcal{F}_1(\omega, Q^2) + O(\lambda^4) \end{aligned}$$

Some Lattice Details

$$\mathcal{L} = \mathcal{L}_0 + 2\lambda \cos(\vec{q}\cdot\vec{x}) J_3(x)$$

- Valence u/d quarks in $S(\lambda)$ only
    - no disconnected terms
    - would require dedicated configs
- Vector current $J_\mu = Z_V \bar{q}\gamma_\mu q$
  Z_V previously determined ~ 0.86
- 4 field strengths $\lambda = (\pm 0.0125, \pm 0.025)$
- 5 different current momenta in range
  $3 \le Q^2 \le 7\,\mathrm{GeV}^2$
- $O(10^4)$ measurements for each Q^2, λ pair
  Inversion for each q, λ,
  varying $\vec{p}$ relatively cheap
- Jacobi smeared sources and sinks,
  rms $\sim 0.5\,\mathrm{fm}$
- errors from 200 bootstrap samples

$m_\pi \sim 470$ MeV, ~SU(3) sym.
$m_\pi L = 5.6$



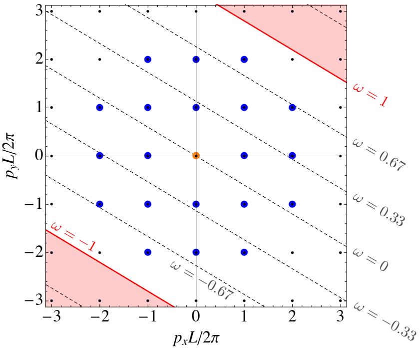
Kinematic coverage

- For example consider fixed

$$\vec{q} = \frac{2\pi}{32}(3,5,0) \qquad L = 32$$

- Can access different ω by varying nucleon momenta $\vec{p} = (2\pi/32)\vec{n}$

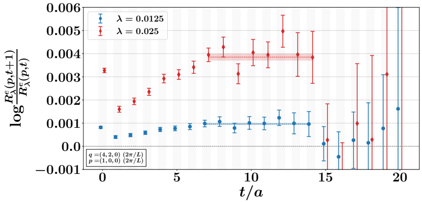
$$\omega = \frac{2\vec{p}\cdot\vec{q}}{\vec{q}^2} = \frac{2}{34}(3n_x + 5n_y)$$



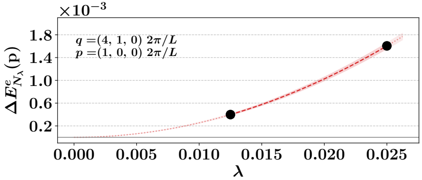
Blue dots: different nucleon Fourier momenta

Extract Energy Shifts: $\Delta E_{N\lambda}$ for each λ

- Effective energy plot for $\Delta E_{N\lambda}$:



- Find the $O(\lambda^2)$ term:



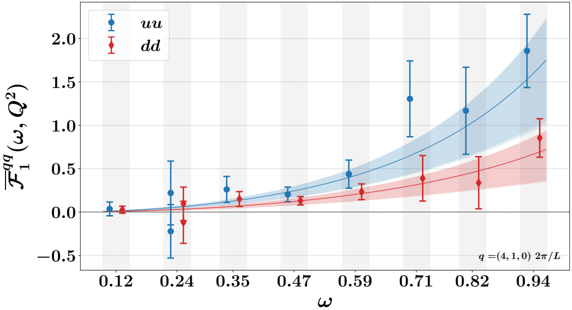
- Ratio of perturbed to unperturbed 2-pt correlation functions

$$R_\lambda = \frac{C_{+\lambda}(t)C_{-\lambda}(t)}{C_0(t)^2}$$

- Slope of curve

Structure Function(s) of the Compton amplitude

- eg fixed $\vec{q} = 2\pi/32\,(4,1,0)$, ie $Q^2 = 4.7\,\mathrm{GeV}^2$
- Varying $\vec{p}$ gives range of $\omega = 2\vec{p}\cdot\vec{q}/\vec{q}^2$ values:



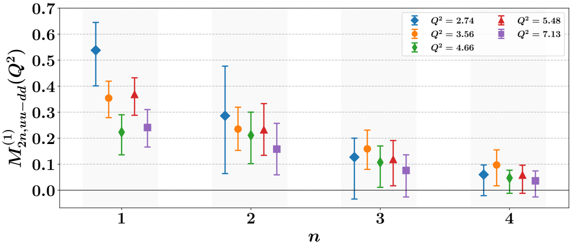
- Now determine moments

$$\overline{\mathcal{F}}_1(\omega, Q^2) = 2\omega^2\left(M_2^{(1)}(Q^2) + \omega^2 M_4^{(1)}(Q^2) + \ldots\right)$$

Fits $\Longrightarrow$ Moments

- Constraints: $M_2^{(1)} \geq M_4^{(1)} \geq \ldots \geq M_{2n}^{(1)} \geq \ldots > 0$ for u, d separately
- Bayesian implementation (likelihood + priors as constraints) $\longrightarrow$ previous curves on $\mathcal{F}_1$ versus ω plot

- $M_2^{(1)\, uu-dd}(Q^2)$:

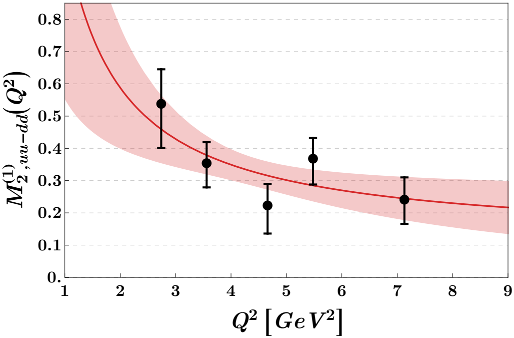


- Fall-off of the moments as expected
- Second moment does not decrease as rapidly as expected from DIS

Scaling – Power corrections

- Now have ability to study the Q^2 dependence of the moments. Not restricted to OPE and large Q^2
- eg naive model – constant + power corrections

$$M_2^{(1)\,uu-dd}(Q^2) = M_2^{(1)\,uu-dd} + \frac{C_2^{uu-dd}}{Q^2}$$



- Need $Q^2 \gtrsim 10\,\mathrm{GeV}^2$ to reliably extract moments to determine a value at $\mu = 2\,\mathrm{GeV}$

Reconstruction of the Form Factor / pdf

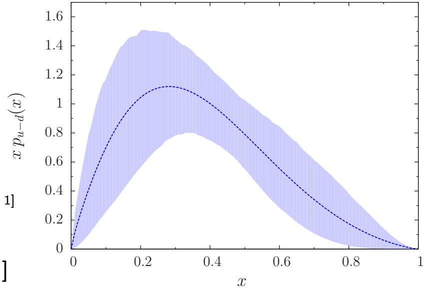
$$
\begin{aligned}
T_{33}(\omega, Q^2) &= \overline{\mathcal{F}}_1(\omega, Q^2) \\
&= 4\omega^2 \int_0^1 dx' \, \frac{x' F_1(x', Q^2)}{1 - x'^2 \omega^2} \\
&= \int_0^1 dx' \, K(x', \omega) \, F_1(x', Q^2)
\end{aligned}
$$

- Inverse problem – ill defined
- Ansatz

$$
\begin{aligned}
F_1(x, Q^2) &\equiv a p_{\mathrm{val}}(x : b, c) \\
&= a \, \frac{\Gamma(b + c + 3)}{\Gamma(b + 2)\Gamma(c + 1)} \, x^b (1 - x)^c
\end{aligned}
$$

$[\int_0^1 dx \, x p_{\mathrm{val}} = 1]$

- Again a Bayesian implementation
- General shape OK $[Q^2 = 2.7 \, \mathrm{GeV}^2]$



The $O(\lambda)$ term

The $O(\lambda)$ terms – Scattering and Form Factors – General Discussion

- We previously showed that the $O(\lambda)$ terms vanish
- Can escape if there is an energy degeneracy
- Replace state $|N(\vec{p})\rangle$ with energy $E(\vec{p})$ by

  $|N(\vec{p}_r)\rangle$, $r = 1, \ldots, d_S$ degenerate energy states, with energy $\bar{E}(\vec{p}, \vec{q})$

- As $\langle N(\vec{p})|\hat{\tilde{O}}(\vec{q})|N(\vec{p})\rangle$ now becomes a $d_S \times d_S$ Hermitian matrix

$$M_{rs} = \langle N(\vec{p}_r)|\hat{\tilde{O}}(\vec{q})|N(\vec{p}_s)\rangle$$

then can diagonalise, to give

$$E^{(i)}_{N\,\lambda}(\vec{p}, \vec{q}) = \bar{E}_N(\vec{p}, \vec{q}) - \lambda\,\mu^{(i)}(\vec{p}, \vec{q}) + O(\lambda^2)\,, \quad i = 1, \ldots, d_S$$

[$\mu^{(i)}$, $i = 1, \ldots, d_S$ eigenvalues]

Specific discussion I

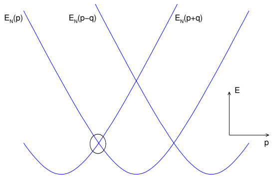
- $d_S = 2$:

$$|N(\vec{p}_1)\rangle = |N(\vec{p})\rangle\,, \qquad |N(\vec{p}_2\rangle = |N(\vec{p}+\vec{q})\rangle$$

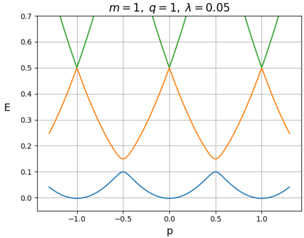
with

$$E_N(\vec{p}) = E_N(\vec{p}+\vec{q}) \Rightarrow 2\vec{p}\cdot\vec{q} = -\vec{q}^{\,2}$$

- eg 1-dimensional sketch, $p = -\frac{1}{2}q$:



$\Rightarrow$



Specific discussion II

- Remember matrix elements step up or down in $\vec{q}$:

$$[\hat{O}(\vec{x}) = e^{-i\hat{\vec{p}}\cdot\vec{x}}\,\hat{O}(\vec{0})\,e^{i\hat{\vec{p}}\cdot\vec{x}}]$$

$$\begin{aligned}
&\langle N(\vec{p}_r)|\hat{\tilde{\mathcal{O}}}(\vec{q})|N(\vec{p}_s)\rangle \\
&\quad = \quad \langle N(\vec{p}_r)|\hat{O}(\vec{0})|N(\vec{p}_s)\rangle\,\delta_{\vec{p}_r,\vec{p}_s+\vec{q}} + \langle N(\vec{p}_r)|\hat{O}^\dagger(\vec{0})|N(\vec{p}_s)\rangle\,\delta_{\vec{p}_r,\vec{p}_s-\vec{q}}
\end{aligned}$$

- So here the 2×2 matrix is anti-diagonal

$$M_{rs} = \begin{pmatrix} 0 & a^* \\ a & 0 \end{pmatrix}_{rs} \qquad \text{with } a = \langle N(\vec{p}+\vec{q})|\hat{O}(\vec{0})|N(\vec{p})\rangle$$

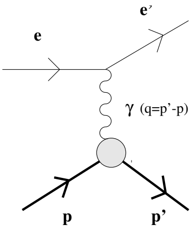
and diagonalisation gives $E^{(\pm)}_{\lambda\,N}$: their difference is

$$\begin{aligned}
\Delta E_{\lambda\,N}(\vec{p},\vec{q}) &= E^{(-)}_{\lambda\,N}(\vec{p},\vec{q}) - E^{(+)}_{\lambda\,N}(\vec{p},\vec{q}) \\
&= 2\lambda\,|\langle N(\vec{p}+\vec{q})|\hat{O}(\vec{0})|N(\vec{p})\rangle| + O(\lambda^2)
\end{aligned}$$

Elastic nucleon scattering

- $$p' = p + q$$

$$\langle N(\vec{p}\,')|J^\mu(\vec{q})|N(\vec{p})\rangle = \overline{u}(\vec{p}\,')\left[\gamma^\mu F_1(Q^2) + i\sigma^{\mu\nu}\frac{q_\nu}{2M_N}F_2(Q^2)\right]u(\vec{p})$$



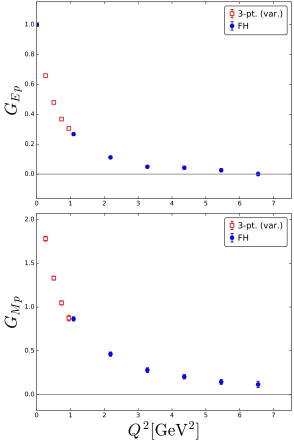
- Sachs form factors

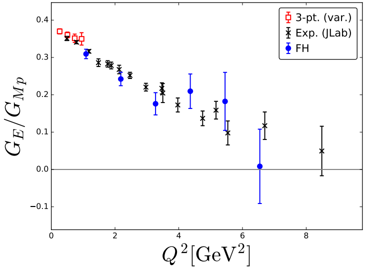
$$G_E(Q^2) = F_1(Q^2) - \frac{Q^2}{(2M_N)^2}F_2(Q^2)$$
$$G_M(Q^2) = F_1(Q^2) + F_2(Q^2)$$

- Feynman–Hellmann:
  - $O = J_4,\ J_3$
  - Breit frame geometry
    [electron bounces from nucleon, $\vec{p}\,' = -\vec{p}$ is a trivial solution of $E_N(\vec{p}\,') = E_N(\vec{p})$]

$$\Delta E_{\lambda\, N} = \begin{cases} \lambda \frac{M_N}{E_N}\, G_E \\ \lambda \frac{(\vec{e}\times\vec{q})_3}{E_N}\, G_M \end{cases}$$

Results

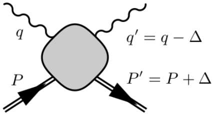




- LH: G_E, G_M also compared to variational 3-point (on same configs)
- RH: As for LH together with JLAB experimental results

Possible future perspectives

- Off-forward Compton Amplitude (OFCA) and GPDs



Hannaford-Gunn, Can et al.,
(CSSM-QCDSF-UKQCD)

arXiv:2110.11532

- Spin dependent Structure functions / Form factors
- Including quark-line-disconnected matrix elements
  - Expensive: Need purpose generated configurations with determinant also containing the λ term
  - (H)MC problem: for probability definition need real determinant so fermion matrix must be γ_5-Hermitian

$$\Longrightarrow \lambda^V, \lambda^A \text{ imaginary} \qquad [\lambda^S, \lambda^P, \lambda^T \text{ real}]$$

so E_λ develops an imaginary part for $O \sim V, A$

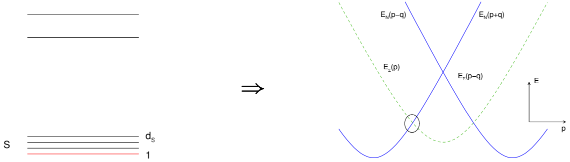
[Not a problem for the valence sector, as just inversion of a matrix]

- Generalisation to transition matrix elements

eg $s \to u$ ie $\Sigma(sdd) \to N(udd)$ decay

$$O \sim \bar{u}\gamma s$$

At $O(\lambda)$: 'quasi-degenerate' states



$$\Delta E_{\lambda\,\Sigma N}(\vec{p},\vec{q}) = \sqrt{(E_N(\vec{p}+\vec{q}) - E_\Sigma(\vec{p}))^2 + 4\lambda^2 \left|\langle N(\vec{p}+\vec{q})|\hat{O}(\vec{0})|\Sigma(\vec{p})\rangle\right|^2}$$

[Also holds when $E_N(\vec{p}) \approx E_N(\vec{p}+\vec{q})$]

Further applications of the Compton amplitude

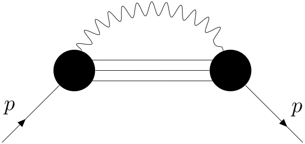
- Electromagnetic correction to proton – neutron mass splitting

[eg Walker-Lourd arXiv:1203.0254, ...]

$$M_p - M_n = \delta M^\gamma + \delta M^{m_d - m_u}$$

- Cottingham formula

$$\delta M^\gamma = \frac{i}{2M} \frac{\alpha_{em}}{(2\pi)^2} \int \frac{\eta^{\mu\nu}}{q^2 + i\epsilon} T_{\mu\nu}(p, q)$$



- Mixed currents: Neutrino-nucleon charged weak current:

$$\begin{aligned} W^{\mu\nu} &\equiv \frac{1}{4\pi} \int d^4z \, e^{iq\cdot z} \rho_{ss'} \, {}_{\mathrm{rel}}\langle p, s' | [J^\mu_{\mathrm{em}}(z), J^\nu_{W,A}(0)] | p, s \rangle_{\mathrm{rel}} \\ &= -i\epsilon^{\mu\nu\alpha\beta} \frac{q_\alpha p_\beta}{2p \cdot q} F_3(x, Q^2) \end{aligned}$$

[$J^\nu_{W,A} = \bar{u}\gamma_\nu\gamma_5 d$ axial part of weak charged current]

[eg Seng arXiv:1903.07969, Feng arXiv:2003.09798, ...]

Conclusions

- A new versatile approach
- Only involves computation of 2-point correlation functions [rather than 3-pt or 4-pt]
- Particularly for $\langle N|JJ|N\rangle$:
  - Longer source-sink separations possible – less excited states contamination,
  - Overcomes fierce operator mixing / renormalisation issues
  - Able to compute Compton Amplitudes and structure function moments

Backup

Dyson Series I

Regarding $\hat{B}$ as 'small', then have operator expansion

$$
\begin{aligned}
e^{t(\hat{A}+\hat{B})} &= e^{t\hat{A}} + \int_0^t dt'\, e^{(t-t')\hat{A}}\, \hat{B}\, e^{t'\hat{A}} \\
&\quad + \int_0^t dt' \int_0^{t'} dt''\, e^{(t-t')\hat{A}}\, \hat{B}\, e^{(t'-t'')\hat{A}}\, \hat{B} e^{t''\hat{A}} + O(\hat{B}^3)
\end{aligned}
$$

Apply to

$$\langle N(\vec{p})| e^{-(\hat{H}_0 - \lambda_\alpha \hat{\hat{O}}_\alpha)t} | Y(\vec{p}_Y)\rangle$$

with $\hat{A} \to -\hat{H}_0$ and $\hat{B} \to \lambda_\alpha \hat{\hat{O}}_\alpha$

Procedure

- $\hat{H}_0 |X(\vec{p}_X)\rangle = E_X(\vec{p}_X)|X(\vec{p}_X)\rangle$
- Insert additional complete set of states (X) in $O(\lambda^2)$ term
- Use integrals

$$
\begin{aligned}
\int_0^t dt'\, e^{-\alpha t'} &= \frac{1}{\alpha}\left(1 - e^{-\alpha t}\right) \\
\int_0^t dt' \int_0^{t'} dt''\, e^{-\alpha t' - \beta t''} &= \frac{1}{\beta - \alpha}\left[\frac{1}{\alpha}\left(1 - e^{-\alpha t}\right) - \frac{1}{\beta}\left(1 - e^{-\beta t}\right)\right]
\end{aligned}
$$

α, β are both differences in energy states: $E_X - E_N$, $E_Y - E_N$, $E_Y - E_X$

Dyson Series II

$$\langle N(\vec{p})| e^{-(\hat{H}_0 - \lambda_\alpha \hat{\tilde{O}}_\alpha)t} |Y(\vec{p}_Y)\rangle$$

$$= e^{-E_N(\vec{p})t}$$

$$\times \Bigg[\delta_{YN} + t\,\lambda_\alpha \langle N(\vec{p})|\hat{\tilde{O}}_\alpha(\vec{q})|N(\vec{p})\rangle \delta_{YN} + \lambda_\alpha \left. \frac{\langle N(\vec{p})|\hat{\tilde{O}}_\alpha(\vec{q})|Y(\vec{p}_Y)\rangle}{E_Y(\vec{p}_Y) - E_N(\vec{p})} \right|_{E_Y > E_N}$$

$$+ \frac{1}{2!} t^2 \lambda_\alpha \lambda_\beta \langle N(\vec{p})|\hat{\tilde{O}}_\alpha(\vec{q})|N(\vec{p})\rangle \langle N(\vec{p})|\hat{\tilde{O}}_\beta(\vec{q})|N(\vec{p})\rangle$$

$$+ t\,\lambda_\alpha \lambda_\beta \left. \frac{\langle N(\vec{p})|\hat{\tilde{O}}_\alpha(\vec{q})|N(\vec{p})\rangle \langle N(\vec{p})|\hat{\tilde{O}}_\beta(\vec{q})|Y(\vec{p}_Y)\rangle}{E_Y(\vec{p}_Y) - E_N(\vec{p})} \right|_{E_Y > E_N}$$

$$+ t\,\lambda_\alpha \lambda_\beta \sum\!\!\!\!\!\!\int_{E_X(\vec{p}_X) > E_N(\vec{p})} \frac{\langle N(\vec{p})|\hat{\tilde{O}}_\alpha(\vec{q})|X(\vec{p}_X)\rangle \langle X(\vec{p}_X)|\hat{\tilde{O}}_\beta(\vec{q})|N(\vec{p})\rangle}{E_X(\vec{p}_X) - E_N(\vec{p})} + O(\lambda^3) \Bigg] + \ldots$$

Comments

- Dropped more damped terms then $e^{-E_N t}$
- Linear and quadratic terms in t arise due to cases when α or $\beta \to 0$ or $\beta \to \alpha$

Comment on Moments – Compare to Conventional Approach I

- Operator Product Expansion [Expand OO in Compton Amplitude as sum of Ops]

$$M_{2n}^{(1)}(Q^2) = \sum_{\mathrm{NS}} \underbrace{C_{q\,2n}^{(1)\overline{\mathrm{MS}}}(Q^2/\mu^2, \alpha_s^{\overline{\mathrm{MS}}})}_{\mathrm{Wil\,coeff}} \underbrace{v_{2n}^{(q)\overline{\mathrm{MS}}}(\mu)}_{\mathrm{Had\,ME}} + O(1/Q^2)$$

- Wilson coefficient perturbatively computable: $Q_q^2(1 + O(\alpha_s^{\overline{\mathrm{MS}}}))$
- Matrix element given by

$$\langle N(\vec{p})| \left[\mathcal{O}_q^{\{\mu_1\cdots\mu_n\}} - \mathrm{Tr}\right] |N(\vec{p})\rangle^{\overline{\mathrm{MS}}} \equiv 2v_n^{(q)\overline{\mathrm{MS}}}[p^{\mu_1}\cdots p^{\mu_n} - \mathrm{Tr}]$$

and

$$\mathcal{O}_q^{\gamma\,\mu_1\cdots\mu_n} = i^{n-1}\overline{q}\gamma^{\mu_1}\overleftrightarrow{D}^{\mu_2}\cdots\overleftrightarrow{D}^{\mu_n}q, \qquad q = u, d$$

- Just need to compute these matrix elements !?

Moments – Conventional approach II

- The problem is that on the lattice, reduced $H(4)$ symmetry means much more mixing, practically only v_2, v_4 possible [QCDSF: hep-ph/0410187]

$$\mathcal{O}_{v_4} = \mathcal{O}^{\gamma}_{\{1144\}} + \mathcal{O}^{\gamma}_{\{2233\}} - \frac{1}{2}\left(\mathcal{O}^{\gamma}_{\{1133\}} + \mathcal{O}^{\gamma}_{\{1122\}} + \mathcal{O}^{\gamma}_{\{2244\}} + \mathcal{O}^{\gamma}_{\{3344\}}\right)$$

- Additional operators mixing with $\mathcal{O}_{v_4}$:

$$\begin{aligned}
\mathcal{O}^{m_1}_{v_4} = & -\mathcal{O}^{\gamma}_{1144} - \mathcal{O}^{\gamma}_{4114} - \mathcal{O}^{\gamma}_{1441} - \mathcal{O}^{\gamma}_{4411} + 2\mathcal{O}^{\gamma}_{1414} + 2\mathcal{O}^{\gamma}_{4141} \\
& -\mathcal{O}^{\gamma}_{2233} - \mathcal{O}^{\gamma}_{3223} - \mathcal{O}^{\gamma}_{2332} - \mathcal{O}^{\gamma}_{3322} + 2\mathcal{O}^{\gamma}_{2323} + 2\mathcal{O}^{\gamma}_{3232} \\
& +\frac{1}{2}\Big(+\mathcal{O}^{\gamma}_{1133} + \mathcal{O}^{\gamma}_{3113} + \mathcal{O}^{\gamma}_{1331} + \mathcal{O}^{\gamma}_{3311} - 2\mathcal{O}^{\gamma}_{1313} - 2\mathcal{O}^{\gamma}_{3131} \\
& \quad +\mathcal{O}^{\gamma}_{1122} + \mathcal{O}^{\gamma}_{2112} + \mathcal{O}^{\gamma}_{1221} + \mathcal{O}^{\gamma}_{2211} - 2\mathcal{O}^{\gamma}_{1212} - 2\mathcal{O}^{\gamma}_{2121} \\
& \quad +\mathcal{O}^{\gamma}_{2244} + \mathcal{O}^{\gamma}_{4224} + \mathcal{O}^{\gamma}_{2442} + \mathcal{O}^{\gamma}_{4422} - 2\mathcal{O}^{\gamma}_{2424} - 2\mathcal{O}^{\gamma}_{4242} \\
& \quad +\mathcal{O}^{\gamma}_{3344} + \mathcal{O}^{\gamma}_{4334} + \mathcal{O}^{\gamma}_{3443} + \mathcal{O}^{\gamma}_{4433} - 2\mathcal{O}^{\gamma}_{3434} - 2\mathcal{O}^{\gamma}_{4343}\Big)\,, \\
\mathcal{O}^{m_2}_{v_4} = & \frac{1}{2}\Big(+\mathcal{O}^{\gamma\gamma_5}_{1234} - \mathcal{O}^{\gamma\gamma_5}_{3214} - \mathcal{O}^{\gamma\gamma_5}_{1432} + \mathcal{O}^{\gamma\gamma_5}_{3412} + \mathcal{O}^{\gamma\gamma_5}_{2143} - \mathcal{O}^{\gamma\gamma_5}_{4123} - \mathcal{O}^{\gamma\gamma_5}_{2341} + \mathcal{O}^{\gamma\gamma_5}_{4321} \\
& \quad -\mathcal{O}^{\gamma\gamma_5}_{1324} + \mathcal{O}^{\gamma\gamma_5}_{2314} + \mathcal{O}^{\gamma\gamma_5}_{1423} - \mathcal{O}^{\gamma\gamma_5}_{2413} - \mathcal{O}^{\gamma\gamma_5}_{3142} + \mathcal{O}^{\gamma\gamma_5}_{4132} + \mathcal{O}^{\gamma\gamma_5}_{3241} - \mathcal{O}^{\gamma\gamma_5}_{4231}\Big) \\
& \quad +\mathcal{O}^{\gamma\gamma_5}_{1243} - \mathcal{O}^{\gamma\gamma_5}_{4213} - \mathcal{O}^{\gamma\gamma_5}_{1342} + \mathcal{O}^{\gamma\gamma_5}_{4312} + \mathcal{O}^{\gamma\gamma_5}_{2134} - \mathcal{O}^{\gamma\gamma_5}_{3124} - \mathcal{O}^{\gamma\gamma_5}_{2431} + \mathcal{O}^{\gamma\gamma_5}_{3421} \\
\mathcal{O}^{m_3}_{v_4} = & \frac{1}{2}i\Big(+\mathcal{O}^{\sigma\gamma_5}_{2413} - \mathcal{O}^{\sigma\gamma_5}_{2431} + \mathcal{O}^{\sigma\gamma_5}_{1324} - \mathcal{O}^{\sigma\gamma_5}_{1342} - \mathcal{O}^{\sigma\gamma_5}_{3412} + \mathcal{O}^{\sigma\gamma_5}_{3421} - \mathcal{O}^{\sigma\gamma_5}_{1234} + \mathcal{O}^{\sigma\gamma_5}_{1243} \\
& \quad +2\mathcal{O}^{\sigma\gamma_5}_{2314} - 2\mathcal{O}^{\sigma\gamma_5}_{2341} + 2\mathcal{O}^{\sigma\gamma_5}_{1423} - 2\mathcal{O}^{\sigma\gamma_5}_{1432}\Big) \qquad \text{lower dim}
\end{aligned}$$

- Feasible !?
- Note: we compute here ‘physical’ moments – everything included, not just ME of local operators

Hadronic matrix elements

In particle physics, need computation of physical quantities of mesons/baryons such as masses and matrix elements:

decay constants, form factors, (moments of) structure functions, . . .

directly from the underlying theory of QCD

Need computation of non-perturbative quantities: [m_H]

$$\langle H' | \widehat{\mathcal{O}} | H \rangle$$

General structure

- $H \sim \overline{\psi}\psi$ (meson) or $H \sim \psi\psi\psi$ (baryon)
- $\mathcal{O} \sim \overline{\psi}\psi \sim J$ or $\mathcal{O} \sim FF$ or even more complicated $\mathcal{O} \sim JJ$

This talk:

Describes determination of matrix elements using the Feynman-Hellmann theorem, with application to the Compton Amplitude and scattering