Rotating Relativistic Matter and Angular Momenta

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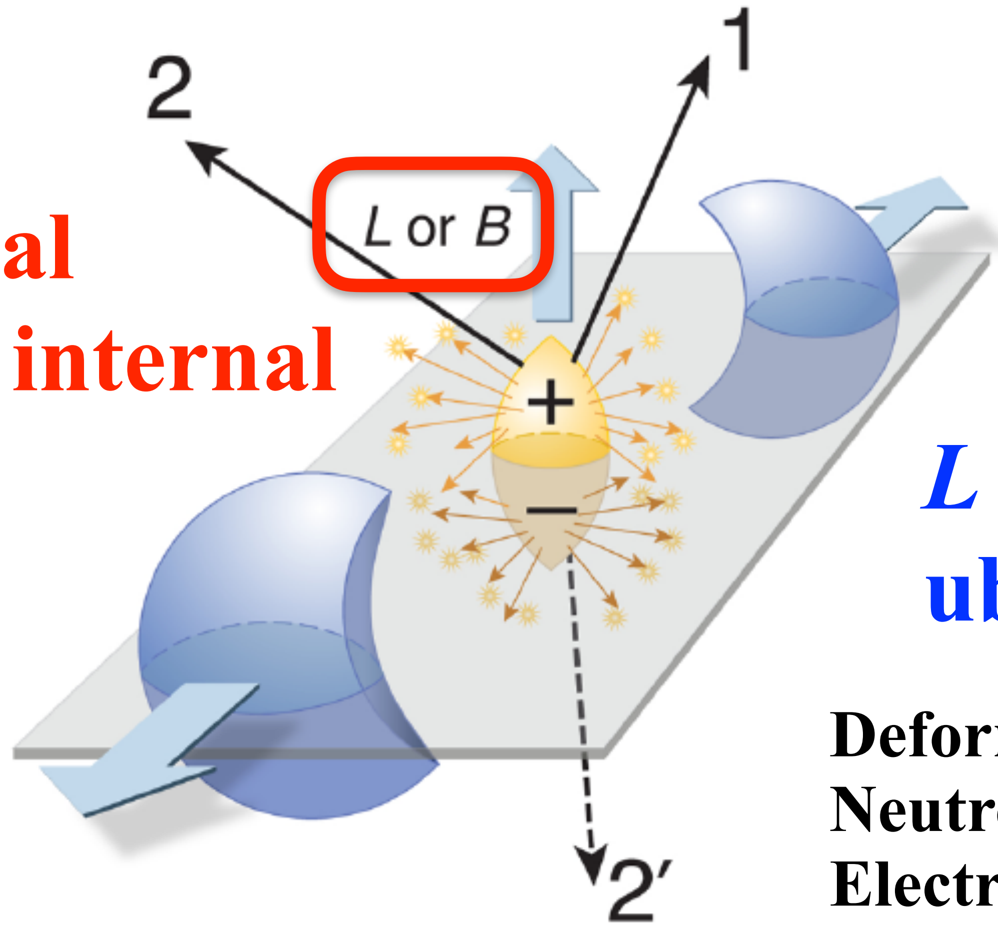
The University of Tokyo

— XXXII International Workshop on High Energy Physics
"Hot problems of Strong Interactions" —

Physics Motivation

External and/or internal



***L* or *B* is ubiquitous**

Deformed Nuclei
Neutron Stars
Electron Vortices …

Topics to be Covered

Rotation + Matter = Current : how to understand

- □ Puzzle and Answer
- □ Polarization surrounding a Blackhole

Technical Difficulties

- □ Necessity to impose a boundary condition (**Mameda**)

Electron Vortices

- □ Mode decomposed analysis — Chiral Magnetic Effect
- □ Unphysical divergences

Angular Momenta of Magnetic Vortices

- □ Theoretically and experimentally ideal objects to treat the Orbital Angular Momenta

ROTATION + MATTER

Rotation-induced Current

$$\boldsymbol{J}_5 = (\text{matter with } T^2, \mu^2, \ldots)\, \boldsymbol{\omega}$$

Rotation can be introduced by the coordinate transformation:

$$\begin{aligned} x &\to x \cos \omega t - y \sin \omega t \\ y &\to y \cos \omega t + x \sin \omega t \end{aligned}$$

If J_5 was zero at ω=0, how is the current induced by rotation?

Vector (tensor) transformations do not change zero to nonzero.

Rotation-induced Current

Flachi-Fukushima (2018)

$$j^\mu_{\rm CS} = \frac{1}{96\pi^2}\epsilon^{\mu\nu\rho\lambda}\underline{\Gamma^\alpha{}_{\nu\beta}}\Big(\partial_\rho\Gamma^\beta{}_{\alpha\lambda} + \frac{2}{3}\Gamma^\beta{}_{\rho\sigma}\Gamma^\sigma{}_{\alpha\lambda}\Big)$$

This is not a tensor

Rotation induces an extra term:

$$\delta\Gamma^x{}_{0y} = -\delta\Gamma^y{}_{0x} = \omega$$

$$\boxed{j^\mu_{\rm CS} = \frac{\omega}{48\pi^2}\big(R^0{}_{x0x} + R^0{}_{y0y} - R^x{}_{yxy} - R^y{}_{xyx}\big)}$$

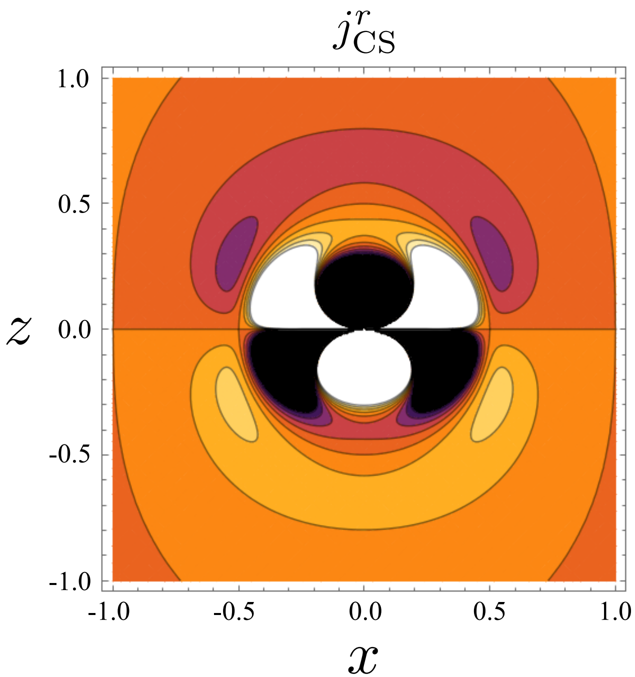
Can be interpreted as a physical current for a given ω

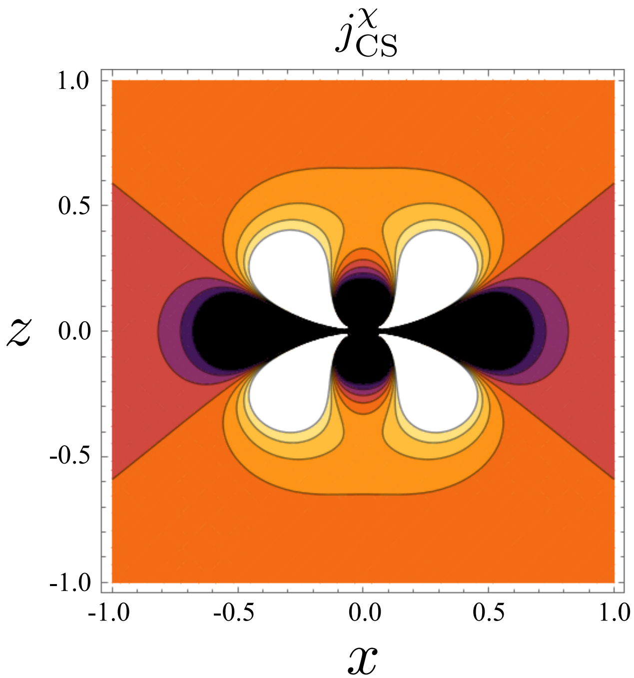
Rotation-induced Current

Axial-current (polarization) surrounding the Kerr metric

Flachi-Fukushima (2018)

$$(t,\, r,\, \chi = \cos\theta,\, \phi)$$





Local polarization? (Implication to QGP physics?)

TECHNICAL DIFFICULTIES

Theoretical Formulation

$$\left[i\gamma^{\mu}(D_{\mu}+\Gamma_{\mu})-m\right]\psi=0$$

Chen-Fukushima-Huang-
-Mameda (2016)

$$\Gamma_{\mu}=-\frac{i}{4}\omega_{\mu ij}\sigma^{ij}$$

$$\omega_{\mu ij}=g_{\alpha\beta}e_i^{\alpha}(\partial_{\mu}e_j^{\beta}+\Gamma_{\nu\mu}^{\beta}e_j^{\mu})$$

$$\sigma^{ij}=\frac{i}{4}[\gamma^i,\gamma^j]$$

$$e_0^t=e_1^x=e_2^y=e_3^z=1 \quad e_0^x=y\omega \quad e_0^y=-x\omega$$

$$\boxed{H_{\mathrm{rot}}=H-\omega J_z}$$

Rotation ~ Density

Theoretical Formulation

Vilenkin (1980)

Here,

$$n_{\omega m} = (e^{\beta(\omega - m\Omega)} - 1)^{-1} \tag{23}$$

is the Bose-Einstein distribution for a rotating system,[5] $\tau = \tau_1 - \tau_2$, the upper and lower lines in

[5]Note that $n_{\omega m}$ has a
larity, however, i
cannot have size g
city at the boundary would exceed the velocity of light), and in a finite system the energy is quantized in such a way that ω is always greater than $m\Omega$. (There are some exceptions in which the field has exponentially growing modes. See Ref. 6.) As an example, consider an infinite cylinder of radius R rotating around its axis. Requiring that Ψ vanishes at the boundary, we find the energy levels $\omega_{nmp} = (p^2 + \mu^2 + \xi_{mn}{}^2 R^{-2})^{1/2}$, where ξ_{mn} is the nth root of $J_m(x)$. It can be shown (Ref. 7) that $\xi_{mn} > m$. Thus, $\omega_{nmp} > \xi_{mn} R^{-1} > m\Omega$. In the present paper we shall assume that the lowest energy modes are unimportant and thus the infinite-space solutions (17) can be used.

Theoretical Formulation

Vilenkin (1980)

Here,

(23)

rotating

r lines in

[5] Note that $n_{\omega m}$ has a singularity at $\omega = m\Omega$. This singularity, however, is unphysical. A rotating system cannot have size greater than Ω^{-1} (otherwise the velocity at the boundary would exceed the velocity of light), and in a finite system the energy is quantized in such a way that ω is always greater than $m\Omega$. (There are some exceptions in which the field has exponentially growing modes. See Ref. 6.) As an example, consider an infinite cylinder of radius R rotating around its axis. Requiring that Ψ vanishes at the boundary, we find the energy levels $\omega_{nmp} = (p^2 + \mu^2 + \xi_{mn}{}^2 R^{-2})^{1/2}$, where ξ_{mn} is the nth root of $J_m(x)$. It can be shown (Ref. 7) that $\xi_{mn} > m$. Thus, $\omega_{nmp} > \xi_{mn} R^{-1} > m\Omega$. In the present paper we shall assume that the lowest energy modes are unimportant and thus the infinite-space solutions (17) can be used.

Theoretical Formulation

Is it really possible to change the QCD vacuum just by rotation ???

The answer is negative (nontrivial for fermions)

Ebihara-Fukushima-Mameda (2017)

Causality — **System size should be finite ~ *R***

$\omega R < 1$ — **Energy dispersion should be gapped ~ *J*/*R***

Induced chemical potential ~ *ωJ*

Gap is always bigger than the energy shift

Where is difficulty?

There are lots of calculations on the phase diagram under magnetic fields.

There are already some calculations on the phase diagram under rotations.

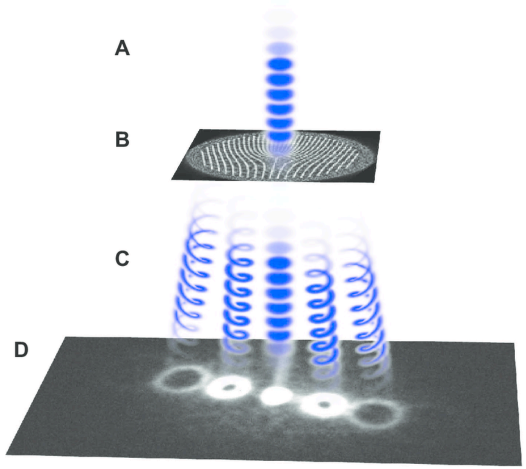
Why no calculations involving both???

Naive calculations contaminated by unphysical divergences…

ELECTRON VORTICES

Mode Solutions

Fukushima-Shimazaki-Wang (2020)



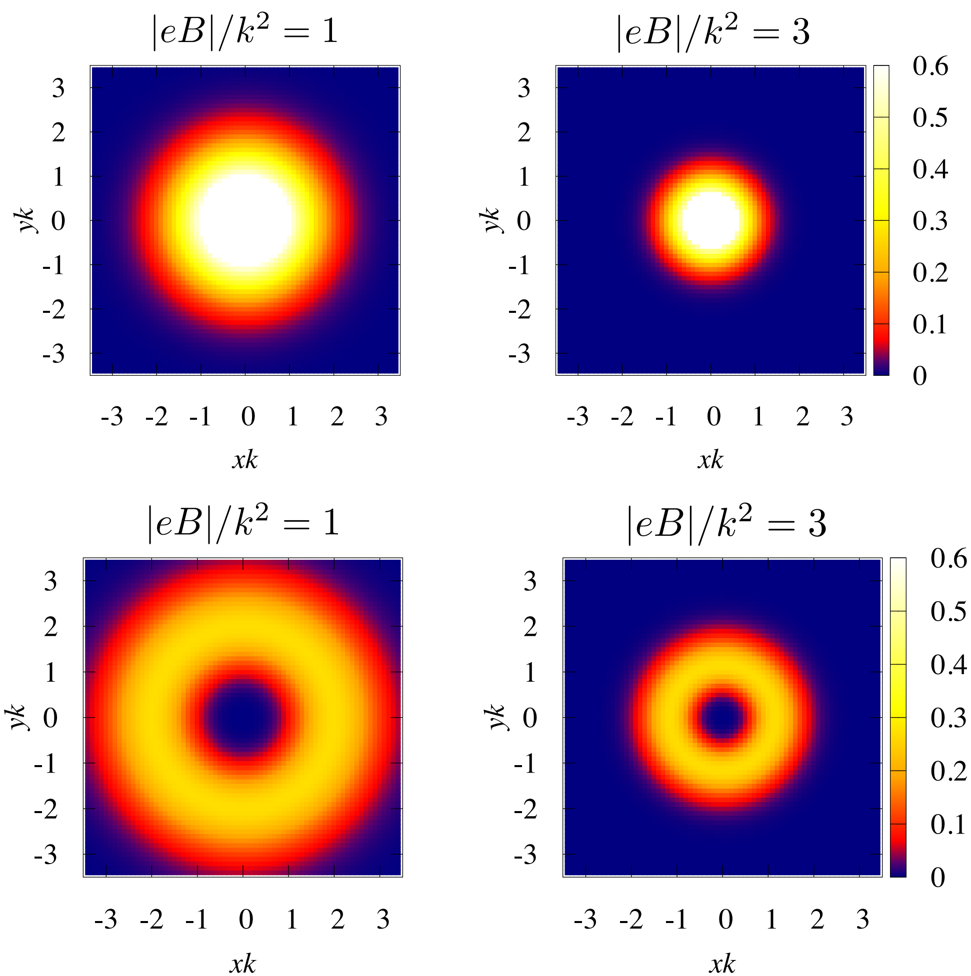
$$u^{(\uparrow)}_{n,l,k}(r,\varphi,z,t)$$

$$= \frac{e^{-i\varepsilon^{(\uparrow)}_{n,l,k}t+ikz}}{\sqrt{\varepsilon^{(\uparrow)}_{n,l,k}+m}} \begin{pmatrix} (\varepsilon^{(\uparrow)}_{n,l,k}+m)\Phi_{n,l}(\chi^2,\varphi) \\ 0 \\ k\Phi_{n,l}(\chi^2,\varphi) \\ i\sqrt{|e|B(2n+|l|+l+2)}\,\Phi_{n,l+1}(\chi^2,\varphi) \end{pmatrix}$$

$$\Phi_{n,l}(\chi^2,\varphi) = \sqrt{\frac{n!}{(n+|l|)!}}\, e^{-\frac{1}{2}\chi^2}\, \chi^{|l|} L^{|l|}_n(\chi^2)\, e^{il\varphi}$$

Polarized wave-function with angular momentum *l*

Mode Solutions



***s*-wave density**

***d*-wave density**

Mode Solutions

$$\rho = \sum\!\!\!\!\!\!\!\int_{n,l,k} \left(\rho^{(\uparrow)}_{n,l,k} + \rho^{(\downarrow)}_{n,l,k} + \bar{\rho}^{(\uparrow)}_{n,l,k} + \bar{\rho}^{(\downarrow)}_{n,l,k}\right)$$

$$\rho_5 = \sum\!\!\!\!\!\!\!\int_{n,l,k} \left(\rho^{(\uparrow)}_{5\,n,l,k} + \rho^{(\downarrow)}_{5\,n,l,k} + \bar{\rho}^{(\uparrow)}_{5\,n,l,k} + \bar{\rho}^{(\downarrow)}_{5\,n,l,k}\right)$$

In the same way, the vector and the axial-vector currents are also mode decomposed… then…

Mode Solutions

Fukushima-Shimazaki-Wang (2020)

$$j^{z(\uparrow)}_{n,l,k} = \rho^{(\uparrow)}_{5\,n,l,k}\,, \qquad j^{z(\downarrow)}_{n,l,k} = -\rho^{(\downarrow)}_{5\,n,l,k}$$

$$\bar{j}^{z(\uparrow)}_{n,l,k} = -\bar{\rho}^{(\uparrow)}_{5\,n,l,k}\,, \qquad \bar{j}^{z(\downarrow)}_{n,l,k} = \bar{\rho}^{(\downarrow)}_{5\,n,l,k}$$

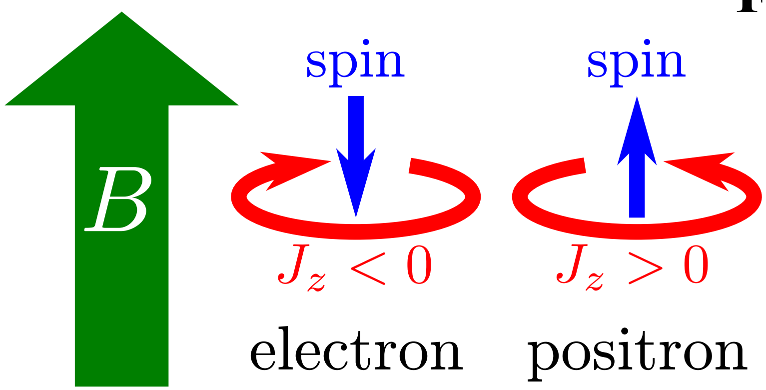
These relations hold mode-by-mode.
No such relations for the axial-vector current!

This is the mode-decomposed origin of CME.

Cannot be seen without mode decomposition.

Applications

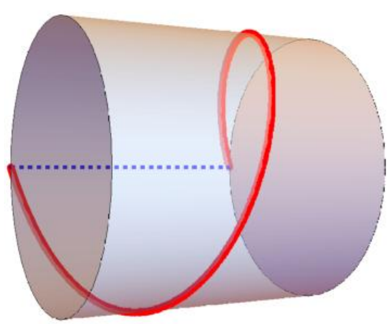
Fukushima-Shimazaki-Wang (2020)

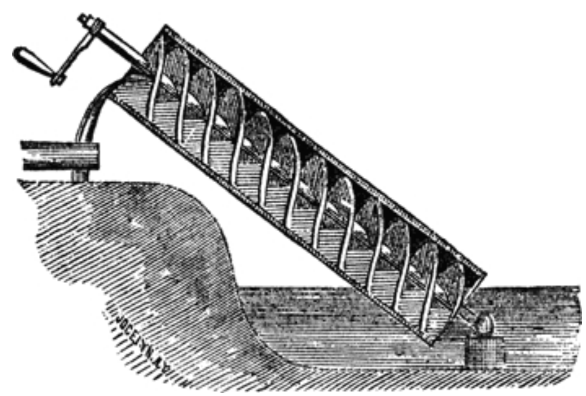


$$\rho_{\mathrm{spin}} = -\frac{\omega|eB|}{4\pi^2}$$

Hattori-Yin (2016)

Mechanism ~ Thouless pumping (Floquet theory)





Applications

Fukushima-Shimazaki-Wang (2020)

$$\rho = -\left(\frac{1}{S_\perp}\frac{2\pi}{|eB|}\right)\frac{|eB|}{2\pi}\sum_{J_z>0}^{S_\perp|eB|/(2\pi)}\int\frac{dk}{2\pi}\,\theta(\omega J_z - |k|)$$

$$= -\frac{\omega}{\pi S_\perp}\sum_{J_z}^{S_\perp|eB|/(2\pi)}\left(l+\frac{1}{2}\right)$$

$$= -\frac{\omega|eB|}{4\pi^2} + \underline{(\text{orbital part})}\,.$$

Thermodynamically unstable!?

Finite-size effects must be considered! (Talk by Mameda)

Applications

Fukushima-Shimazaki-Wang (2020)

Axial-vector current in the LLL approximation:

$$j^z_{5,\mathrm{LLL}} = \frac{1}{S_\perp}\Bigg[-\sum_{J_z<0}^{-S_\perp|eB|/(2\pi)} \int \frac{dk}{2\pi}\theta(\omega J_z + \mu - |k|) + \sum_{J_z>0}^{S_\perp|eB|/(2\pi)} \int \frac{dk}{2\pi}\theta(\omega J_z - \mu - |k|)\Bigg]$$

$$= \Big(\omega - 2\mu\Big)\frac{|eB|}{4\pi^2} + \underline{(\text{orbital part})}$$

Unphysical

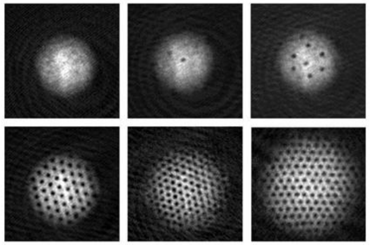
Finite-size effects must be considered! (Talk by Mameda)

MAGNETIC VORTICES

Magnetic Vortices

Fukushima-Hidaka-Yee: 2011.xxxxx

Angular momentum of superfluid vortex is topological, but that of magnetic vortex is *not*.



dimensionless static potential ($\sim E$)

$$(1-a)\Psi + \frac{1}{m_H^2}(\boldsymbol{\nabla} - ien_e\boldsymbol{A})^2\Psi - |\Psi|^2\Psi = 0$$

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{A}) + m_V^2\left[\boldsymbol{A}|\Psi|^2 - \frac{i}{2en_e}\left(\Psi\boldsymbol{\nabla}\Psi^\dagger - \Psi^\dagger\boldsymbol{\nabla}\Psi\right)\right] = 0$$

$$\boldsymbol{\nabla}^2 a + 2m^2\frac{m_V^2}{m_H^2}(|\Psi|^2 + \tilde{q}) = 0$$

electric charge density

Magnetic Vortices

Fukushima-Hidaka-Yee: 2011.xxxxx

$$(1-a)\Psi + \frac{1}{m_H^2}(\boldsymbol{\nabla} - ien_e\boldsymbol{A})^2\Psi - |\Psi|^2\Psi = 0$$

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{A}) + m_V^2\left[\boldsymbol{A}|\Psi|^2 - \frac{i}{2en_e}\left(\Psi\boldsymbol{\nabla}\Psi^\dagger - \Psi^\dagger\boldsymbol{\nabla}\Psi\right)\right] = 0$$

$$\boldsymbol{\nabla}^2 a + 2m^2\frac{m_V^2}{m_H^2}(|\Psi|^2 + \tilde{q}) = 0$$

Global Neutrality

$$\boxed{\int_{\boldsymbol{x}}\tilde{q} = -\int_{\boldsymbol{x}} f^2}$$

Vortex Solution

$$\Psi = f(r)\,e^{i\nu\varphi}\,, \qquad \boxed{a = a(r)}\,, \qquad A^i = -\frac{\nu}{en_e}\,\varepsilon^{ij}\,\frac{x^j}{r^2}\,[1-h(r)]$$

Electric fields from local charge density

Magnetic Vortices

Fukushima-Hidaka-Yee: 2011.xxxxx

$$L_z^{\text{can}} = \int_{\boldsymbol{x}} \psi^\dagger \Big(\frac{\hbar}{i}\,\partial_\varphi\Big)\psi = \nu(2\pi\hbar)\frac{\mu}{g}\int_0^R dr\, r\, f^2(r) = \nu\hbar\, N$$

Canonical angular momentum looks a quantized AM of superfluid vortex…?

$$L_z^{\text{kin}} = \int_{\boldsymbol{x}} \psi^\dagger \Big(\frac{\hbar}{i}\,D_\varphi\Big)\psi = \nu(2\pi\hbar)\frac{\mu}{g}\int_0^R dr\, r\, h(r)\, f^2(r)$$

Belinfante gives a smaller value…?

Magnetic Vortices

Fukushima-Hidaka-Yee: 2011.xxxxx

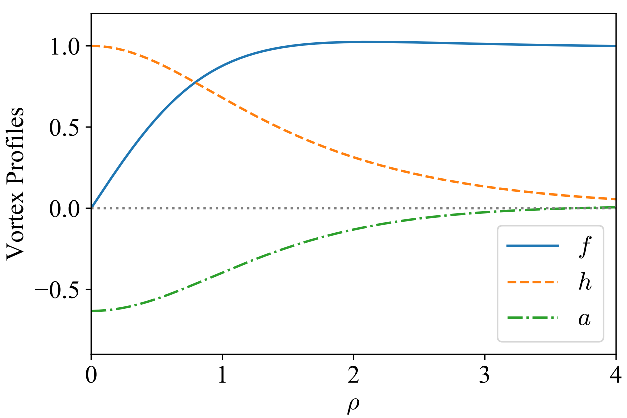
$$L_z^{\mathrm{kin}} = \int_{\boldsymbol{x}} \psi^\dagger \Big(\frac{\hbar}{i} D_\varphi\Big)\psi = \nu(2\pi\hbar)\frac{\mu}{g}\int_0^R dr\, r\, h(r)\, f^2(r)$$

$$L_z^{\mathrm{EM}} = \int_{\boldsymbol{x}} \big[\boldsymbol{x}\times(\boldsymbol{E}\times\boldsymbol{B})\big]_z$$

$$= -\nu(2\pi\hbar)\frac{\mu}{g}\int_0^R dr\, r\, h(r)\, \big[f^2(r) + \underline{\tilde{q}(r)}\big]$$

Cancel out

Finite AM remains from the background charge needed for neutrality



Magnetic Vortices

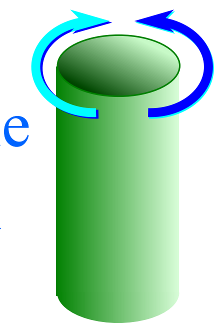
Fukushima-Hidaka-Yee: 2011.xxxxx

Relativistic Nielsen-Olesen vortices

Vortices exist in the vacuum (zero chemical potential)

Neutral Nielsen-Olesen Vortex

Antiparticle
Antivortex



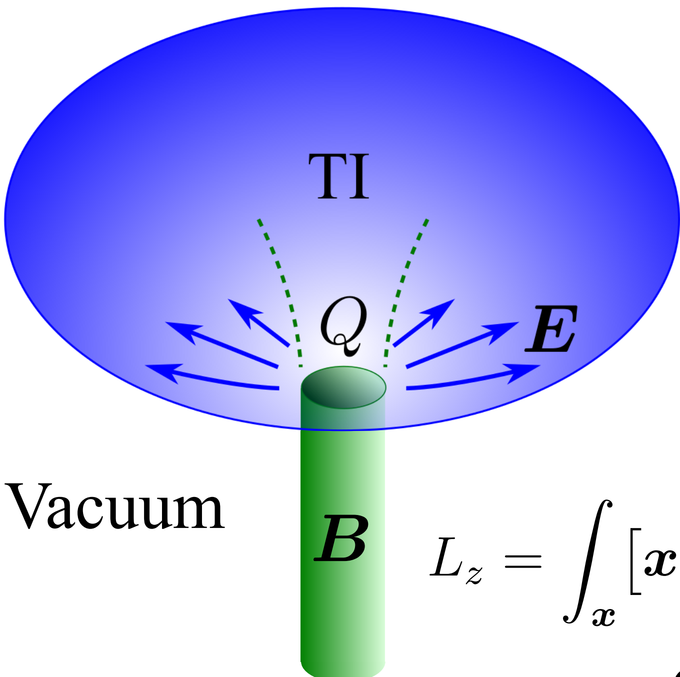
Particle
Vortex

Relativistic vortices do not carry any angular momentum.

Magnetic Vortices

Fukushima-Hidaka-Yee: 2011.xxxxx

Topological Insulator



$$j^\mu = -\frac{e^2}{8\pi\hbar}\,\epsilon^{\mu\nu\alpha}F_{\nu\alpha}$$

$$\frac{e^2}{16\pi^2\hbar}\Phi_0^2 \qquad -\frac{e^2}{8\pi^2\hbar}\Phi_0^2$$

$$L_z = \int_{\boldsymbol{x}}\big[\boldsymbol{x}\times(\boldsymbol{E}\times\boldsymbol{B})\big]_z = \int_{\boldsymbol{x}}(\boldsymbol{\nabla}\cdot\boldsymbol{E})\,A_\varphi - \int A_\varphi(\boldsymbol{E}\cdot d\boldsymbol{S})$$

$$= -\frac{e^2}{16\pi^2\hbar}\Phi_0^2$$

Nogueira-Nussinov-
-van den Brink (2018)

Classification

Fukushima-Hidaka-Yee: 2011.xxxxx

Magnetic vortices carry a finite angular momentum

Class Ia (spinful vortices)

Class Ib (topological vortices)

Magnetic vortices carry zero angular momentum

Class II (spinless vortices)

Vortices are twisted with "intrinsic" angular momentum

Class III (exotic vortices) [Abraham's vortex]

Summary

- **Rotation + *B* makes rich phase structures!**
  - □ Phenomenologically important
- **Finite size systems with boundary are crucial!**
  - □ Unphysical divergences cured by finite size
  - □ Calculations cumbersome with many special funcs.
- **Mode decomposed analysis very useful!**
  - □ More differential information on CME, CSE, CVE,…
- **Magnetic vortex is a well-defined object!**
  - □ Boundary is imposed physically
  - □ Electromagnetic contributions to the angular momenta