Finite Isospin Chiral Perturbation Theory in a Uniform Magnetic Field

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November 12, 2020

XXXII International Workshop on High Energy Physics - HOT PROBLEMS OF STRONG INTERACTIONS

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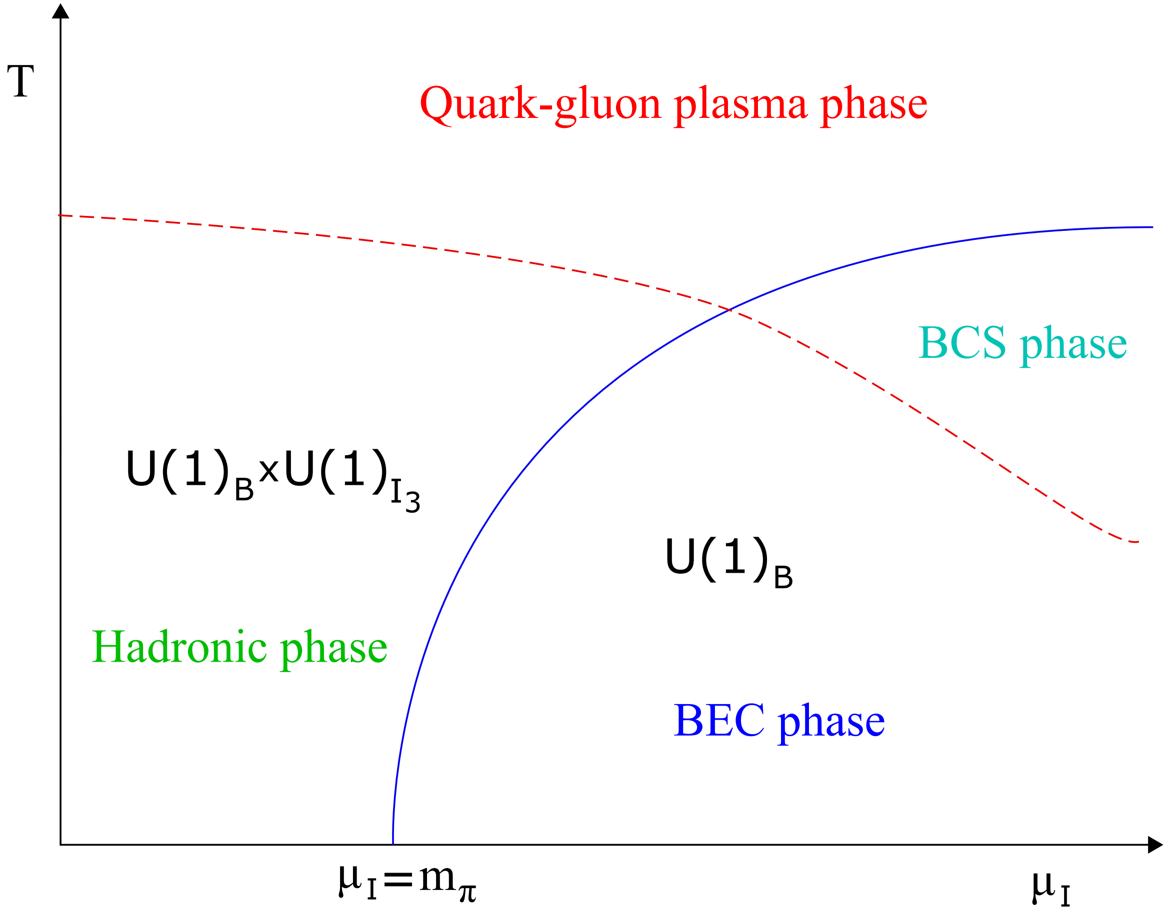
Outline

- Background and Motivation
  - Finite Isospin Phase Diagram
  - Chiral Perturbation Theory (ChPT): Brief Overview
- Nature of Superconductivity in ChPT: type-I and type-II
- Magnetic Vortex Lattice
  - Method of Success Approximation
  - Condensation Energy
- Summary

Schematic Phase Diagram



Adopted from Brandt, Endrodi, Schmalzbauer, Phys. Rev. D **97**, 054514 (2018)
PA, Andersen, Mojahed, arXiv: 2010.13655 (2020)

Chiral Perturbation Theory

- In the chiral limit, the QCD Lagrangian has an $SU(2)_L xSU(2)_R$ flavor symmetry
- The ground state breaks the symmetry down to the $SU(2)_{L+R}$, resulting in three pion degrees of freedom
- Since ChPT is an effective field theory, the observables are **model-independent**
  - The systematic corrections for observable (p_χ) is small if

$$\frac{p_\chi}{4\pi f_\pi} \ll 1$$

$$\mathcal{L}_{\chi\mathrm{PT}} = \frac{f_\pi^2}{4}\mathrm{Tr}(D^\mu \Sigma D_\mu \Sigma^\dagger) + \frac{m_\pi^2 f_\pi^2}{2}\mathrm{Tr}(\Sigma + \Sigma^\dagger)\ .$$

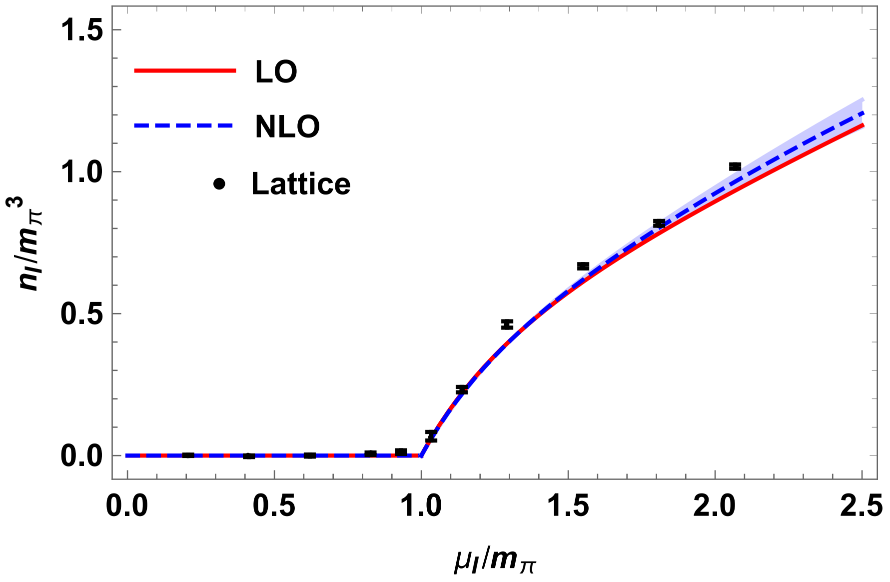
$$D^\mu \Sigma = \partial^\mu \Sigma + i\left[\mu_I \delta_{\mu 0} + qA^\mu, \Sigma\right]$$

$$\Sigma = \exp\left(i\frac{\phi}{f_\pi}\right)$$

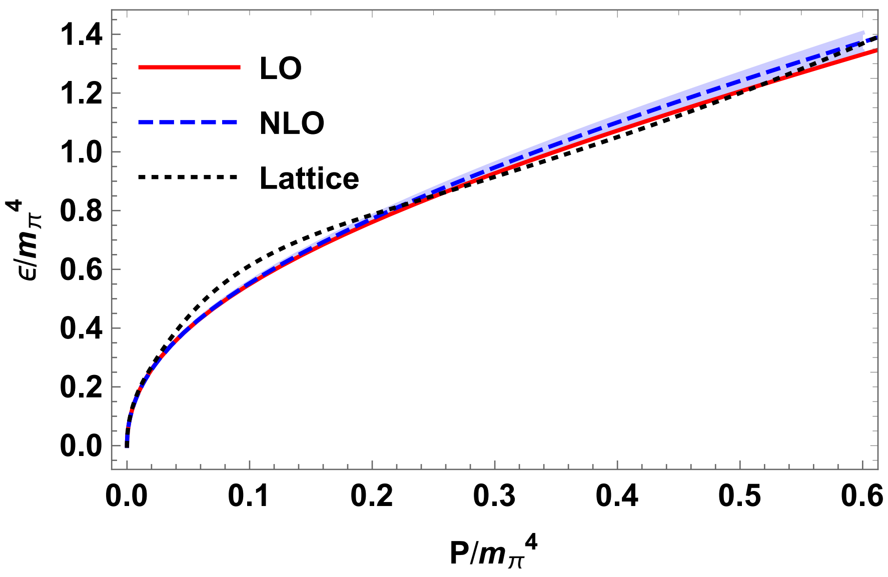
$$\phi = \phi_i \tau_i = \begin{pmatrix} \pi_0 & \sqrt{2}\pi_+ \\ \sqrt{2}\pi_- & -\pi_0 \end{pmatrix}\ ,$$

ChPT at next-to-leading-order

Isospin Density



Equation of State



PA, Andersen and Kneschke, Eur. J. Phys. C **79**, 874 (2019)
Brandt, et. al. Phys. Rev. D **98**, 094510 (2018)

ChPT in a magnetic field

- The isospin chemical potential is tuned to a value larger than the pion mass before the uniform magnetic field is turned on.
- In the thermodynamic limit, electrostatic forces between pions dominate the strong force
- In our analysis, we ignore the electrostatic energy associated with the condensed pions.

$$\mathcal{L}_{\chi\mathrm{PT}} = \frac{f_\pi^2}{4}\mathrm{Tr}(D^\mu \Sigma D_\mu \Sigma^\dagger) + \frac{m_\pi^2 f_\pi^2}{2}\mathrm{Tr}(\Sigma + \Sigma^\dagger) \ .$$

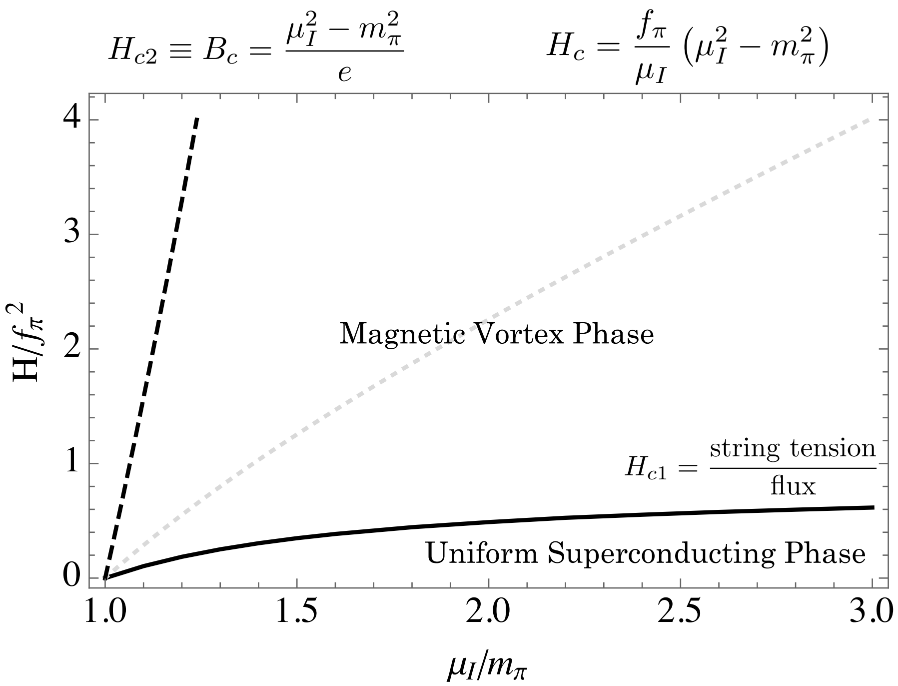
$$\mathcal{G} = \mathcal{H} - \vec{M} \cdot \vec{H}$$

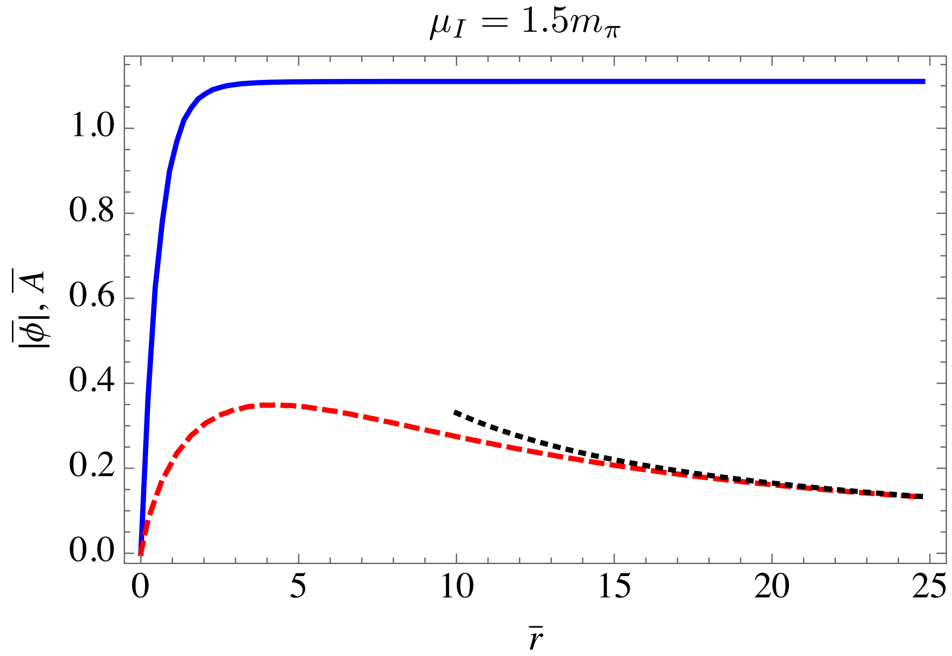
$$E_{\mathrm{electrostatic}} \sim e^2 n_I^2 R^5$$

$$E_{\chi\mathrm{PT}} \sim f_\pi^4 g\left(\frac{p_\chi}{f_\pi}\right) R^3$$

PA, Thomas D. Cohen, Julia Sakowitz, Phys. Rev. C **91**, 045202 (2015)

ChPT in a magnetic field

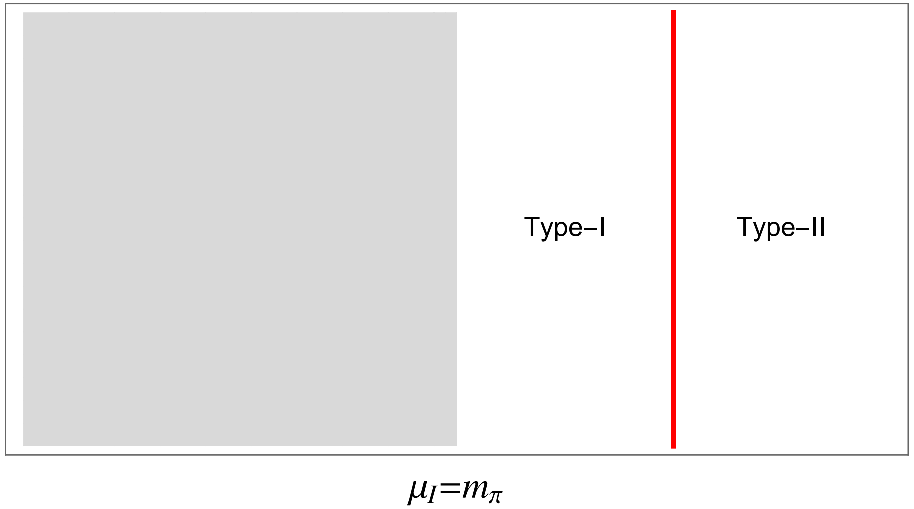


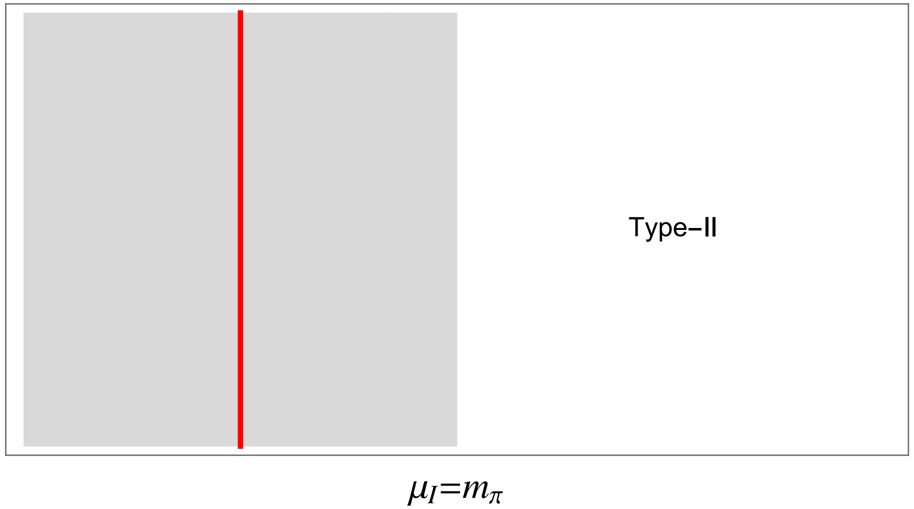


The units are dimensionless and normalized using the pion decay constant.

PA, Physics Letters B **790**, 211 (2019)

ChPT in a magnetic field





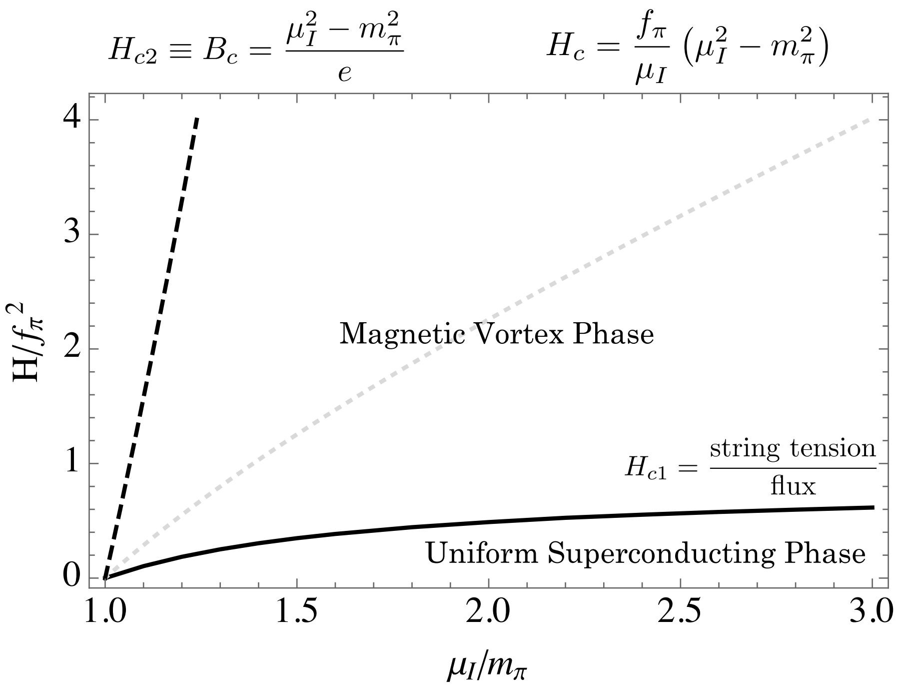
- Chiral perturbation theory exhibits both type-I and type-II superconductivity
- Superconductivity is type-II if $H_{c2} > H_c$

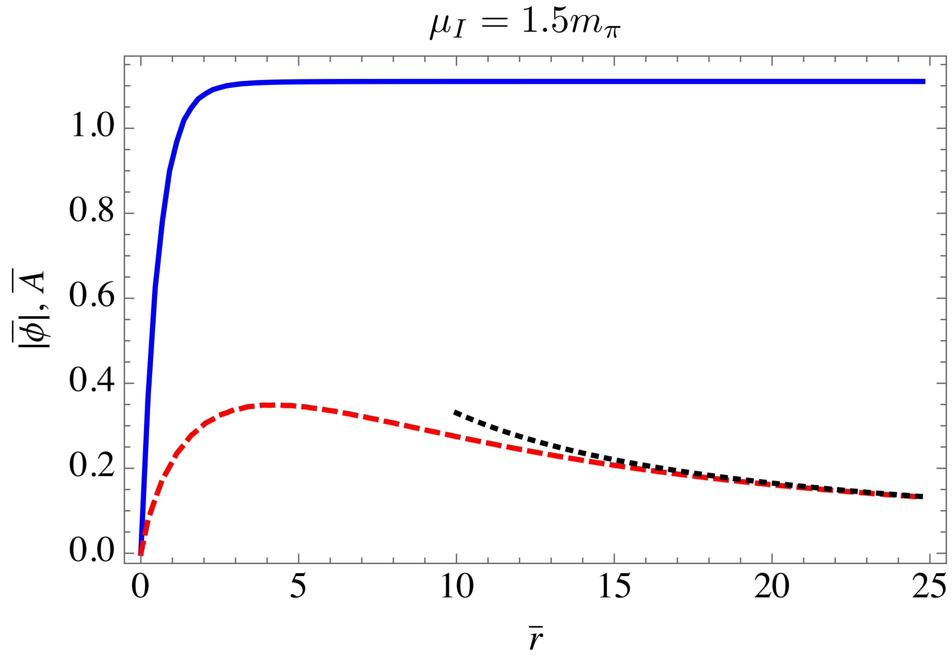
$$H_{c2} \equiv B_c = \frac{\mu_I^2 - m_\pi^2}{e}$$

$$H_c = \frac{f_\pi}{|\mu_I|}\left(\mu_I^2 - m_\pi^2\right)$$

  - i.e. $|\mu_I| > e f_\pi$

ChPT in a magnetic field





The units are dimensionless and normalized using the pion decay constant.

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Magnetic Vortex Condensation Energy

- We use the method of successive approximations.
- The condensation energy of the vortex lattice is
- The Hamiltonian density assuming ***neutral pions do not condense***
  - We revisit this assumption later
- The problem is different than the standard Abelian Higgs Model or Ginzburg-Landau Theory
  - Derivative Interactions
  - Neutral Pion Degrees of Freedom

$$\frac{B_c - B_{\text{ext}}}{B_c} \ll 1 \qquad H \equiv B_{\text{ext}}$$

$$\mathcal{E} \equiv \langle \mathcal{H} \rangle - \frac{1}{2} B_{\text{ext}}^2 \qquad \langle \mathcal{O} \rangle = \frac{1}{A_\perp} \int dxdy\; \mathcal{O}$$

$$\begin{aligned}\mathcal{H}_{4\pi} &= \frac{1}{2}B^2 + \frac{f_\pi^2}{4} D_i\pi_- D_i\pi_+ - \frac{4\mu_I^2 - m_\pi^2}{6f_\pi^2}(\pi_-\pi_+)^2 \\ &+ \frac{1}{3f_\pi^2}(D_i^\dagger \pi_-)(D_i\pi_+)\pi_-\pi_+ \\ &- \frac{1}{6f_\pi^2}\left[(D_i^\dagger\pi_-)^2\pi_+^2 + (D_i\pi_+)^2\pi_-^2\right] + \cdots\end{aligned}$$

$$D_\mu\pi_+ = (\partial_\mu + ieA_\mu)\pi_+$$

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Method of Successive Approximation

$$\frac{B_c - B_{\text{ext}}}{B_c} \ll 1$$

- In the method of successive approximation, the power counting scheme is

$$B = B_0 + \epsilon^2 \delta B \qquad z = x + iy,\ \bar{z} = x - iy$$

$$\pi_+ = \epsilon \tilde{\pi}_+ + \epsilon^3 \delta\pi_+ \qquad \partial = \frac{1}{2}(\partial_x - i\partial_y),\ \bar{\partial} = \frac{1}{2}(\partial_x + i\partial_y)$$

- The full equations of motion are

$$D_\mu D^\mu \pi_+ - (\mu_I^2 - m_\pi^2)\pi_+ + \frac{4\mu_I^2 - m_\pi^2}{3f_\pi^2}(\pi_-\pi_+)\pi_+$$

$$- \frac{1}{3f_\pi^2} D_\mu \left[(D^\mu \pi_+)\pi_-\pi_+\right] + \frac{1}{3f_\pi^2} D_\mu \left[(D^{\mu\dagger}\pi_-)\pi_+\pi_+\right]$$

$$+ \frac{1}{3f_\pi^2}(D_\mu^\dagger \pi_-)(D^\mu \pi_+)\pi_+ - \frac{1}{3f_\pi^2}(D_\mu \pi_+)(D^\mu \pi_+)\pi_- = 0$$

$$\partial_\mu F^{\mu\nu} = e\left[\pi_- D^\mu \pi_+ - (D^{\mu\dagger}\pi_-)\pi_+\right]\left[1 - \frac{2}{3f_\pi^2}\pi_-\pi_+\right]$$

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Method of Successive Approximation

$$\frac{B_c - B_{\text{ext}}}{B_c} \ll 1 \qquad L_{B_0} = \sqrt{\frac{2\pi}{B_0}}$$

- In the method of successive approximation, the power counting scheme is

$$B = B_0 + \epsilon^2 \delta B \qquad z = x + iy, \; \bar{z} = x - iy$$

$$\pi_+ = \epsilon \tilde{\pi}_+ + \epsilon^3 \delta\pi_+ \qquad \partial = \frac{1}{2}(\partial_x - i\partial_y), \; \bar{\partial} = \frac{1}{2}(\partial_x + i\partial_y)$$

- The leading order equation of motion for the charged pion is

$$\left(2\bar{\partial} + \frac{eB_0}{2} z\right) \tilde{\pi}_+ = 0 \qquad \tilde{\pi}_+ = \sum_{n=-\infty}^{\infty} C_n \phi_n(\nu, z, \bar{z})$$

$$\phi_n(\nu, z, \bar{z}) = e^{-\pi\nu^2 n^2 - \frac{\pi}{2L_{B_0}}(|z|^2 + z^2) + \frac{2\pi}{L_{B_0}}\nu n z}$$

- The sub-leading order equation of motion for B

$$\bar{\partial} B = \frac{j^2 - ij^1}{2} = -e\bar{\partial}(\tilde{\pi}_- \tilde{\pi}_+)\left[1 - \frac{2}{3f_\pi^2}\tilde{\pi}_- \tilde{\pi}_+\right]$$

$$B = B_{\text{ext}} - e\left(\tilde{\pi}_- \tilde{\pi}_+ - \langle \tilde{\pi}_- \tilde{\pi}_+ \rangle\right) + \frac{e}{3f_\pi^2}\left((\tilde{\pi}_- \tilde{\pi}_+)^2 - \langle (\tilde{\pi}_- \tilde{\pi}_+)^2 \rangle\right)$$

- The sub-leading order equation of motion for the charged pion gives

$$\langle (\tilde{\pi}_- \tilde{\pi}_+)^2 \rangle = \frac{e(B_c - B_{\text{ext}})\langle \tilde{\pi}_- \tilde{\pi}_+ \rangle - e\langle \tilde{\pi}_- \tilde{\pi}_+ \rangle^2}{\frac{2\mu_I^2 + m_\pi^2}{3f_\pi^2} - e^2} + \mathcal{O}((\tilde{\pi}_- \tilde{\pi}_+)^3)$$

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Condensation Energy

- In the final step, we convert derivative interactions into non-derivative ones using the eom
- Then we have the average Hamiltonian density, which can be minimized to get the final expression for the condensation energy
- Abrikosov ratio determines the lattice structure

$$\langle (D_i^\dagger \tilde{\pi}_-)(D_i \tilde{\pi}_+)\tilde{\pi}_- \tilde{\pi}_+ \rangle = (\mu_I^2 - m_\pi^2)\langle (\tilde{\pi}_- \tilde{\pi}_+)^2 \rangle + \mathcal{O}((\tilde{\pi}_- \tilde{\pi}_+)^3)$$

$$\langle (D_i^\dagger \tilde{\pi}_-)^2 \tilde{\pi}_+^2 + (D_i \tilde{\pi}_+)^2 \tilde{\pi}_-^2 \rangle = \mathcal{O}((\tilde{\pi}_- \tilde{\pi}_+)^3)$$

$$\langle \mathcal{H} \rangle = \frac{1}{2} B_{\text{ext}}^2 - e(B_c - B_{\text{ext}})\langle \tilde{\pi}_- \tilde{\pi}_+ \rangle + \frac{e^2}{2}\langle \tilde{\pi}_- \tilde{\pi}_+ \rangle^2 + \frac{1}{2}\left(\frac{2\mu_I^2 + m_\pi^2}{3 f_\pi^2} - e^2 \right) \langle (\tilde{\pi}_- \tilde{\pi}_+)^2 \rangle + \mathcal{O}((\tilde{\pi}_- \tilde{\pi}_+)^3)$$

$$\beta_A \equiv \frac{\langle (\tilde{\pi}_- \tilde{\pi}_+)^2 \rangle}{\langle \tilde{\pi}_- \tilde{\pi}_+ \rangle^2} = 1.159595\ldots$$

$$\mathcal{E} = -\frac{e^2 (B_c - B_{\text{ext}})^2}{2e^2 + \beta_A \left(\frac{2\mu_I^2 + m_\pi^2}{3 f_\pi^2} - e^2 \right)} + \mathcal{O}\left((B_c - B_{\text{ext}})^4 \right)$$

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Vortex Lattice Solution

- The Abrikosov parameter that minimizes the condensation energy requires a triangular (hexagonal) lattice

$$\beta_A \equiv \frac{\langle(\tilde{\pi}_-\tilde{\pi}_+)^2\rangle}{\langle\tilde{\pi}_-\tilde{\pi}_+\rangle^2} = 1.159595\ldots$$

- The lattice vectors are

$$\vec{d}_1 = \frac{L_B}{\nu}(0,1)$$

$$\vec{d}_2 = \frac{L_B}{\nu}\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$$

$$\tilde{\pi}_+ = \frac{C}{2\nu}\left[e^{-\frac{\pi y(-ix+y)}{L_B^2}}\theta_3\left(\frac{-\pi(x+iy)}{2L_B\nu}, e^{-\frac{\pi}{4\nu^2}}\right)\right. \\ \left. + ie^{-\frac{\pi y(-ix+y)}{L_B^2}}\theta_3\left(\frac{-\pi(x+iy-L_B\nu)}{2L_B\nu}, e^{-\frac{\pi}{4\nu^2}}\right)\right]$$

$$\nu = \frac{\sqrt[4]{3}}{\sqrt{2}},\ \ L_B = \sqrt{\frac{2\pi}{B_{\text{ext}}}}$$

$$|C| = \sqrt{\frac{3^{\frac{1}{4}}e(B_c - B_{\text{ext}})}{\beta_A\left(\frac{2\mu_I^2+m_\pi^2}{3f_\pi^2} - e^2\right) + e^2}}\ , \text{ for } B_{\text{ext}} \leq B_c$$

$$|C| = 0\ , \text{ for } B_{\text{ext}} \geq B_c$$

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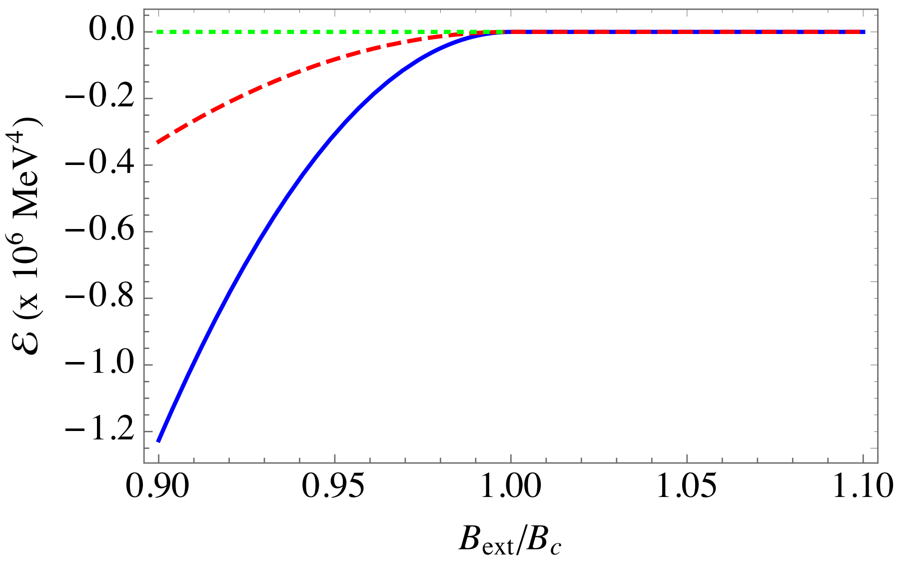
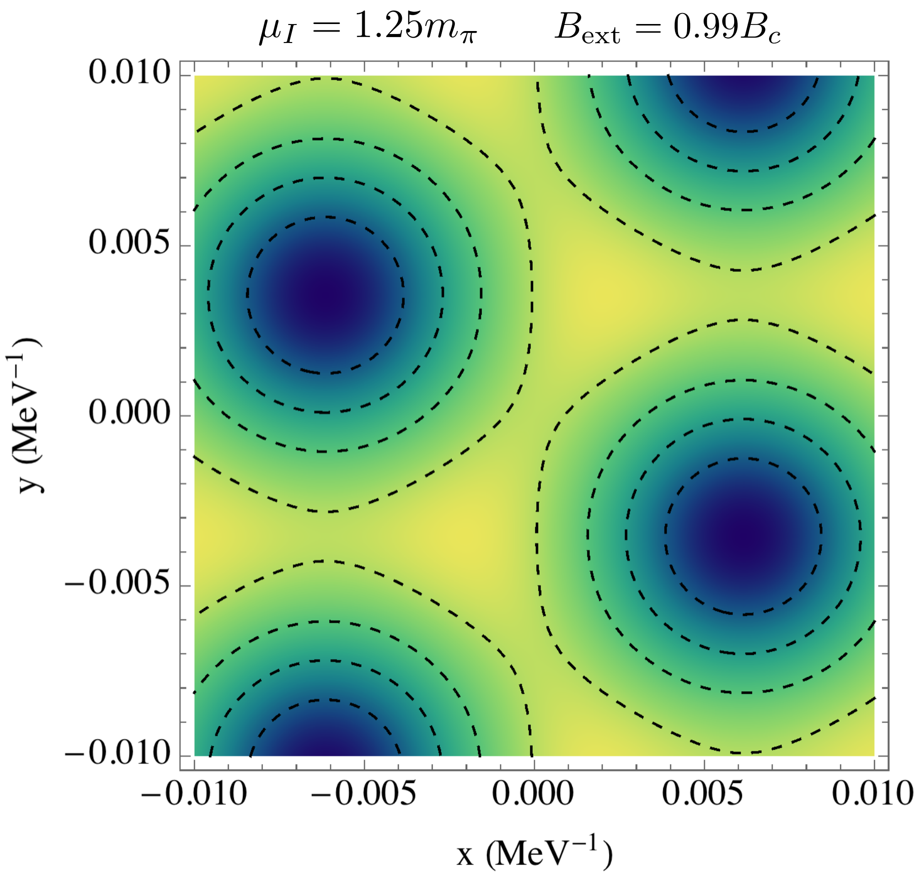
Condensation Energy and Vortex Lattice



$\mu_I = m_\pi,\ 1.25m_\pi,\ 1.50m_\pi$



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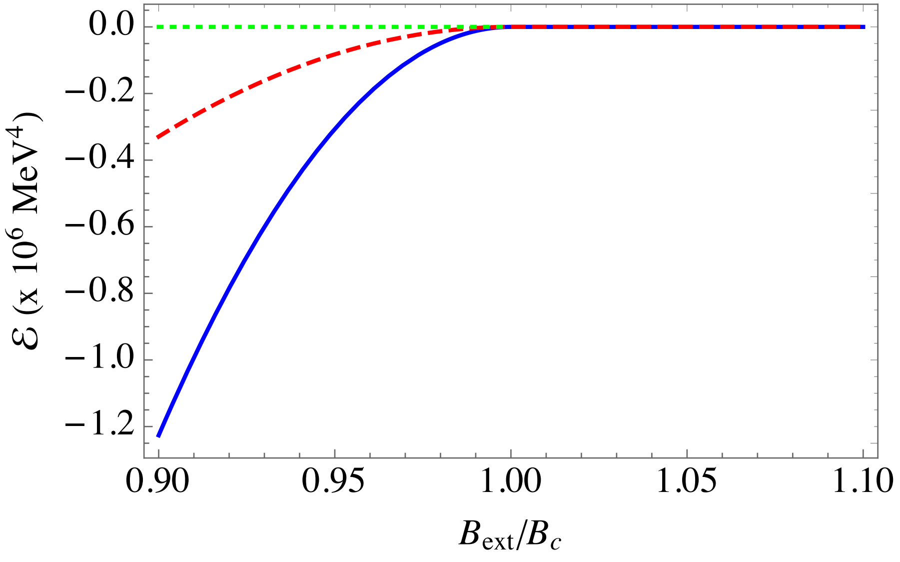
Condensation Energy and Vortex Lattice



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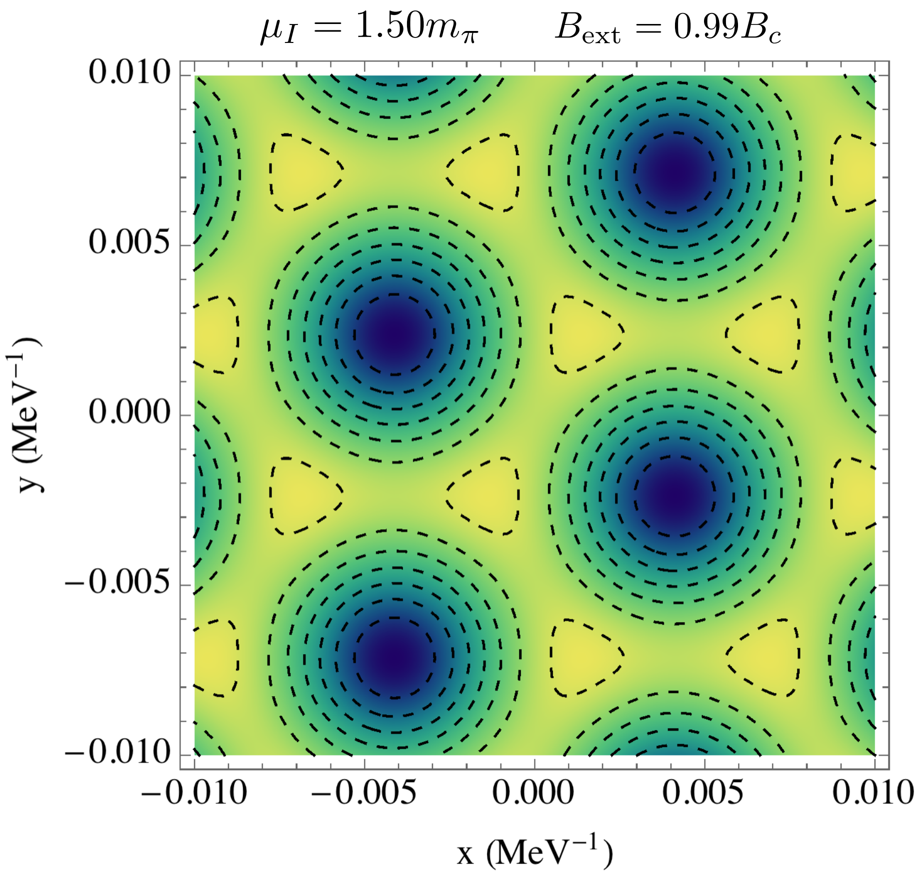
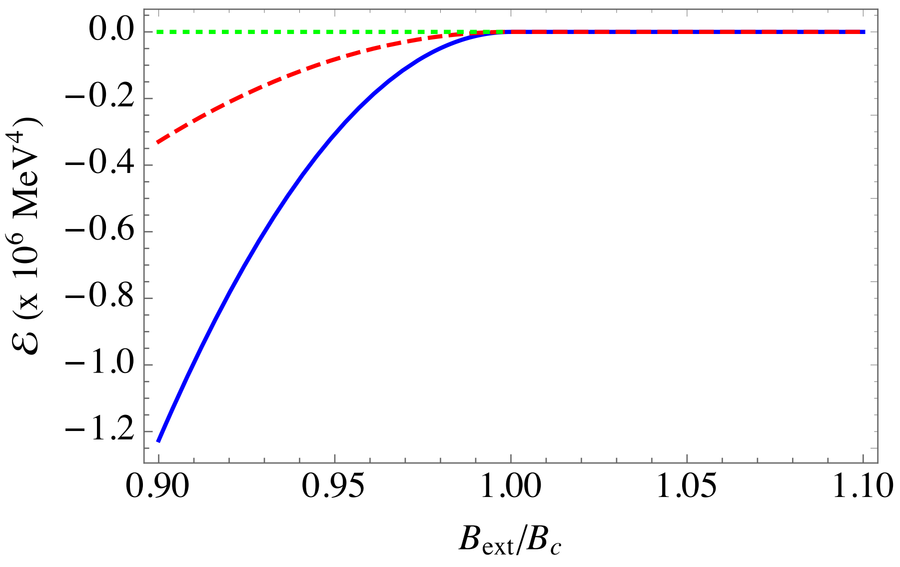
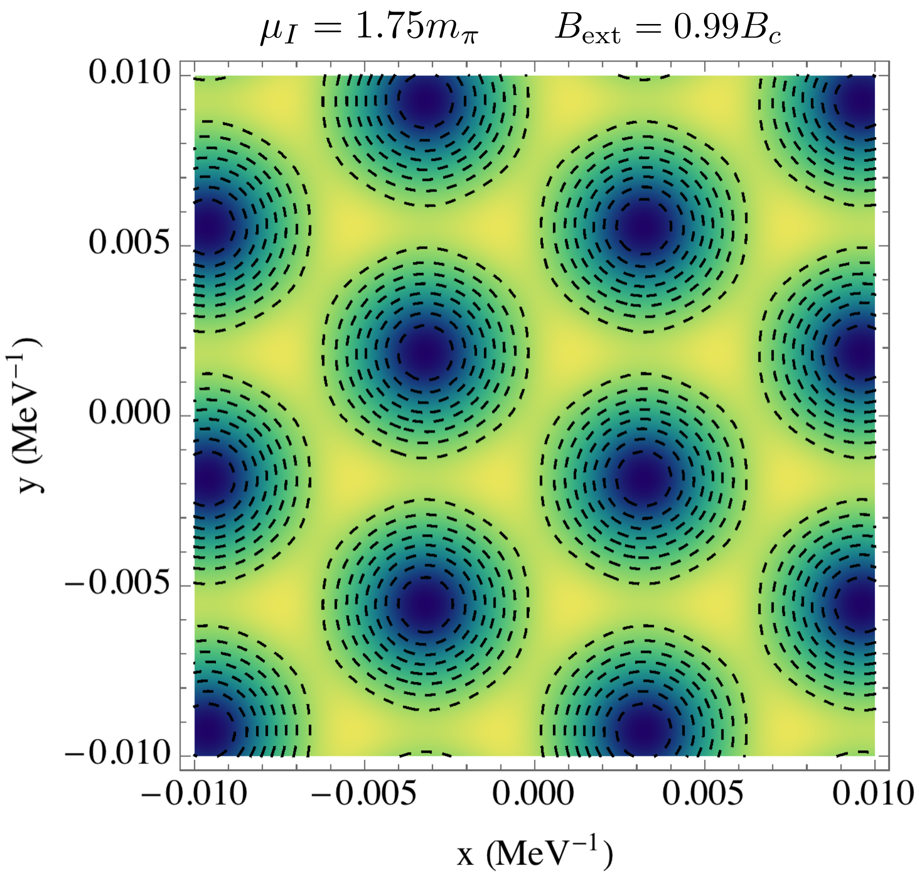


PA, Physics Letters B **790**, 211 (2019)

Condensation Energy and Vortex Lattice



$\mu_I = m_\pi,\ 1.25m_\pi,\ 1.50m_\pi$



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Neutral Pion Fluctuations

- Consider fluctuations of the neutral pions around the vortex lattice solutions

$$\pi_0(t,\vec{x}) = \sum_n e^{iE_n t}\tilde{\pi}_0(\vec{x})$$

- The linearized "Schrodinger" equation is

$$\left(\partial_i\partial^i + m_\pi^2 + \frac{m_\pi^2}{f_\pi^2}\pi_-\pi_+\right)\tilde{\pi}_0(\vec{x}) = E_n^2\tilde{\pi}_0$$

- The dispersion relation is

$$E_0^2 = p_z^2 + m_\pi^2 + \delta$$

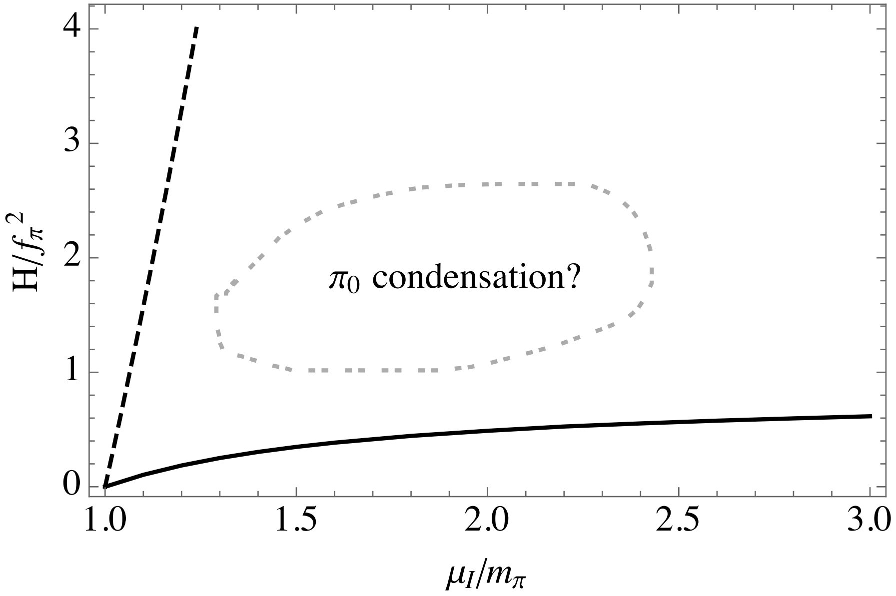
- The potential is positive definite

$$\delta \geq 0$$

S. Coleman, Aspects of Symmetry: Selected Erice Lectures (1985)

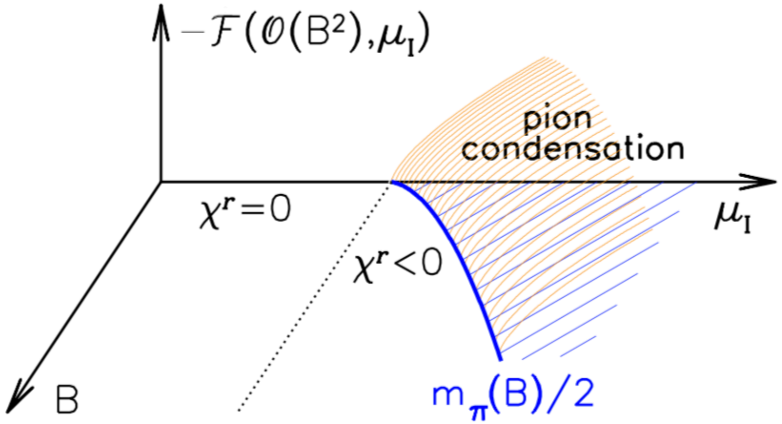
Summary

- **Neutral pion condensation** isn't ruled out away from the phase transition lines
- Pions at finite isospin chemical potential (in QCD) behave as type-II superconductors for physical parameters
  - This result is **model-independent**
- The vortex lattice is triangular (hexagonal) and the condensation energy is negative near the upper



Summary

- Lattice QCD observes a diamagnetic phase and a different phase diagram
- Pions condense when the chemical potential is equal to the **magnetic mass** of the charged pion, not the zero field mass
- **“Ordering of limits”** is different: in our calculation, chemical potential is first increased to go into the superfluid phase and then the external magnetic field is turned on



Endrodi, Phys. Rev. D **90**, 094501 (2014)