Inhomogeneous phases of the Gross-Neveu model with a finite number of flavors

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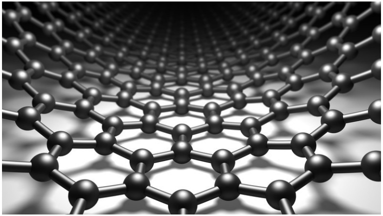
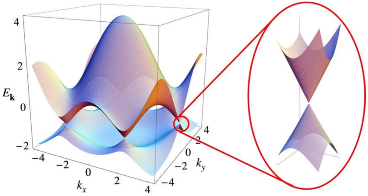
- $\mathrm{N_f}$ flavors of relativistic fermions

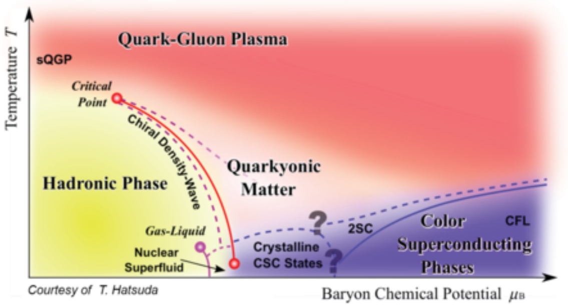
$$\mathcal{L}_\psi = \bar\Psi \partial\!\!\!/ \Psi + \frac{g^2}{2\mathrm{N_f}} (\bar\Psi M_1 \Psi)(\bar\Psi M_2 \Psi), \quad \Psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_{\mathrm{N_f}} \end{pmatrix}$$

- low energy description of tight binding model
- relativistic dispersion relation, e.g. on honeycomb lattice (graphen)
- long-range order $\Rightarrow$ AF, CDW, QAHS
- increase coupling $\Rightarrow$ PT semi-metal $\rightarrow$ (Mott) insulator





possible QCD-phase diagram



- QCD: crystalline high density CSC phase
- problem with complex fermion determinant

  → Taylor-expansion in μ, complex Langevin, thimble method, dual variables, . . . wednesday session 16:00 - 20:40
- effective models for CPT (mean field, lattice, functional methods, . . .)
- gauge theories without sign-problem (SU(2), G_2) Kogut, Hands, . . . , Smekal,. . . , Jena-collaboration

massless GN model:

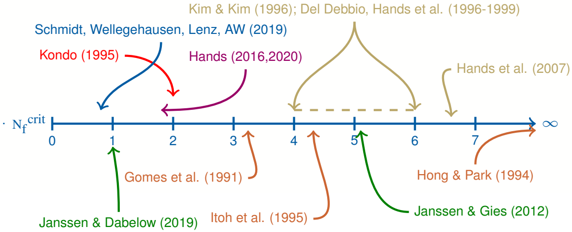
$$\mathcal{L}_{\mathrm{GN}} = \bar{\Psi}(\partial\!\!\!/ + \mu\gamma^0)\Psi - \frac{g^2}{2\mathrm{N_f}}(\bar{\Psi}\Psi)^2, \quad \bar{\Psi}\Psi = \sum_{a=1}^{\mathrm{N_f}} \bar{\psi}_a\psi_a$$

$1+1$ spacetime dimensions

- conducting polymers (Trans- and Cis-polyacetylen) — Su, Schrieffer, Heeger
- quasi 1-dimensional inhomogeneous superconductors — Mertsching, Fischbeck
- dimensional reduction $d=4 \to d=2$ in strong B-field
- $\langle\bar{\Psi}\Psi\rangle$ order parameter for $\mathbb{Z}_2$ chiral symmetry
- number of flavors and CSB:
  - $\mathrm{N_f} = 1 \;\Rightarrow$ no SSB (CFT) — Sachs, AW
  - $\mathrm{N_f} = \infty \Rightarrow$ SSB at low T, μ; inhomogeneous phase — Wolf; Thies et al.
  - $1 < \mathrm{N_f} < \infty \Rightarrow$ asymptotically free, integrable — Gross-Neveu, Coleman, ...

$1-2$ spacetime dimensions

- quasi 2-dimensional Dirac materials, superconductivity, cold atoms
- interacting UV fixed points — Gawedzki, Kupiainen; Park, Rosenstein, Warr, …
  $\to$ simple realization of asymptotically safety — H. Gies et al.
- $\langle\bar{\Psi}\Psi\rangle$ order parameter for $\mathbb{Z}_2$ parity
  - all $\mathrm{N_f} \Rightarrow$ SSB at low T, μ
  - in contrast: SSB in Thirring only for $1 \le \mathrm{N_f} \le N_{\rm crit}$
  - 30 years efforts on $N_{\rm crit}$
    $\to$ staggered lattice fermions problematic — Lenz, Schmidt, Wellegenausen, AW; Hands



- SD equations
- $1/\mathrm{N_f}$-expansion
- FRG
- lattice, staggered
- lattice, domain wall
- lattice, SLAC

- back to Gross-Neveu: $\mathcal{L}_{\mathrm{GN}}$ equivalent to

$$\mathcal{L}_\sigma = \bar{\Psi}\mathcal{D}\Psi + \frac{\mathrm{N_f}}{2g^2}\sigma^2, \quad \mathcal{D} = \partial\!\!\!/ + \sigma + \mu\gamma^0$$

- grand canonical ensemble (anti-periodic ψ_a)

$$\langle\mathcal{O}\rangle = \frac{1}{Z}\int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi\mathcal{D}\sigma\, \mathrm{e}^{-S_\sigma}\mathcal{O}, \quad Z = \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi\mathcal{D}\sigma \mathrm{e}^{-S_\sigma}$$

- order parameter

$$\langle\bar{\Psi}(x)\Psi(x)\rangle = -\frac{\mathrm{N_f}}{g^2}\langle\sigma(x)\rangle$$

- fermion integration $\Rightarrow$ induced action

$$Z = \int \mathcal{D}\sigma\, e^{-\mathrm{N_f}S_{\mathrm{eff}}[\sigma]}, \quad S_{\mathrm{eff}}[\sigma] = \frac{1}{2g^2}\int \sigma^2 - \log\det\mathcal{D}$$

- saddle point approximation

$$Z \longrightarrow \mathrm{e}^{-\mathrm{N_f}\min_\sigma S_{\mathrm{eff}}[\sigma]}, \quad \text{exact for } \mathrm{N_f} \to \infty$$

- homogeneous phases $\Rightarrow \sigma$ constant, renormalized $S_{\text{eff}} = \beta V_s U_{\text{eff}}$

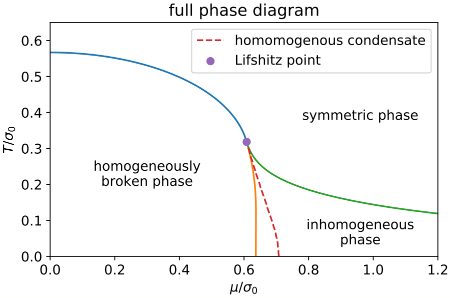
$$U_{\text{eff}} = U_{\text{eff}}^{(0)} - \frac{\mathcal{C}_d}{\beta}\int_0^\infty \mathrm{d}p\, p^{d-2}\left(\log\left(1+\mathrm{e}^{\beta(\varepsilon_p+\mu)}\right) + (\mu \to -\mu)\right)$$

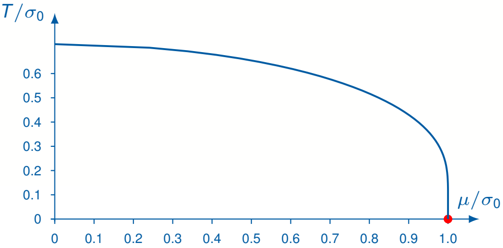
- renormalization condition: SSB at $\mu = T = 0$:

$$\langle\sigma\rangle_{T=\mu=0} = \sigma_0 \Rightarrow \text{energy scale } \sigma_0$$

- effective potential for $T = 0$ and $\mu = 0$:

$$U_{\text{eff}}^{(0)} = \begin{cases} \frac{\sigma^2}{4\pi}\left(\log\frac{\sigma^2}{\sigma_0^2} - 1\right) & d = 2 \\ \frac{\sigma^2}{6\pi}\left(\sigma - \frac{3}{2}\sigma_0\right) & d = 3 \end{cases}$$





symmetric phase at large T, μ

broken phase at small T, μ

- 2 dimensions

  second and first order line

  Lifschitz-point at $(T, \mu_0) \approx (0.318, 0.608)$

  inhomogeneous phase

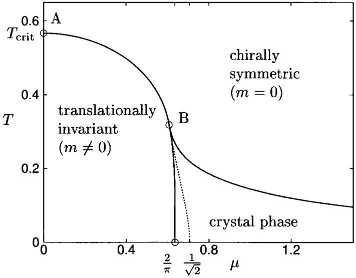
- 3 dimensions

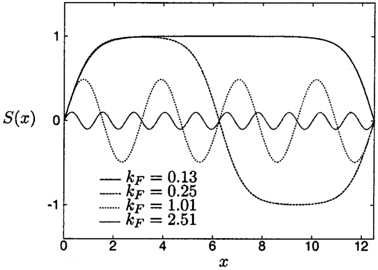
  second order line

$$\cosh \frac{\mu}{2T} = \frac{1}{2} e^{\sigma_0/kT}$$

only jump at $(\mu, T) = (1, 0)$

- analytic solution of gap equation for $\mathrm{N_f} \to \infty$ and $L \to \infty$ Thies et al.
- GN & NJL: elliptic functions, inverse scattering methods, ...
- inhomogeneous $\langle\bar{\psi}\psi\rangle \to$ SSB of translation invariance
  $\to$ massless Goldstone-excitations
- is phase diagram = artifact of $\mathrm{N_f} \to \infty$ and/or $d = 2$?
- inhomogeneous condensates for $\mathrm{N_f} < \infty$, $d \geq 2$?





- induced action

$$S_{\text{eff}} = \frac{1}{2g^2}\sum_x \sigma_x^2 - \log\det\mathcal{D}, \qquad \mathcal{D} = \gamma^\mu\partial_\mu + \mu\gamma^\mu + \sigma$$

- charge conjugation and hermitean conjugation $\Rightarrow$

even dimensions: $\det\mathcal{D}$ real, even in μ, even in

- true for anti-hermitean and real ∂_μ
  naive fermions, chiral SLAC fermions
- SLAC fermions: $\mathrm{N_f} = 2, 4, \ldots$, naive fermions $\mathrm{N_f} = 8, 16, \ldots$
- Jena-Frankfurt collaboration: first simulations with chiral fermions for
  - 3d Thirring model $\rightarrow$ critical flavour number — Wellegehausen, Schmidt, Lenz, AW
  - 2d GN model at finite μ, T — Lenz, Panullo, Wagner, Wellegehausen, AW
  - 3d GN model at finite μ, T (preliminary)

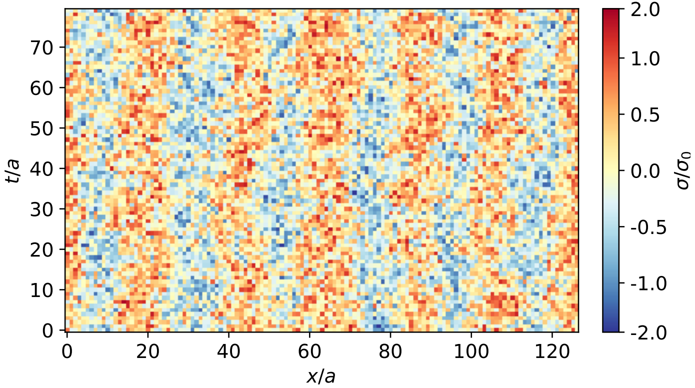
- chiral SLAC fermions
  - no additional sign-problem due to discretization
  - same internal symmetries as continuum theory
  - no doublers
- simulations: line of constant physics: g^2 such that

$$\lim_{L\to\infty} \langle|\bar{\sigma}|\rangle\Big|_{\mu=T=0} = \sigma_0$$

$\rightarrow$ dimensionful quantities in units of fixed σ_0

- ensembles on (μ, T)-grid
- $31 \leq N_s \leq 128 \rightarrow$, lattice spacings $a \approx 0.41, 0.25, 0.20, 0.13$
- rational HMC with $N_{PF} = 2\mathrm{N_f}$
- compare SLAC fermions $\leftrightarrow$ naive femions ($\mathrm{N_f} = 8$)
- TD and UV limit under control
- longe-range correlators $\mathrm{N_f} = 2$

- typical configuration



- cp. kink-antikink (staggered, smeared σ, $\mathrm{N_f} = 12$)

Karsch, Kogut, Wyld

- differentiate homogeneously broken $\leftrightarrow$ symmetric & inhomogeneous

$$\Sigma^2 = \frac{\langle \sigma^2 \rangle}{\sigma_0^2}$$

- differentiate all phases:

$$C(x) = \langle \sigma(t_0, x)\sigma(t_0, 0)\rangle = \frac{1}{N_t N_s} \sum_{t,y} \langle \sigma(t, y + x)\sigma(t, y)\rangle$$

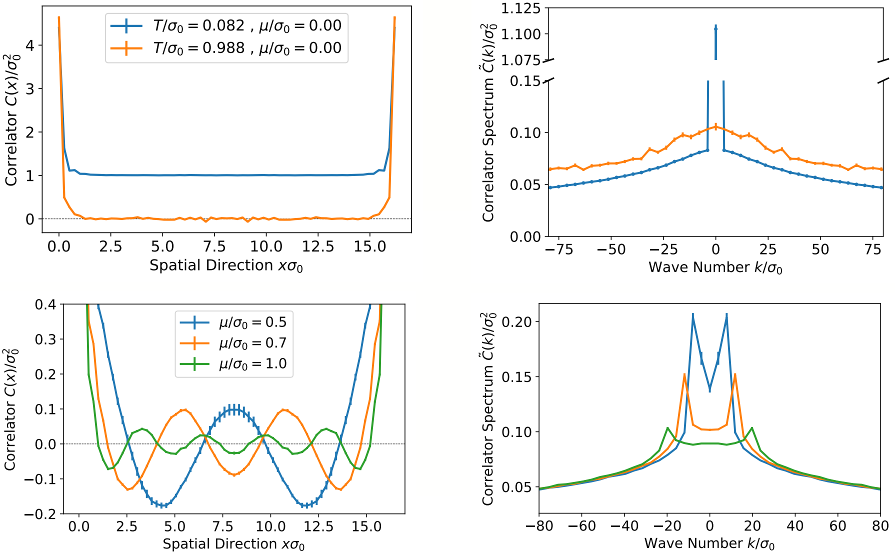
- Fourier transform

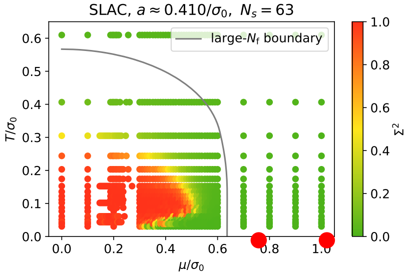
$$\tilde{C}(k) = \mathcal{F}_x(C)(k)$$

symmetric phase: small amplitude

homogeneously broken phase: peak at $k = 0$

inhomogeneous phase: peaks at $\pm q$ (dominant wavelength)


$T/\sigma_0 = 0.082$, $\mu/\sigma_0 = 0.00$
$T/\sigma_0 = 0.988$, $\mu/\sigma_0 = 0.00$
Correlator $C(x)/\sigma_0^2$
Spatial Direction $x\sigma_0$
Correlator Spectrum $\tilde{C}(k)/\sigma_0^2$
Wave Number k/σ_0
$\mu/\sigma_0 = 0.5$
$\mu/\sigma_0 = 0.7$
$\mu/\sigma_0 = 1.0$
Correlator $C(x)/\sigma_0^2$
Spatial Direction $x\sigma_0$
Correlator Spectrum $\tilde{C}(k)/\sigma_0^2$
Wave Number k/σ_0



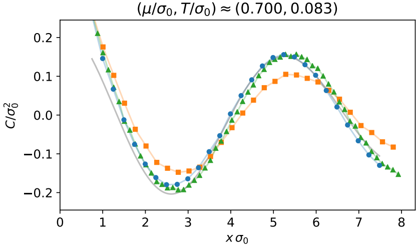
- left: $\Sigma^2 \rightarrow$ symmetric or broken

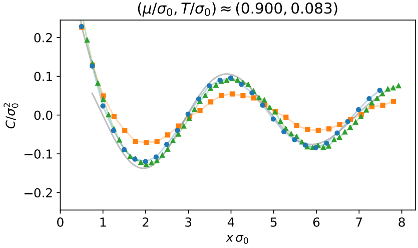
  insensitive to inhom. phase
- below:

  wavelength increases with μ

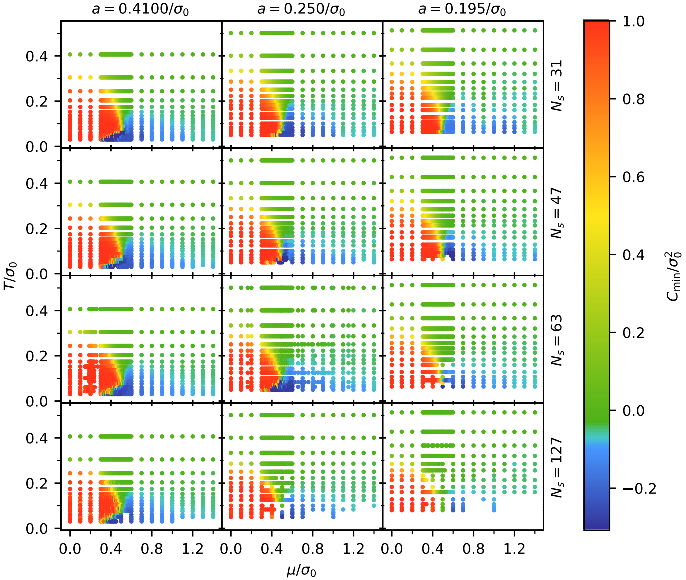
  amplitude decreases with μ

  similar to large N_f limit

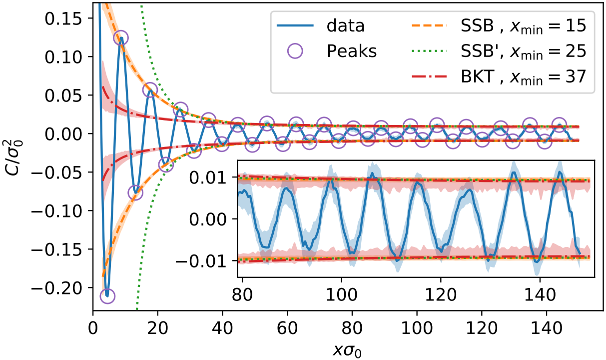




blue dots: SLAC, $(a, N_s) = (0.25, 63)$, orange: naive $(0.252, 64)$, green: naive $(0.126, 128)$

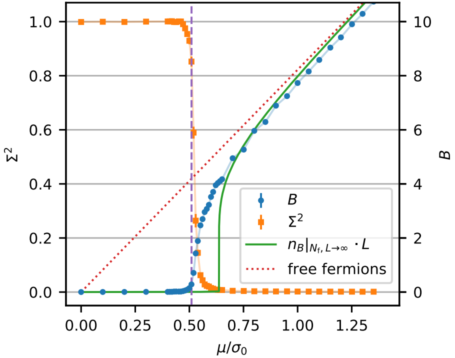


$a = 0.4100/\sigma_0$
$a = 0.250/\sigma_0$
$a = 0.195/\sigma_0$
$N_s = 31$
$N_s = 47$
$N_s = 63$
$N_s = 127$
T/σ_0
μ/σ_0
$C_{\min}/\sigma_0^2$
0.0
0.2
0.4
0.8
1.2
1.0
0.6
−0.2



- longe range correlations, $N_s = 725$
- SSB $C(x) \sim (\alpha + \beta e^{-mx}) \cdot C_{periodic}(x)$
- BKT (Berezinskii, Kosterlitz, Thouless) $C(x) \sim |x|^{-\beta} \cdot C_{periodic}(x)$
- analysis not conclusive yet!

baryon number at low temperature & finite density



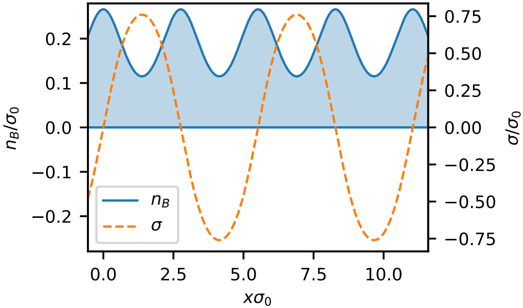
- low temperature $T = 0.076$
- Baryon number:

$$B = \frac{\mathrm{i}}{\mathrm{N_f}} \left\langle \int \mathrm{d}x\, \bar{\Psi}(x)\gamma^0\Psi(x) \right\rangle$$

- transition at $\mu \approx 0.51$
  symmetric $\to$ inhomogeneous
- green: analytic large-$\mathrm{N_f}$ result

Schnetz, Thies, Urlichs; talk of M. Wagner

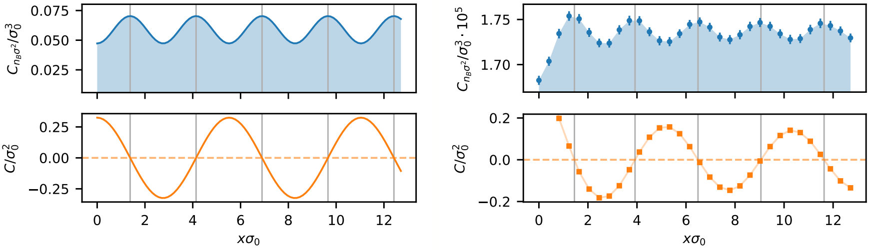
- analytic $\mathrm{N_f} \to \infty$ result for $(\mu, T) = (0.7, 0)$



- baryonic crystal: $\mathrm{N_f}$ fermions for each cycle of oscillation
- fermions located at nodes of condensate field σ
  $\to$ correlation $n_B(x)$ and $\sigma^2(x)$

- correlation condensate $\leftrightarrow$ baryon density

$$C_{n_B\sigma^2}(x) = \frac{\mathrm{i}}{\mathrm{N_f}}\left\langle n_B(0,x)\,\sigma^2(0,0)\right\rangle$$



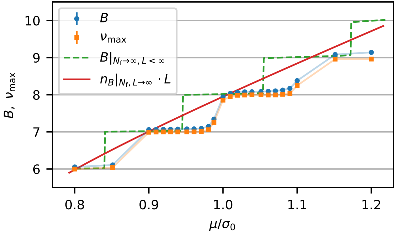
- left: analytical results for $\mathrm{N_f} \to \infty$ Schnetz, Thies
- right: simulation results (SLAC, $\mathrm{N_f} = 8$)

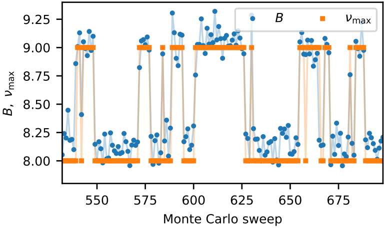
- $k_{\max}$ maximizes $|\tilde{c}|$, with $\tilde{c} = \mathcal{F}(c)$,

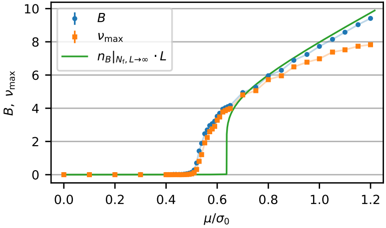
$$c(x) = \frac{1}{N_t N_s} \sum_{t,y} \sigma(t, y + x)\sigma(t, y), \quad C(x) = \langle c(x) \rangle$$

- dominant frequency

$$\nu_{\max} = \frac{L\langle |k_{\max}| \rangle}{2\pi}$$
$$= \langle \text{number of oscillations} \rangle$$
$$\approx \text{ baryon numbers}$$







conclusions

- first simulations with chiral fermions finite $\mu, T, \mathrm{N_f}$
- comparable results for $\mathrm{N_f} = 8$ and $\mathrm{N_f} = 16$, naive and SLAC fermions
- phase diagram very similar to $\mathrm{N_f} \to \infty$ Thies
  broken phases shrink with increasing $\mathrm{N_f}$
  baryon density $\leftrightarrow$ condensate
- long-range correlations for $\mathrm{N_f} = 2$ ($L \lessapprox 1024$) Witten

outlook

- what happens to Goldstone-modes for $\mathrm{N_f} < \infty$
- dispersion relations of lightest states in $1 + 1$ dimensions
- $1 + 2$ dimensions, first numerical results for GN
  cutoff-dependence in large $\mathrm{N_f}$ limit $\to$ Marc Wagner
- external magnetic fields ($\to$ first order transitions)