XXX International(ONLINE) Workshop on High Energy Physics
"Hard Problems of Hadron Physics: Non-Perturbative QCD & Related Quests", 11 Nov. 2021

Confinement, mass gap and gauge symmetry in the Yang-Mills theory – restoration of residual local gauge symmetry –

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based on arXiv:2111.xxxxx [hep-th], to appear soon.

§ Introduction

Quark confinement is well understood based on the dual superconductivity picture where condensation of magnetic monopoles and antimonopoles occurs. For a review, see e.g., K.-I. Kondo, S. Kato, T. Shinohara and A. Shibata, Phys. Rept **579**, 1–226 (2015), arXiv:1409.1599 [hep-th]
However, gluon confinement is less understood. [For recent developments, see e.g. Hayashi's talk in the next session.] Even if the dual superconductor picture is true, however, it is not an easy task to apply this picture to various composite particles composed of quarks and/or gluons.

In view of these, we recall the color confinement due to Kugo and Ojima (1979). If the Kugo and Ojima (KO) criterion is satisfied, all colored objects cannot be observed. Then quark confinement and gluon confinement immediately follow as special cases of color confinement.

However, the KO criterion was derived only in the Lorenz gauge $\partial^\mu \mathscr{A}_\mu = 0$, even if the issue on the existence of the nilpotent BRST symmetry is put aside for a while.

The KO criterion is written in terms of a specific correlation function called the KO function which is clearly gauge-dependent and is not directly applied to the other gauge fixing conditions.

From this point of view, the maximally Abelian (MA) gauge is the best gauge to be investigated because the dual superconductor picture for quark confinement was intensively investigated in the MA gauge.

Nevertheless,
Suzuki and Shimada (1983) pointed out that the KO criterion cannot be applied to the MA gauge and the KO criterion is violated in the model for which quark confinement is shown to occur by Polyakov (1977) due to magnetic monopole and antimonopole condensation.

Hata and Niigata (1993) claimed that the MA gauge is an exceptional case to which the KO color confinement criterion cannot be applied.

We wonder how the color confinement criterion of the KO type is compatible with the dual superconductor picture for quark confinement.

We reconsider the color confinement criterion of the KO type in the Lorenz gauge and give an explicit form to be satisfied in the MA gauge within the same framework as the Lorenz gauge in the manifestly Lorentz covariant operator formalism with the unbroken BRST symmetry.

For this purpose, we make use of the method of Hata (1982) saying that the KO criterion is equivalent to the condition for the residual local gauge symmetry to be restored.

We show that singular topological gauge field configurations play the role of restoring the residual local gauge symmetry violated in the MA gauge.

This result implies that color confinement phase is a disordered phase which is realized by non-perturbative effect due to topological configurations.

What is the residual gauge symmetry in gauge theory?

§ The residual gauge symmetry in Abelian gauge theory

Consider QED, or any local U(1) gauge-invariant system with the total Lagrangian density

$$\mathscr{L} = \mathscr{L}_{\mathsf{inv}} + \mathscr{L}_{\mathsf{GF+FP}}. \tag{1}$$

Here the gauge-invariant part $\mathscr{L}_{\mathsf{inv}}$ is invariant under the local gauge transformation:

$$A_\mu(x) \to A_\mu^\omega(x) := A_\mu(x) + \partial_\mu \omega(x). \tag{2}$$

To fix this gauge degrees of freedom, we introduce the Lorenz gauge fixing condition:

$$\partial_\mu A^\mu(x) = 0. \tag{3}$$

Then the GF+FP term is given by

$$\mathscr{L}_{\mathsf{GF+FP}} = B\partial_\mu A^\mu + \frac{1}{2}\alpha B^2 - i\partial^\mu \bar{c}\partial_\mu c. \tag{4}$$

However, this gauge-fixing still leaves the invariance under the transformation $\omega(x)$ linear in x^μ:

$$\omega(x) = a + \epsilon_\rho x^\rho, \tag{5}$$

since this is a solution of the equation

$$\partial^\mu \partial_\mu \omega(x) = 0 \Longrightarrow \partial^\mu A_\mu^\omega(x) = \partial^\mu A_\mu(x) + \partial^\mu \partial_\mu \omega(x) = 0. \tag{6}$$

This symmetry is an example of the residual local gauge symmetry.

There are two conserved charges, the usual charge Q and the vector charge Q^μ, as generators of the transformation:

$$\delta^\omega A_\mu(x) := A^\omega_\mu(x) - A_\mu(x) = [i(aQ + \epsilon_\rho Q^\rho), A_\mu(x)] = \partial_\mu \omega(x) = \epsilon_\mu. \tag{7}$$

This relation must hold for arbitrary a and ϵ_μ, leading to the commutator relations:

$$[iQ, A_\mu(x)] = 0, \quad [iQ^\rho, A_\mu(x)] = \delta^\rho_\mu. \tag{8}$$

The first eq.: the usual Q symmetry, i.e., global gauge symmetry is not spontaneously broken:

$$\langle 0|[iQ, A_\mu(x)]|0\rangle = 0, \tag{9}$$

The second eq.: Q^μ symmetry, i.e., the residual local gauge symmetry is always spontaneously broken:

$$\langle 0|[iQ^\rho, A_\mu(x)]|0\rangle = \delta^\rho_\mu. \tag{10}$$

Ferrari and Picasso (1971) argued from this observation that photon is understood as the massless Nambu-Goldstone (NG) vector boson associated with the spontaneous breaking of Q^μ symmetry according to the Nambu-Goldstone theorem...

The restoration of the residual local gauge symmetry does not occur in the ordinary Abelian case.

§ Color confinement and residual local gauge symmetry

Proposition 1: [Kugo-Ojima color confinement criterion(1979)] Choose the Lorenz gauge fixing $\partial^\mu \mathscr{A}_\mu = 0$. Suppose that the BRST symmetry exists. Let $\mathcal{V}_{\rm phys}$ be the physical state space with $\langle {\rm phys}|{\rm phys}\rangle \geq 0$ as a subspace of an indefinite metric state space $\mathcal{V}$ defined by the BRST charge operator $Q_{\rm B}$ as

$$\mathcal{V}_{\rm phys} = \{|{\rm phys}\rangle \in \mathcal{V}; Q_{\rm B}|{\rm phys}\rangle = 0\} \subset \mathcal{V}. \tag{1}$$

Introduce the function $u^{AB}(p^2)$ called the **Kugo-Ojima (KO) function** defined by

$$u^{AB}(p^2)\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) = \int d^D x \; e^{ip(x-y)} \langle 0|{\rm T}[(\mathscr{D}_\mu \mathscr{C})^A(x) g(\mathscr{A}_\mu \times \bar{\mathscr{C}})^B(y)|0\rangle. \tag{2}$$

If the condition called **Kugo-Ojima (KO) color confinement criterion** is satisfied in the Lorenz gauge

$$\lim_{p^2 \to 0} u^{AB}(p^2) = -\delta^{AB}, \tag{3}$$

then the color charge operator Q^A is well defined, namely, the color symmetry is not spontaneously broken, and Q^A vanishes for any physical state $\Phi, \Psi \in \mathcal{V}_{\rm phys}$,

$$\langle \Phi | Q^A | \Psi \rangle = 0, \quad \Phi, \Psi \in \mathcal{V}_{\rm phys}, \tag{4}$$

The BRST singlets as physical particles are all color singlets, while colored particles belong to the BRST quartet representation. Therefore, all colored particles cannot be observed and only color singlet particles can be observed.

Proposition 2: [Hata (1982)] Consider the residual "local gauge symmetry" specified by $\omega(x) \in su(N)$ linear in x^μ:

$$\omega(x) = T_A \omega^A(x),\ \omega^A(x) = \epsilon^A_\rho x^\rho, \tag{5}$$

where ϵ^A_ρ is x-independent constant parameters. Then there exists the Noether current

$$\mathscr{J}^\mu_\omega(x) = gJ^{\mu A}(x)x^\rho \epsilon^A_\rho + \mathscr{F}^{\mu\rho A}(x)\epsilon^A_\rho := \mathscr{J}^{\mu A}_{\ \ \rho}(x)\epsilon^{\rho A}, \tag{6}$$

which is conserved only in the physical subspace $\mathcal{V}_{\rm phys}$ of the state vector space $\mathcal{V}$:

$$\langle \Phi | \partial_\mu \mathscr{J}^\mu_\omega(x) | \Psi \rangle = 0, \quad \Phi, \Psi \in \mathcal{V}_{\rm phys}, \tag{7}$$

where $J^{\mu A}(x)$ is the Noether current associated with the global gauge symmetry which is conserved in $\mathcal{V}$. Then the Ward-Takahashi (WT) relation holds for $\mathscr{J}^{\mu A}_{\ \ \rho}(x)$ communicating to $\mathscr{A}^B_\sigma(y)$:

$$\int d^D x\ e^{ip(x-y)} \partial^x_\mu \langle 0 | \mathrm{T}[\mathscr{J}^{\mu A}_{\ \ \rho}(x) \mathscr{A}^B_\sigma(y)] | 0 \rangle = i \left(g_{\rho\sigma} - \frac{p_\rho p_\sigma}{p^2} \right) [\delta^{AB} + u^{AB}(p^2)]. \tag{8}$$

Thus, if the KO condition in the Lorenz gauge is satisfied

$$\lim_{p^2 \to 0} u^{AB}(p^2) = -\delta^{AB}, \tag{9}$$

then the massless pole between $\mathscr{J}^{\mu A}_{\ \ \rho}$ and $\mathscr{A}^B_\sigma$ contained in perturbation theory disappears.
The restoration condition coincides exactly with the Kugo and Ojima color confinement criterion! This means that the residual local gauge symmetry is restored if the KO condition is satisfied.

§ Residual gauge symmetry in the Lorenz gauge

The total Lagrangian density is given by

$$\mathscr{L} = \mathscr{L}_{\text{inv}} + \mathscr{L}_{\text{GF+FP}}. \tag{1}$$

The first term $\mathscr{L}_{\text{inv}}$ is the gauge-invariant part for the gauge field $\mathscr{A}_\mu$ and the matter field φ given by

$$\mathscr{L}_{\text{inv}} = -\frac{1}{4}\mathscr{F}_{\mu\nu} \cdot \mathscr{F}^{\mu\nu} + \mathscr{L}_{\text{matter}}(\psi, D_\mu \psi), \tag{2}$$

with $\mathscr{F}_{\mu\nu} := \partial_\mu \mathscr{A}_\nu - \partial_\nu \mathscr{A}_\mu + g\mathscr{A}_\mu \times \mathscr{A}_\nu = -\mathscr{F}_{\nu\mu}$ and $D_\mu \psi := \partial_\mu \psi - ig\mathscr{A}_\mu \psi$.
The second term $\mathscr{L}_{\text{GF+FP}}$ is the sum of the the gauge-fixing (GF) term and the Faddeev-Popov (FP) ghost term where the GF term includes the Nakanishi-Lautrup field $\mathscr{B}(x)$ which is the Lagrange multiplier field to incorporate the gauge fixing condition and the FP ghost term includes the ghost field $\mathscr{C}$ and the antighost field $\bar{\mathscr{C}}$.

For the gauge field and the matter field, we consider the local gauge transformation with the Lie algebra-valued transformation function $\omega(x) = \omega^A(x)T_A$

$$\begin{aligned}
\delta^\omega \mathscr{A}_\mu(x) &= \mathscr{D}_\mu \omega(x) := \partial_\mu \omega(x) - ig\mathscr{A}_\mu \times \omega(x), \\
\delta^\omega \varphi(x) &= ig\omega(x)\varphi(x), \\
\delta^\omega \mathscr{B}(x) &= g\mathscr{B}(x) \times \omega(x), \\
\delta^\omega \mathscr{C}(x) &= g\mathscr{C}(x) \times \omega(x), \\
\delta^\omega \bar{\mathscr{C}}(x) &= g\bar{\mathscr{C}}(x) \times \omega(x).
\end{aligned} \tag{3}$$

Now we proceed to write down the Ward-Takahashi relation to examine the appearance or disappearance of the massless pole. We consider the condition for the restoration of the residual local gauge symmetry for a general ω. We focus on the WT relation

$$
\begin{aligned}
&\int d^D x e^{ip(x-y)} \partial_\mu^x \langle \mathrm{T}\, \mathscr{J}_\omega^\mu(x) \mathscr{A}_\lambda^B(y) \rangle \\
=& i \langle \delta^\omega \mathscr{A}_\lambda^B(y) \rangle + \int d^D x \; e^{ip(x-y)} \langle \mathrm{T} \partial_\mu \mathscr{J}_\omega^\mu(x) \mathscr{A}_\lambda^B(y) \rangle \\
=& i \langle \partial_\lambda \omega^B(y) + g(\mathscr{A}_\lambda \times \omega)^B(y) \rangle + \int d^D x e^{ip(x-y)} \langle \mathrm{T} \delta^\omega \mathscr{L}_{\mathrm{GF+FP}}(x) \mathscr{A}_\lambda^B(y) \rangle \\
=& i \partial_\lambda \omega^B(y) + \int d^D x e^{ip(x-y)} \langle \mathrm{T} \delta^\omega \mathscr{L}_{\mathrm{GF+FP}}(x) \mathscr{A}_\lambda^B(y) \rangle \,,
\end{aligned}
\tag{4}
$$

where we have used $\langle 0 | \mathscr{A}_\lambda(x) | 0 \rangle = 0$ in the final step. Note that this relation is valid for any choice of the gauge fixing condition.

For the Lorenz gauge $\partial_\mu \mathscr{A}^\mu = 0$, the GF+FP term is given by

$$
\mathscr{L}_{\text{GF+FP}} = \mathscr{B} \cdot \partial_\mu \mathscr{A}^\mu + \frac{1}{2} \alpha \mathscr{B} \cdot \mathscr{B} - i \partial^\mu \bar{\mathscr{C}} \cdot \mathscr{D}_\mu \mathscr{C} = -i \boldsymbol{\delta}_\mathrm{B} \left[\bar{\mathscr{C}} \cdot \left(\partial^\mu \mathscr{A}_\mu + \frac{\alpha}{2} \mathscr{B} \right) \right] , \tag{5}
$$

where α is the gauge-fixing parameter. The change under the generalized local gauge transformation is given by α-independent expression:

$$
\delta^\omega \mathscr{L}_{\mathrm{GF+FP}}(x) = i \boldsymbol{\delta}_\mathrm{B} \bar{\boldsymbol{\delta}}_\mathrm{B} \mathscr{A}_\mu(x) \cdot \partial^\mu \omega(x) = i \boldsymbol{\delta}_B (\mathscr{D}_\mu \bar{\mathscr{C}}(x))^A \partial^\mu \omega^A(x). \tag{6}
$$

In the Lorenz gauge, the above WT relation (4) reduces to

$$\begin{aligned}&\int d^D x e^{ip(x-y)} \partial_\mu^x \langle \mathrm{T}\, \mathscr{J}_{\omega\ \nu}^{\ \mu A}(x) \partial^\nu \omega^A(x) \mathscr{A}_\lambda^B(y) \rangle \\ =& i\partial_\lambda \omega^B(y) + \int d^D x e^{ip(x-y)} \partial^\mu \omega^A(x) \langle \mathrm{T} i \boldsymbol{\delta}_B (\mathscr{D}_\mu \bar{\mathscr{C}}(x))^A \mathscr{A}_\lambda^B(y) \rangle \, .\end{aligned} \tag{7}$$

The second term of (7) is rewritten using $\boldsymbol{\delta}_B(\mathscr{D}_\mu \bar{\mathscr{C}}) = \boldsymbol{\delta}_B(\partial_\mu \bar{\mathscr{C}} + g(\mathscr{A}_\mu \times \bar{\mathscr{C}})) = -\partial_\mu \mathscr{B} + g\boldsymbol{\delta}_B(\mathscr{A}_\mu \times \bar{\mathscr{C}})$ as

$$\begin{aligned}&\int d^D x e^{ip(x-y)} \partial^\mu \omega^A(x) \langle \mathrm{T} i \boldsymbol{\delta}_B (\mathscr{D}_\mu \bar{\mathscr{C}}(x))^A \mathscr{A}_\lambda^B(y) \rangle \\ =& - \int d^D x e^{ip(x-y)} \partial^\mu \omega^A(x) \partial_\mu^x i \frac{\partial_\lambda^x}{\partial_x^2} \delta^D(x-y) \delta^{AB} \\ &+ i \int d^D x e^{ip(x-y)} \partial^\mu \omega^A(x) \left(g_{\mu\lambda} - \frac{\partial_\mu^x \partial_\lambda^x}{\partial_x^2} \right) u^{AB}(x-y)\end{aligned} \tag{8}$$

where we have used $\langle \boldsymbol{\delta}_{\mathrm{B}} F \rangle = 0$ for any functional F due to the physical state condition, the exact form of the propagator in the Lorenz gauge

$$\langle 0 | \mathrm{T} \mathscr{A}_\mu^A(x) \mathscr{B}^B(y) | 0 \rangle = \langle 0 | \mathrm{T}^* (\mathscr{D}_\mu \mathscr{C})^A(x) i \bar{\mathscr{C}}^B(y) | 0 \rangle = i \frac{\partial_\mu^x}{\partial_x^2} \delta^D(x-y) \delta^{AB}, \tag{9}$$

and the definition of the **Kugo-Ojima (KO) function** u^{AB} in the configuration space

$$\langle 0|\mathrm{T}(\mathscr{D}_\mu \mathscr{C})^A(x)(g\mathscr{A}_\nu \times \bar{\mathscr{C}})^B(y)|0\rangle = \left(g_{\mu\nu} - \frac{\partial_\mu^x \partial_\nu^x}{\partial_x^2}\right) u^{AB}(x-y). \tag{10}$$

Thus, we arrive at the desired general condition in the Lorenz gauge written in the Euclidean form:

$$\boxed{\lim_{p\to\ 0} \int d^D x e^{ip(x-y)} \partial_\mu \omega^A(x) \left(\delta_{\mu\lambda} - \frac{\partial_\mu^x \partial_\lambda^x}{\partial_x^2}\right) \left[\delta^D(x-y)\delta^{AB} + u^{AB}(x-y)\right] = 0}, \tag{11}$$

This confinement criterion can be applied to the Abelian and non-Abelian gauge theory as well irrespective of the compact or non-compact formulation, and is able to understand confinement in all the cases.

In the non-compact gauge theory formulated in terms of the Lie-algebra valued gauge field, the choice of $\omega^A(x) = \mathrm{const.} + \epsilon_\mu^A x_\mu$ linear in x is allowed. Indeed, for this choice, the criterion (11) is reduced to

$$\epsilon_\mu^A \lim_{p\to\ 0} \left(\delta_{\mu\lambda} - \frac{p_\mu p_\lambda}{p^2}\right) \left[\delta^{AB} + \tilde{u}^{AB}(p)\right] = 0. \tag{12}$$

This reproduces the KO condition $\tilde{u}^{AB}(0) = -\delta^{AB}$ as first shown by Hata.
For the Abelian gauge theory, the KO function is identically zero $u^{AB}(x-y) \equiv 0$, i.e., $u(0) = 0$. Therefore, the KO condition is not satisfied, which means no confinement in the Abelian gauge theory. In the compact gauge theory, however, confinement does occur even in the Abelian gauge theory, as is well known in the lattice gauge theory. This case is also understood by using the above criterion.

§ Restoration of residual local symmetry in MA gauge

We decompose the Lie-algebra valued quantity to the diagonal Cartan part and the remainig off-diagonal part, e.g., the gauge field $\mathscr{A}_\mu = \mathscr{A}_\mu^A T_A$ with the generators T_A $(A = 1, \ldots, N^2 - 1)$ of the Lie algebra $su(N)$ has the decomposition:

$$\mathscr{A}_\mu(x) = \mathscr{A}_\mu^A(x) T_A = a_\mu^j(x) H_j + A_\mu^a(x) T_a, \tag{1}$$

where H_j are the Cartan generators and T_a are the remaining generators of the Lie algebra $su(N)$. In what follows, $j, k, \ell, \ldots$ label the diagonal components and the index $a, b, c, \ldots$ labels the off-diagonal components. The maximal Abelian (MA) gauge is given by

$$(\mathscr{D}^\mu[a] A_\mu(x))^a := \partial^\mu A_\mu^a(x) + g f^{ajb} a^{\mu j}(x) A_\mu^b(x) = 0, \tag{2}$$

The MA gauge is a partial gauge which fix the off-diagonal components, but does not fix the diagonal components. Therefore, we further impose the Lorenz gauge for the diagonal components

$$\partial^\mu a_\mu^j(x) = 0. \tag{3}$$

The GF+FP term for the gauge-fixing condition (2) and (3) is given using the BRST transformation as

$$\mathscr{L}_{\text{GF+FP}} = -i\boldsymbol{\delta}_{\rm B} \left\{ \bar{C}^a \left(\mathscr{D}^\mu[a] A_\mu + \frac{\alpha}{2} B \right)^a \right\} - i\boldsymbol{\delta}_{\rm B} \left\{ \bar{c}^j \left(\partial^\mu a_\mu + \frac{\beta}{2} b \right)^j \right\}, \tag{4}$$

$$
\begin{aligned}
\mathscr{L}_{\mathsf{GF+FP}} =& - (\mathscr{D}^\mu[a]^{ba}B^a)A^b_\mu + \frac{\alpha}{2}B^aB^a - i(\mathscr{D}^\mu[a]^{ba}\bar{C}^a)\mathscr{D}_\mu[a]^{bc}C^c \\
& - ig(\mathscr{D}^\mu[a]^{ba}\bar{C}^a)f^{bcd}A^c_\mu C^d - ig(\mathscr{D}^\mu[a]^{ba}\bar{C}^a)f^{bcj}A^c_\mu c^j \\
& + ig\bar{C}^a f^{ajb}\partial_\mu c^j A^{\mu b} + ig^2\bar{C}^a f^{ajb}f^{jcd}A^c_\mu C^d A^{\mu b} \\
& - \partial^\mu b^j a^j_\mu + \frac{\beta}{2}b^jb^j - i\partial^\mu\bar{c}^j\partial_\mu c^j - ig\partial^\mu\bar{c}^j f^{jab}A^a_\mu C^b.
\end{aligned} \tag{5}
$$

The local gauge transformation of the Lagrangian has the following form

$$
\begin{aligned}
\delta^\omega\mathscr{L} = \delta^\omega\mathscr{L}_{\mathsf{GF+FP}} = \partial_\mu\mathscr{J}^\mu_\omega &= g\partial_\mu\mathscr{J}^\mu\cdot\omega + [\partial_\nu\mathscr{F}^{\mu\nu} + g\mathscr{J}^\mu]\cdot\partial_\mu\omega \\
&= g\partial^\mu J^j_\mu\omega^j + \left[\partial^\nu f^j_{\mu\nu} + gJ^j_\mu\right]\partial_\mu\omega^j + g\partial^\mu J^a_\mu\omega^a + \left[\partial^\nu F^a_{\mu\nu} + gJ^a_\mu\right]\partial_\mu\omega^a \\
&= i\boldsymbol{\delta}_B\partial_\mu\bar{c}^j\partial^\mu\omega^j + i\boldsymbol{\delta}_{\mathrm{B}}\partial^\mu(\mathscr{D}_\mu[\mathscr{A}]\bar{\mathscr{C}})^a\omega^a + i\boldsymbol{\delta}_{\mathrm{B}}(\mathscr{D}_\mu[\mathscr{A}]\bar{\mathscr{C}})^a\partial^\mu\omega^a.
\end{aligned} \tag{6}
$$

This is BRST exact, showing that the local gauge current $\mathscr{J}^\mu_\omega$ is conserved in the physical state space. The WT relation in the MA gauge can be calculated in the similar way to the Lorenz gauge by using (6) as follows. We focus on the diagonal gauge field a^k_λ. Consequently, we obtain the condition for the restoration of the residual local gauge symmetry for the diagonal gauge field

$$
\begin{aligned}
&\lim_{p\to 0}\int d^Dx\; e^{ip(x-y)}\partial^x_\mu\langle\mathrm{T}\,\mathscr{J}^\mu_\omega(x)a^k_\lambda(y)\rangle \\
=& \boxed{\lim_{p\to 0} i\int d^Dx\; e^{ip(x-y)}\partial^\mu\omega^k(x)(\delta_{\mu\lambda}\Box_D - \partial_\mu\partial_\lambda)\Box_D^{-1}(x,y) = 0},
\end{aligned} \tag{7}
$$

where $\Box_D^{-1}(x,y)$ denotes the Green function of the Laplacian $\Box_D = \partial_\mu \partial_\mu$ in the D-dimensional Euclidean space.
If we choose $\omega^j(x) = \epsilon^j_\nu x^\nu$, this indeed reproduces non-vanishing divergent result.
However, this choice must be excluded in the MA gauge, since the maximal torus subgroup $U(1)^{N-1}$ for the diagonal components is a compact subgroup of the compact $SU(N)$ group. In some sense, $\omega^j(x)$ must be angle variables reflecting the compactness of the gauge group.

For concreteness, we consider the $SU(2)$ case with singular configurations coming from the angle variables. In what follows, we work in the Euclidean space and use subscripts instead of the Lorentz indices. As the residual gauge transformation, we take the following examples which satisfy both the Lorenz gauge condition $\partial_\mu \mathscr{A}_\mu^A = 0$ and the MA gauge condition $(\mathscr{D}_\mu[a]A_\mu)^a = 0$ (and $\partial^\mu a_\mu^j = 0$).

- For $D = 2$, a collection of vortices of Abrikosov-Nielsen-Olesen type (1979)

$$\partial_\mu \omega^j(x) = \sum_{s=1}^{n} C_s \varepsilon_{j\mu\nu} \frac{(x-a_s)_\nu}{|x-a_s|^2} \ (j=3,\ \mu,\nu=1,2)\ (x, a_s \in \mathbb{R}^2), \tag{8}$$

where C_s $(s = 1, \ldots, n)$ are arbitrary constants. This type of $\omega(x)$ is indeed an angle variable θ going around a point $a = (a_1, a_2) \in \mathbb{R}^2$, because

$$\omega(x) = \theta(x) =: \arctan \frac{x_2 - a_2}{x_1 - a_1} \Longrightarrow \partial_\mu \omega(x) = -\varepsilon_{\mu\nu} \frac{x_\nu - a_\nu}{(x_1-a_1)^2 + (x_2-a_2)^2} \ (\mu = 1,2). \tag{9}$$

This is a topological configuration which is classified by the winding number of the map from the circle in the space to the circle in the target space: $S^1 \to U(1) \cong S^1$, i.e., by the first Homotopy group $\pi_1(S^1) = \mathbb{Z}$.

• For $D = 3$, a collection of magnetic monopoles of the Wu-Yang type (1975) which corresponds to the zero size limit of the 't Hooft-Polyakov magnetic monopole (1974)

$$\partial_\mu \omega^j(x) = \sum_{s=1}^{n} C_s \varepsilon_{j\mu\nu} \frac{(x-a_s)_\nu}{|x-a_s|^2} \; (j=3, \; \mu,\nu = 1,2,3) \; (x, a_s \in \mathbb{R}^3). \tag{10}$$

A magnetic monopole is a topological configuration which is classified by the winding number of the map from the sphere in the space to the sphere in the target space: $S^2 \to SU(2)/U(1) \cong S^2$, i.e., by the second Homotopy group $\pi_2(S^2) = \mathbb{Z}$.

• For $D = 4$, a collection of merons of Alfaro-Fubini-Furlan (1976) instantons of the Belavin-Polyakov-Shwarts-Tyupkin (BPST) type (1975) in the non-singular gauge with zero size,

$$\partial_\mu \omega^j(x) = \sum_{s=1}^{n} C_s \eta^j_{\mu\nu} \frac{(x-a_s)_\nu}{|x-a_s|^2} \; (j=3, \; \mu,\nu = 1,2,3,4) \; (x, a_s \in \mathbb{R}^4). \tag{11}$$

Meron and instanton are topological configuration which are classified by the winding number of the map from the 3-dimensional sphere in the space to the sphere in the target space: $S^3 \to SU(2) \cong S^3$, i.e., by the third Homotopy group $\pi_3(S^3) = \mathbb{Z}$.

By taking into account $\varepsilon^j_{\mu\nu} = -\varepsilon^j_{\nu\mu}$, $\eta^j_{\mu\nu} = -\eta^j_{\nu\mu}$, it is easy to show that all these configurations satisfy the Laplace equation $\Box \omega^j(x) = 0$ almost everywhere except for the locations $a_s \in \mathbb{R}^D$of the singularities: $\Box \omega^j(x) = \sum_{s=1}^{n} C_s \delta^D(x-a_s)$. These configurations are examples of the classical solutions of the Yang-Mills field equation with non-trivial topology.

We can show that the restoration condition is satisfied for these singular configurations:

$$\boxed{\lim_{p\to 0}\int d^D x\ e^{ip(x-y)}\frac{(x-a_s)_\nu}{|x-a_s|^2}\left(\delta_{\mu\lambda}\Box_D-\partial_\mu\partial_\lambda\right)\frac{\frac{\Gamma\left(\frac{D}{2}-1\right)}{4\pi^{D/2}}}{(|x-y|^2)^{\frac{D-2}{2}}}=0}. \tag{12}$$

where we have used the expression of the Green function $\Box_D^{-1}(x,y)$ of the Laplacian $\Box_D=\partial_\mu\partial_\mu$ in the D-dimensional Euclidean space given by

$$\Box_D^{-1}(x,y)=\int\frac{d^Dp}{(2\pi)^D}e^{ip(x-y)}\frac{1}{-p^2}=-\frac{\Gamma\left(\frac{D}{2}-1\right)}{4\pi^{D/2}}\frac{1}{|x-y|^{D-2}}, \tag{13}$$

where Γ is the gamma function with the integral representation given by

$$\Gamma(z)=\int_0^\infty dt\ t^{z-1}e^{-t}\ (z>0). \tag{14}$$

For any $D\geq 2$, this integral goes to zero linearly in p in the limit $p\to 0$. Therefore, the restoration of the residual local gauge symmetry occurs.

§ Conclusion and discussion

▷ Conclusions

• We have reexamined the restoration of the residual local gauge symmetry left even after imposing the gauge fixing condition in quantum gauge field theories. This leads to a generalization of the color confinement criterion.

• We have found an important lesson to understand color confinement in quantum gauge theories that the compactness and non-compactness must be discriminated for the gauge transformation of the gauge field.

• The Kugo-Ojima color confinement criterion can be applied only to the non-compact gauge theory. This is a reason why the Kugo-Ojima criterion obtained in the Lorenz gauge cannot be applied to the Maximal Abelian gauge (maximal torus group is a compact group).

• In the Maximal Abelian gauge we have shown that the restoration of the residual local gauge symmety indeed occurs for the SU(N) Yang-Mills theory in two-, three- and four-dimensional Euclidan spacetime once the singular topological configurations of gauge fields are taken into account.

• This result indicates that the color confinement phase is a disordered phase caused by non-trivial topological configurations irrespective of the gauge choice.

• As a byproduct, we find that the compact U(1) gauge theory can have the disordered confinement phase, while the non-compact U(1) gauge theory has the deconfined Coulomb phase.

▷ Future perspectives

• Gribov copies, existence of BRST symmetry,

• Higgs phase, Brount-Englert-Higgs (BEH) mechanism,

• Finite temperatures,

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▷ Future perspectives

• Gribov copies, existence of BRST symmetry,

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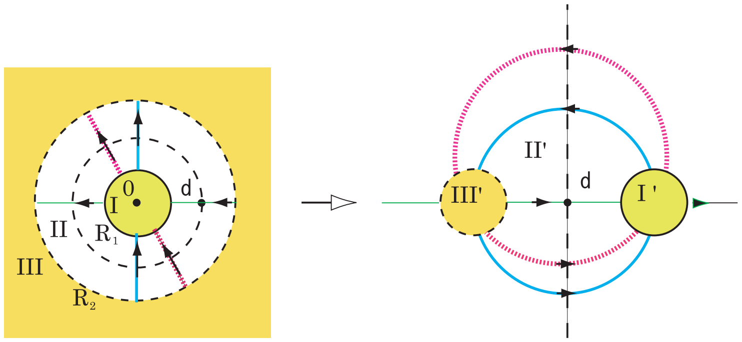
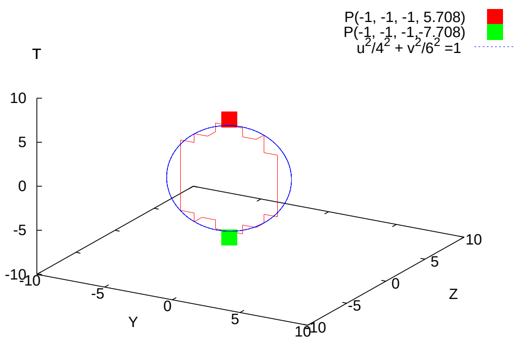
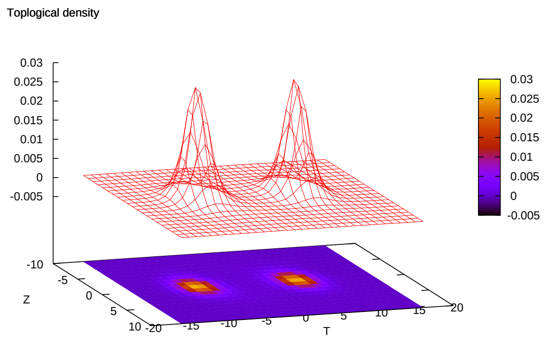
Thank you very much for your attention.

§ Backup slides







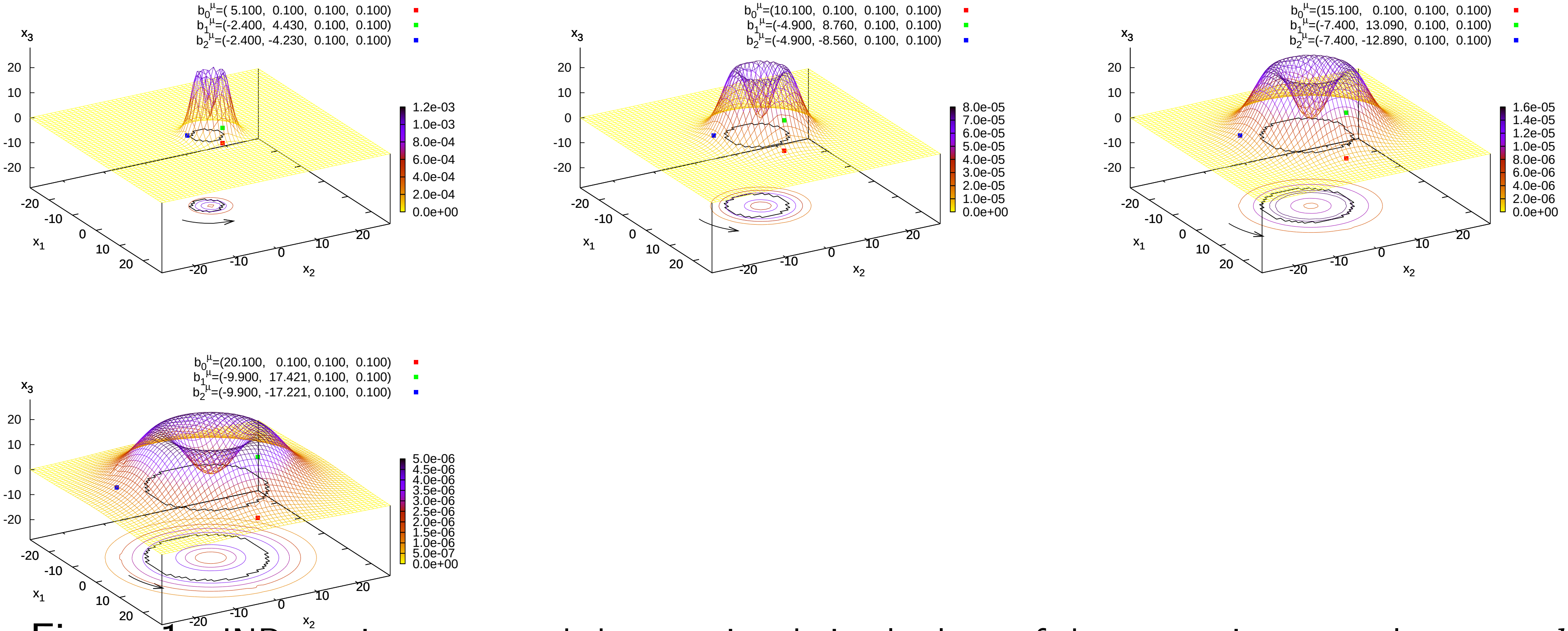


Figure 1: JNR two-instanton and the associated circular loop of the magnetic-monopole current $k_{x,\mu}$. The JNR two-instanton is defined by fixing three scales $\rho_0 = \rho_1 = \rho_2 = 3\epsilon$ and three pole positions $b_0^\mu, b_1^\mu, b_2^\mu$ which are arranged to be three vertices of an equilateral triangle specified by r: (a) $r = 5\epsilon$, (b) $r = 10\epsilon$, (c) $r = 15\epsilon$ and (d) $r = 20\epsilon$. The grid shows an instanton charge density D_x on x_1-x_2 $(x_3 = x_4 = 0)$ plane. The associated circular loop of the magnetic-monopole current is located on the same plane as that specified by three poles. The black line on the base shows the magnetic monopole loop projected on the x_1-x_2 plane and the arrow indicates the direction of the magnetic-monopole current, while colored lines on the base show the contour plot for the equi-D_x lines. Figures are drawn in units of the lattice spacing ϵ.