The mixed 0-form/1-form anomaly in Hilbert space

(pouring the new wine into old bottles)

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hep th 2106.11442, JHEP

subject is, ultimately, the IR phases of strongly coupled gauge theories

- goal: learn more about QFTs, an eye towards (future) beyond-SM applications

difficult, any theoretical constraints welcome!

- ’t Hooft anomalies are a remarkable set of RG-invariant numbers: compute in UV (free) theory, must be matched by IR physics

- ’t Hooft anomalies in QCD well known (massless pions or baryons required in chiral limit)

- “new ’t Hooft anomalies” involving higher form symmetry

Gaiotto et al,+…, 2014-…

this talk:

- discrete parity in pure YM at $\theta = \pi$
- discrete chiral symmetry of QCD(adj)

"0-form"

"1-form"

gauge group can by any of the simple ones with a center

$SU(N)$ (Z_N), $Sp(N)$ (Z_2), $Spin(N)$ $(Z_2, Z_4$, or $Z_2 \times Z_2)$, E_6 (Z_3), E_7 (Z_2)

the fact that these discrete symmetries have a mixed (0-form/ 1-form) anomaly is well known by now; it turns out, however, that a Hilbert space interpretation brings some new insight

"new wine": the mixed anomaly interpretation

Gaiotto et al,+…, 2014-…

"old bottles": quantization of YM in 't Hooft flux backgrounds

't Hooft '81; van Baal '82,'84; Witten '82,'00; Gonzalez-Arroyo, Korthals Altes '88 …

“new wine”: the mixed anomaly interpretation

Gaiotto et al,+…, 2014-…

“old bottles”: quantization of YM in ’t Hooft flux backgrounds

’t Hooft ‘81; van Baal ’82,’84; Witten ’82,’00; Gonzalez-Arroyo, Korthals Altes ’88 …

(the connection appears to not be well appreciated)

pure YM: mixed anomaly 0-form parity at $\theta = \pi$ and 1-form center symmetry

QCD(adj), YM + n_f Weyl adjoints: mixed anomaly between 0-form discrete chiral and 1-form center symmetry

Upshot:

Anomalies’ consequence in Hilbert space on T^3: exact degeneracies **at finite V!**

$\downarrow$

SSB in TD limit

Explain **briefly** how these come about and discuss implications.

For many, a trip back to the ’80s - but with novel interpretation!

To see these anomalies, one has to consider a space (time) with noncontractible two-cycles (a torus).

Torus quantization was used by ’t Hooft (van Baal, Luscher…) in the 1980s to study confinement in pure YM.

’t Hooft introduced the notion of twisted b.c. on the torus. In modern language, these are nondynamical backgrounds gauging the 1-form center symmetry.

$$A(L_1, y, z) = \Gamma_1 A(0,y,z)\Gamma_1^{-1}$$
$$A(x, L_2, z) = \Gamma_2 A(x,0,z)\Gamma_2^{-1}$$
$$A(x, y, L_3) = \Gamma_3 A(x,y,0)\Gamma_3^{-1}$$

$$\Gamma_k \Gamma_l = \Gamma_l \Gamma_k e^{i\frac{2\pi}{N}\epsilon_{klm} m_m}$$

’t Hooft fluxes

$$\oint_{kl} C^{(2)} = \frac{2\pi}{N}\epsilon_{klm} m_m$$ in Kapustin, Seiberg ’14

twist matrices obeying cocycle conditions

Center symmetry, acting on noncontractible Wilson loops in nonzero N-ality representations that wrap around the cycles of the torus was understood as a global symmetry (eg by Luscher, despite some apparent confusion in literature at the time).

new **old**

1-form symmetries … "improper gauge trsfs.", "central conjugations" (Luscher)

$$C[\vec{k},\nu](L_1,y,z) = e^{i\frac{2\pi k_1}{N}}\ \Gamma_1 C[\vec{k},\nu](0,y,z)\Gamma_1^{-1}$$
$$C[\vec{k},\nu](x,L_2,z) = e^{i\frac{2\pi k_2}{N}}\ \Gamma_2 C[\vec{k},\nu](x,0,z)\Gamma_2^{-1}$$
$$C[\vec{k},\nu](x,y,L_3) = e^{i\frac{2\pi k_3}{N}}\ \Gamma_3 C[\vec{k},\nu](x,y,0)\Gamma_3^{-1}$$

$\xrightarrow{\text{basis}}$

$$\hat{T}_1 \, |A\rangle = |C[(1,0,0),0] \circ A$$
$$\hat{T}_2 \, |A\rangle = |C[(0,1,0),0] \circ A$$
$$\hat{T}_3 \, |A\rangle = |C[(0,0,1),0] \circ A$$

winding

$\hat{T}_i$ generate global symmetries: act on Wilson loops in i-th direction

well known from studies of deconfinement transition; here: in all directions of space, "1-form"

Here, 'm' is fixed (we don't sum over values of 'm'). A background field for the discrete 1-form center symmetry.

Crucial insight ('t Hooft +...):

$\hat{T}_i$, *(or their C's)* when $\vec{m} \neq 0$, have fractional $T^3 \to SU(N)$ winding number

$$Q[C] = \frac{1}{24\pi^2}\int_{\mathbb{T}^3} \mathrm{tr}\,(C dC^{-1})^3$$ we define $\hat{T}_i$ s.t. $Q[T_l] = \frac{m_l}{N}$

then, in $\mathcal{H}_\theta^{phys.}$: $\hat{T}_l^N \, |\psi\rangle = |\psi\rangle \, e^{-i\theta m_l}$

and $\hat{T}_l |e_l\rangle = |e_l\rangle \, e^{i\frac{2\pi}{N}e_l - i\theta\frac{m_l}{N}} = |e_l\rangle \, e^{i\frac{2\pi}{N}(e_l - \frac{\theta}{2\pi}m_l)}$

Hamiltonian and T simultaneously diagonal

boundary conditions on T^3

$\vec{m}$ (mod N) ...

discrete magnetic flux

eigenvalues of $\hat{T}_l$, generating 1-form Z_N

$\vec{e}$ (mod N) ...

discrete electric flux

3d (spatial) CS form; V shifts θ angle by α

$$\hat{V}_\alpha[\hat{A}] = e^{i\alpha \int_{\mathbb{T}^3} K_0(\hat{A})}$$

$\oint K_0$ shifts by 1 under a unit winding gauge trf.

can be shown

$$\hat{T}_l \; \hat{V}_{2\pi} = e^{i2\pi \frac{m_l}{N}} \; \hat{V}_{2\pi} \; \hat{T}_l$$

2π shifts of θ and center in the $\vec{m}$-direction do not commute

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APPLICATION 1: SU(N) pure YM, P: parity

$\hat{P}_0 \, \hat{T}_j \, \hat{P}_0 = \hat{T}_j^\dagger$ = dihedral group of order 2N, at $\theta = 0$

at $\theta = \pi$, however, the parity generator is $\hat{P}_\pi = \hat{V}_{2\pi} \hat{P}_0$

$\hat{P}_\pi \, \hat{T}_j \, \hat{P}_\pi = e^{\frac{2\pi i}{N} m_j} \, \hat{T}_j^\dagger$ so the algebra at $\theta = \pi$ is extended

from now on $\vec{m} = (0,0,1)$ **ignore** T_1, T_2 **and labels** e_1, e_2

$$[\hat{T}_3, \hat{H}_{\theta=\pi}] = 0 \ , \quad [\hat{P}_\pi, \hat{H}_{\theta=\pi}] = 0 \ , \quad \hat{T}_3 \hat{P}_\pi = e^{i\frac{2\pi}{N}} \hat{P}_\pi \hat{T}_3^\dagger$$

$$\hat{H}_{\theta=\pi} |E, e_3\rangle = |E, e_3\rangle E$$

$$\hat{T}_3 |E, e_3\rangle = |E, e_3\rangle e^{i\frac{2\pi}{N} e_3}$$

$$\hat{T}_3(\hat{P}_\pi |E, e_3\rangle) = (\hat{P}_\pi |E, e_3\rangle) e^{i\frac{2\pi}{N}(1-e_3)}$$

$$\hat{P}_\pi : |E, e_3\rangle \to |E, 1 - e_3 \ (\mathrm{mod} N)\rangle$$

$$\hat{P}_0 : |E, e_3\rangle \to |E, -e_3\rangle$$

$\theta = 0, e_3 = 0$ is parity invariant for all N

- "global inconsistency" for odd-N

$\theta = \pi$, odd-N: $e_3 = \dfrac{N+1}{2}$ invariant

$$\hat{P}_\pi : |E, e_3\rangle \to |E, 1 - e_3 \ (\mathrm{mod} N)\rangle$$

$\theta = \pi$, even-N all states doubly degenerate

- **mixed center/parity anomaly for even-N SU(N)**
- **mixed center/parity anomaly for all groups … if center is of even order (i.e., for all but E_6, where "global inconsistency")**

$$\vec{m} = (0,0,1) \qquad [\hat{T}_3, \hat{H}_{\theta=\pi}] = 0 \ , \quad [\hat{P}_\pi, \hat{H}_{\theta=\pi}] = 0 \ , \quad \hat{T}_3 \hat{P}_\pi = e^{i\frac{2\pi}{N}} \hat{P}_\pi \hat{T}_3^\dagger$$

- **even N, exact parity degeneracy at any size torus (already seen in anomalies)**

  delicate cancellations of tunneling: semiclassics

  phases due to $\vec{m}$/*contours/thimbles*... **dYM ?**

- **as $L_i \to \infty$, expect lowest energy e-flux states => parity breaking vacua**

  other N-2 higher energy fluxes: metastable/unstable pairs of vacua... seen in dYM

old vs new: ... 1980 vs now:

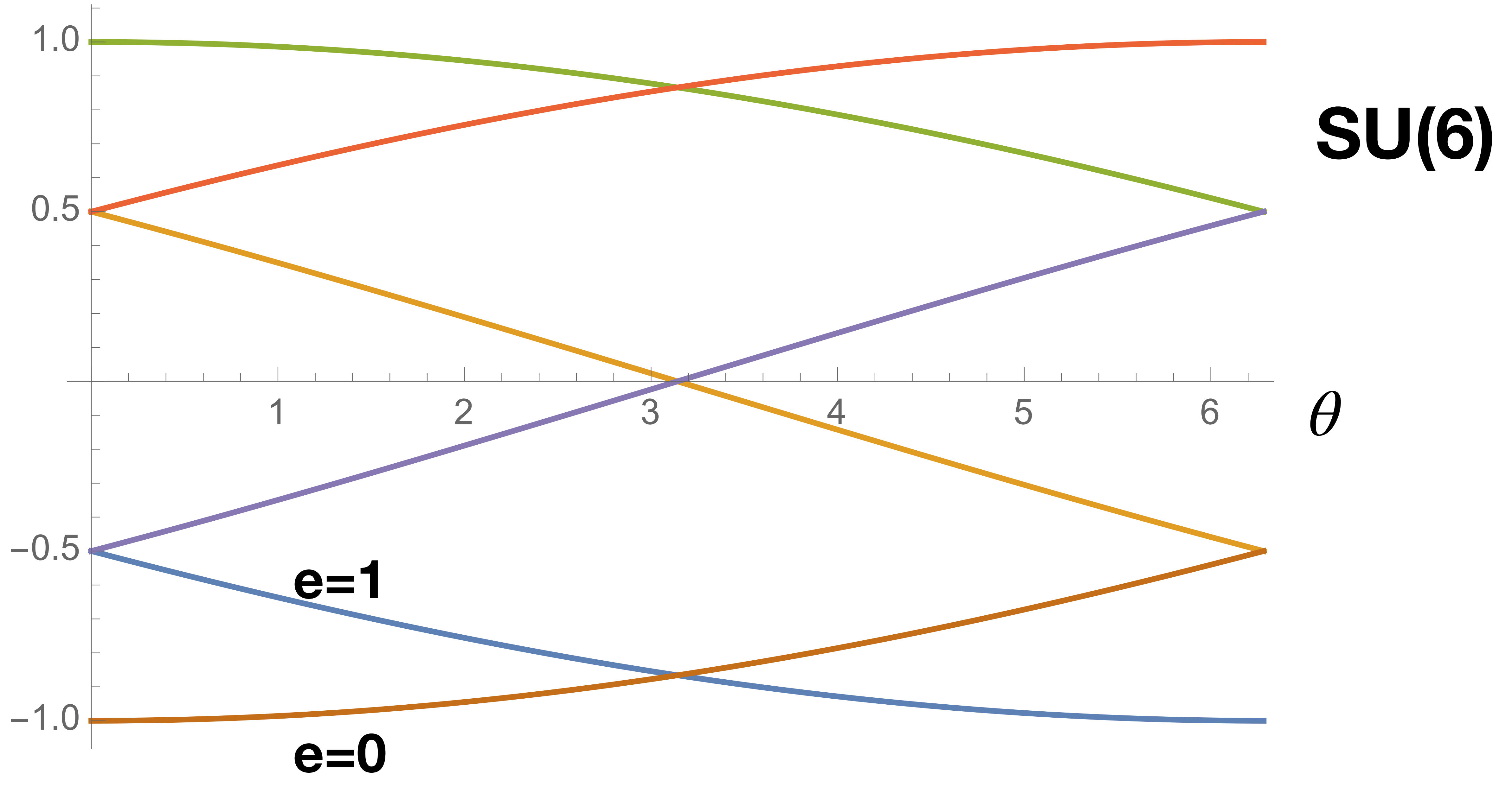
interpretation as anomaly and the centrally-extended algebra new

$$\vec{m} = (0,0,1) \qquad [\hat{T}_3, \hat{H}_{\theta=\pi}] = 0\ , \quad [\hat{P}_\pi, \hat{H}_{\theta=\pi}] = 0\ , \quad \hat{T}_3 \hat{P}_\pi = e^{i\frac{2\pi}{N}} \hat{P}_\pi \hat{T}_3^\dagger$$

- as $L_i \ll \Lambda^{-1}$ **"femtouniverse"** [van Baal in 1999 review, $\vec{m} = (0,0,1)$]

$$E(\theta, e_3) = -\frac{C e^{-\frac{8\pi^2}{g^2 N}}}{L g^4} \cos\left(\frac{2\pi}{N} e_3 - \frac{\theta}{N} m_3\right)$$

fractional I's on $T^3 \times R$; no analytic soltns



for the experts, compare with dYM vacuum energies: $L_1, L_2 \to \infty, L_3 = L \ll 1/(N\Lambda)$

extended algebra seen in IR of dYM *[Aitken, Cherman, Unsal 2018]*: **no m … *precise* relation**

Summary: **mixed parity/center anomaly in $\vec{m}$ leads to extended algebra**

$$[\hat{T}_3, \hat{H}_{\theta=\pi}] = 0 \ , \quad [\hat{P}_\pi, \hat{H}_{\theta=\pi}] = 0 \ , \quad \hat{T}_3 \hat{P}_\pi = e^{i\frac{2\pi}{N}} \hat{P}_\pi \hat{T}_3^\dagger \qquad \vec{m} = (0,0,1)$$

anomaly:	**SU(2k), Sp(2k+1), Spin(2k), E_7**
global inconsistency:	**SU(2k+1), E_6**
neither:	**Sp(2k), Spin(2k+1)**

the most unusual feature is the exact degeneracy at finite volume in the $\vec{m}$ background implied by the extension

would like to better understand in a calculable framework valid in a (partially) infinite volume, dYM

APPLICATION 2: SU(N) QCD(adj):

SU(N) QCD(adj) with $n_f \leq 5$ massless Weyl adjoint fermions

$\frac{\mathbb{Z}_{2n_f N} \times SU(n_f)}{\mathbb{Z}_{n_f}}$ **anomaly free chiral** $\hat{X}_{\mathbb{Z}^{(0)}_{2n_f N}} = e^{i \frac{2\pi}{2n_f N} \hat{Q}_5} = e^{i \frac{2\pi}{2n_f N} \int d^3x \hat{j}^0_f} \; \hat{V}_{2\pi}^{-1}$

mixed chiral/center anomaly in $\overrightarrow{m} = (0,0,1)$ leads to extended algebra

[more general twists possible - not here!]

$$[\hat{T}_3, \hat{H}] = 0, \quad [\hat{X}_{\mathbb{Z}^{(0)}_{2n_f N}}, \hat{H}] = 0, \quad \hat{T}_3 \, \hat{X}_{\mathbb{Z}^{(0)}_{2n_f N}} = e^{-i \frac{2\pi}{N}} \; \hat{X}_{\mathbb{Z}^{(0)}_{2n_f N}} \, \hat{T}_3$$

$\hat{X}_{\mathbb{Z}^{(0)}_{2n_f N}} |E, e_3\rangle = |E, e_3 - 1\rangle$ **- implies N-fold degeneracy**

$\hat{X}_{\mathbb{Z}^{(0)}_{2n_f N}} |E, e_3\rangle = |E, e_3 - 1\rangle$ **- implies N-fold degeneracy**

- exact at any size T^3... at least $Z_{2n_f N} \to Z_{2n_f}$

- yes, for $n_f = 1$ (SYM)
- yes, for any $n_f < 6$ on $R^3 \times S^1$ — Unsal 2007
- various R^3 proposals for $n_f > 1$ — Cordova, Dumitrescu 2018; Anber, EP 2018; Ryttov, EP 2019; most recent lattice work: Athenodorou, Bennett, Bergner, Lucini 20

- other groups: 2-fold , 3-fold, 4-fold degeneracies on T^3

whose center is $Z_2, Z_2 \times Z_2$ $\quad Z_3$ $\quad Z_4$

- min. breaking with multi-fermion condensates (or bilinear)

What I told you:

- **the mixed anomaly between 0-form parity/chiral/ and 1-form center can be seen as an extension of the symmetry operator algebra on T^3 with twisted b.c.** (= 2-form background for the 1-form center symmetry)

- **these central extensions imply exact degeneracies between appropriate electric flux states on T^3**

Lingering questions:

how is tunneling at finite volume avoided?
(learn more about semiclassics?)

may be useful for lattice studies ($\theta = \pi$, especially)?

what happens in theories (e.g. $n_f = 4{,}5$ QCD(adj))
thought to flow to CFTs in R^3 limit?

do more general anomalies involving 0-form and 1-form symmetries have Hilbert space implications?