Exploring the existence of inhomogeneous phases in the $2+1$-dimensional Gross-Neveu model in the limit of infinitely many flavors (part 2 – lattice field theory)

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Workshop "Hot problems of Strong Interactions"

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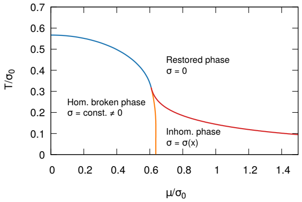
Introduction

- **Long-term goal** (for many researchers in our field): Compute the phase diagram of QCD.
  - Extremely difficult ...
  - ... e.g. "sign problem" in lattice QCD for chemical potential $\mu \neq 0$.
- **QCD-inspired models in the $N_f \to \infty$ limit**:

  QCD-inspired = symmetries similar as in QCD, e.g. chiral symmetry
  $N_f \to \infty$ limit = infinite number of flavors
  - Inhomogeneous phases at large μ and small temperature T.

    inhomogeneous phase = phase with a spatially non-constant order parameter
  - Analytical results for the Gross-Neveu (GN) model in 1+1 dimensions.
    [O. Schnetz, M. Thies and K. Urlichs, Annals Phys. **314**, 425 (2004) [hep-th/0402014]]
  - This talk: lattice field theory investigation of the GN model in 2+1 dimensions.
    - Do inhomogeneous phases exist in 2+1 dimensions?
    - Phase diagram?
    - Shape of the inhomogeneous condensate?



Outline

1. GN model in 2+1 dimensions, $N_f \to \infty$
2. Fermion discretization
3. Scale setting
4. Numerical results

GN model in 2+1 dimensions, $N_f \to \infty$

- Action:

$$S = \int d^3x \left(\sum_{f=1}^{N_f} \bar{\psi}_f \Big(\gamma_\nu \partial_\nu + \gamma_0 \mu\Big) \psi_f - \frac{g^2}{2} \left(\sum_{f=1}^{N_f} \bar{\psi}_f \psi_f \right)^2 \right).$$

- After introducing a scalar field σ and performing the integration over fermionic fields

$$S_{\text{eff}} = N_f \left(\frac{1}{2\lambda} \int d^3x\, \sigma^2 - \ln\Big(\det(\underbrace{\gamma_\nu \partial_\nu + \gamma_0 \mu + \sigma}_{=Q})\Big) \right)$$

$$Z = \int D\sigma\, e^{-S_{\text{eff}}},$$

where $\lambda = N_f g^2$.

- One can show

$$\langle \sigma \rangle - \frac{\lambda}{N_f} \left\langle \sum_{f=1}^{N_f} \bar{\psi}_f \psi_f \right\rangle.$$

- $N_f \to \infty$:
  - Only global minimum of S_{eff}/N important in $\int D\sigma\, e^{-S_{\text{eff}}}$.
  - Assume t independence of this field configuration, i.e. $\sigma = \sigma(x, y)$.

Fermion discretization (1)

- For numerical treatment the degrees of freedom have to be reduced to a finite number.
  $\rightarrow$ Finite volume and discretization needed.
  - For example lattice field theory.
    [P. de Forcrand, U. Wenger, PoS **LAT2006**, 152 (2006) [arXiv:hep-lat/0610117]]
    [M. Winstel, J. Stoll, M. Wagner, arXiv:1909.00064]
    [R. Narayanan, Phys. Rev. D **101**, 096001 (2020) [arXiv:2001.09200]]
  - There are other possibilities to discretize, e.g. finite mode discretization, discretization by piecewise polynomial functions, etc.
    [M. Wagner, Phys. Rev. D **76**, 076002 (2007) [arXiv:0704.3023]]
    [A. Heinz, F. Giacosa, M. Wagner, D. H. Rischke, Phys. Rev. D **93**, 014007 (2016) [arXiv:1508.06057]]

Fermion discretization (2)

- Plane wave expansion in t direction.
  - $(+)$ Easy analytical simplifications possible, because $\sigma = \sigma(x, y)$, i.e. $\det(Q)$ factorizes.
- Lattice discretization with so-called naive fermions in x, y directions.
  - $(+)$ Fermion doubling not a problem in the large-N limit ("$2 \times \infty = \infty$").
  - $(+)$ Chiral symmetry intact (explicitely broken in other discretizations, e.g. Wilson).
  - $(+)$ Rather cheap to evaluate (in contrast to other discretizations, e.g. overlap).

$$S_{\text{free}} = \int d^3x\, \bar{\psi}(x_0, \mathbf{x}) \Big(\gamma_0(\partial_0 + \mu) + \gamma_1 \partial_1 + \gamma_2 \partial_2\Big) \psi(x_0, \mathbf{x}) \rightarrow$$

$$\rightarrow \sum_{n_0=-N_0+1}^{N_0} \sum_{\mathbf{x}} \bar{\chi}(n_0, \mathbf{x}) \bigg(\gamma_0(\mathrm{i}\omega_{n_0} + \mu)\chi(n_0, \mathbf{x}) + \sum_{\nu=1,2} \gamma_\nu \frac{\chi(n_0, \mathbf{x}+\mathbf{e}_\nu) - \chi(n_0, \mathbf{x}-\mathbf{e}_\nu)}{2}\bigg)$$

- Due to fermion doubling χ represents four fermion flavors and, thus, cannot be interpreted as a standard fermion field ψ. The relation between the components of χ and of ψ is non-trivial.

  [Y. Cohen, S. Elitzur, E. Rabinovici, Nucl. Phys. B **220**, 102-118 (1983)]
  [J. Lenz, L. Pannullo, M. Wagner, B. Wellegehausen, A. Wipf, Phys. Rev. D **101**, 094512 (2020) [arXiv:2004.00295]]

Fermion discretization (3)

- Discretized GN action:

$$S_\sigma = \int d^3x \left(\bar{\psi}(x_0, \mathbf{x})\Big(\gamma_\nu \partial_\nu + \gamma_0 \mu + \sigma(\mathbf{x})\Big)\psi(x_0, \mathbf{x}) + \frac{N_f}{2\lambda}\sigma^2(\mathbf{x})\right) \rightarrow$$

$$\rightarrow \sum_{f=1}^{N_f/4} \left(S_{\text{free}} + \sum_{n_0=-N_0+1}^{N_0} \sum_{\mathbf{x},\mathbf{y}} \bar{\chi}_f(n_0, \mathbf{x}) W_2(\mathbf{x}-\mathbf{y})\sigma(\mathbf{y})\chi_f(n_0, \mathbf{x})\right) + \frac{N_f N_0}{\lambda} \sum_{\mathbf{x}} \sigma^2(\mathbf{x})$$

  where W_2 is the Fourier transform of a function $\tilde{W}_2$ with
  - $\tilde{W}_2(\mathbf{k}) \rightarrow 1$ for $\mathbf{k} \approx (0, 0)$,
  - $\tilde{W}_2(\mathbf{k}) \rightarrow 0$ for $\mathbf{k} \approx (\pi, 0)$, $\mathbf{k} \approx (0, \pi)$, $\mathbf{k} \approx (\pi, \pi)$.

- We study two different discretizations, which have both the correct continuum limit:
  - Cosinus function in momentum space:

$$W_2'(\mathbf{x}-\mathbf{y}) = \prod_{\nu=1,2} \left(\frac{1}{4}\delta_{x_\nu-1,y_\nu} + \frac{1}{2}\delta_{x_\nu,y_\nu} + \frac{1}{4}\delta_{x_\nu+1,y_\nu}\right).$$

  - Step function in momentum space:

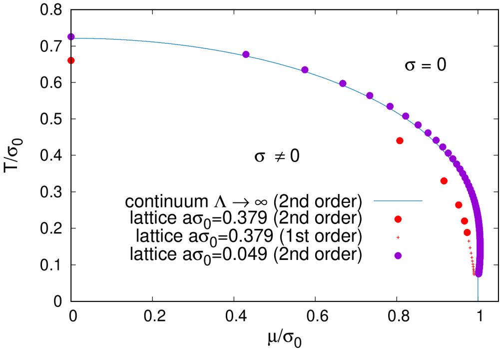
$$W_2''(\mathbf{x}-\mathbf{y}) = \prod_{\nu=1,2} \left(\frac{1}{N_s}\left(1 + \sum_{n=1}^{N_s/4-1} 2\cos\left(\frac{2\pi n(x_\nu - y_\nu)}{L}\right) + \cos\left(\frac{\pi}{2}(x_\nu - y_\nu)\right)\right)\right).$$

Scale setting

- At $\mu = 0$ and for $1/T \to \infty$ the condensate tends to a constant, $\sigma \to \sigma_0$.
- Set the scale via σ_0, i.e. express all dimensionful quantities in units of σ_0, e.g. μ/σ_0, T/σ_0, $a\sigma_0$, ...
- The value of the lattice spacing $a\sigma_0$ depends on the coupling constant λ, i.e. can be changed by adjusting λ

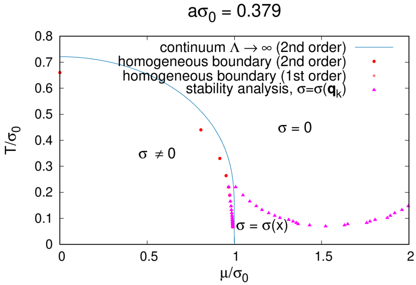
Numerical results, 2+1-dimensional GN model (1)

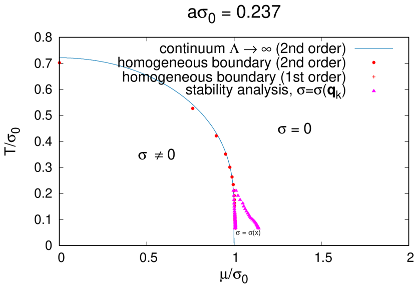
- **Phase diagram with restriction to homogeneous condensate** σ:
  - A Test of our method and implementation:
    - Blue line: K. G. Klimenko / K. Urlichs / continuum approach (part 1 of the talk by L. Kurth).
      [K. G. Klimenko, Z. Phys. C **37**, 457 (1988)]
      [K. Urlichs, PhD thsis, University of Erlangen-Nuremberg (2007)]
    - Red points: coarse lattice spacing.
    - Purple points: fine lattice spacing.



Numerical results, 2+1-dimensional GN model (2)

- **Stability analysis**:
  - Starting point: homogeneous Minimum of S_{eff}.
  - Are there inhomogeneous perturbations, which decrease S_{eff}?
    - Lattice discretization W_2':
      No, "phase diagram" identical to the "homogeneous phase diagram".
    - Lattice discretization W_2'':
      Yes, but the region of instability strongly depends on the lattice spacing a and seems to vanish in the continuum limit $a \to 0$.
      [M. Winstel, J. Stoll, M. Wagner, arXiv:1909.00064]
      [R. Narayanan, Phys. Rev. D **101**, 096001 (2020) [arXiv:2001.09200]]





Numerical results, 2+1-dimensional GN model (3)

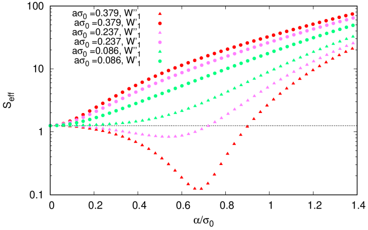
- **Comparison of S_{eff} for the discretizations W_2' and W_2'':**
  - $(\mu/\sigma_0, T/\sigma_0) = (1.035, 0.110)$, i.e. inside the region of instability for W_2''.
  - Ansatz for the condensate:

$$\sigma(\mathbf{x}) = \sigma(x_1) = \alpha \cos\left(\frac{2\pi x_1}{\tilde{\lambda}}\right).$$

  - The plot shows S_{eff} as a function of α for W_2' (circles) and W_2'' (triangles):
    - Red points: corase lattice spacing.
    - Pink points: intermediate lattice spacing.
    - Green points: fine lattice spacing.

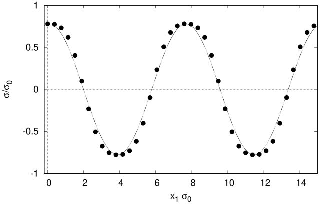
  → Significantly different for coarse lattice spacing.

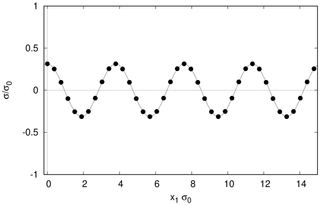
  → Seems to converge in the the continuum limit $a \to 0$.

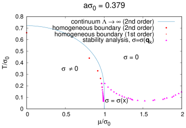


Numerical results, 2+1-dimensional GN model (4)

- **Minimization of S_{eff} (at the moment restriction to $\sigma(\mathbf{x}) = \sigma(x_1)$):**
  - Discretization W_2'' inside the region of instability:
    - Left plot: $(\mu/\sigma_0, T/\sigma_0) = (0.98, 0.132)$, i.e. smaller μ.
      $\rightarrow$ Larger wavelength and amplitude, kink-antikink shape.
    - Right plot: $(\mu/\sigma_0, T/\sigma_0) = (1.11, 0.132)$, i.e. larger μ.
      $\rightarrow$ Smaller wavelength and amplitude, cosinus shape.
    - Qualitative behavior as in the $1+1$-dimensional GN model.
      [O. Schnetz, M. Thies and K. Urlichs, Annals Phys. **314**, 425 (2004) [hep-th/0402014]]



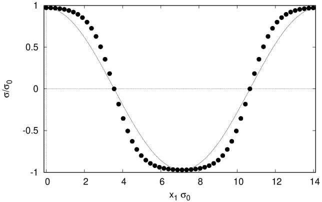


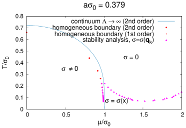


Numerical results, 2+1-dimensional GN model (5)

- **Minimization of S_{eff} (at the moment restriction to $\sigma(\mathbf{x}) = \sigma(x_1)$):**
  - Discretization W_2' and W_2'' inside the homogeneously broken region:
    - $(\mu/\sigma_0, T/\sigma_0) = (0.60, 0.176)$.
    - Global minimum $\sigma = \text{const.}$
    - There are inhomogeneous local minima (example shown in plot).
    - For decreasing T the values of S_{eff} at the global and the local minima approach each other.
    - Consistent with a previous study at $T = 0$ and $\sigma(\mathbf{x}) = \sigma(x_1)$ restricted to an ansatz based on Jacobi elliptic functions.

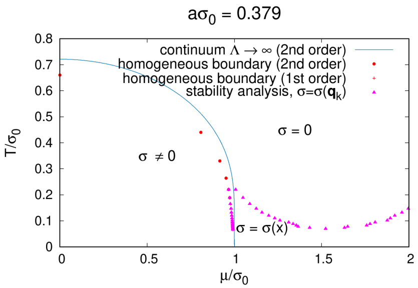
      [K. Urlichs, PhD thsis, University of Erlangen-Nuremberg (2007)]

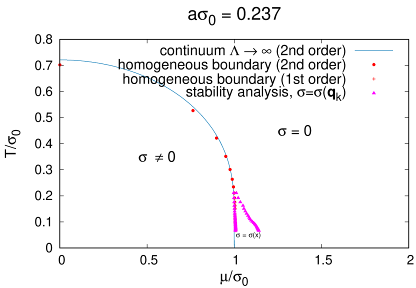




Numerical results, 2+1-dimensional GN model (6)

- **Minimization of S_{eff} (at the moment restriction to $\sigma(\mathbf{x}) = \sigma(x_1)$):**
  - Discretization W_2' and W_2'' inside the symmetric region:
    - Global minimum $\sigma = 0$.
  - Summary:
    - No inhomogeneous phase for discretization W_2'.
    - Inhomogeneous phase for discretization W_2'' at finite a, which coincides with the instability region.
    - In the continuum limit only a symmetric and a homogeneously broken phase, but no inhomogeneous phase.





Next steps

- Are there inhomogeneous phases with 2-dimensional modulations?
  (Minimization of S_{eff} at the moment restricted to $\sigma(\mathbf{x}) = \sigma(x_1)$.)
- Extend studies to 3+1 dimensions.
- Study the phase diagram of more realistic QCD-inspired models (Nambu-Jona-Lasinio (NJL), quark-meson model, ...) with particular focus on inhomogeneous phases.
  $\rightarrow$ Phase structure not only with respect to μ and T but also isospin and strangeness chemical potential μ_I, μ_S ...?