NTNU
Norwegian University of Science and Technology

χPT and pion condensation at finite isospin

Workshop on hot problems of strong interactions

Jens O. Andersen[1]
Department of physics, NTNU

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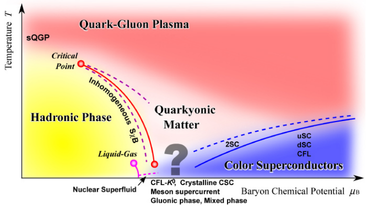
[1] Collaborators: Prabal Adhikari, St. Olavs College/NTNU and Martin Mojahed, NTNU
References: EPJC **79** 879 (2019), JHEP **06**, 170 (2020), Phys. Lett. B **804**, 135352 (2020), EPJC **80**, 1028, (2020)

Introduction

— **QCD phase diagram**



Fukushima and Hatsuda ’11

— **Only a few exact results**

- Quark-gluon plasma at asymptotically high temperature
- Color-flavor locked phase at asymptotically large baryon density

— **Many results are model dependent**

— **More axes in the phase diagram: external magnetic field and independent quark chemical potential for each flavor**

— **Sign problem at finite baryon chemical potential**

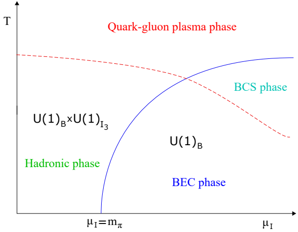
$$\mathcal{Z} = \int \mathcal{D}A_\mu e^{-S_g} \det\left(\not{D} + m_f - \mu_f\gamma^0\right) ,$$
$$\det\left(\not{D} + m_f - \mu_f\gamma^0\right) = \det\left[(iX + \mu_f)(iX^\dagger + \mu_f) + m_f^2\right] ,$$

— **No sign problem at finite isospin** μ_I **with** $\mu_B = \mu_S = 0$

$$\mu_I = \tfrac{1}{2}(\mu_u - \mu_d) ,$$
$$\mu_B = \tfrac{3}{2}(\mu_u + \mu_d) ,$$
$$\mu_S = \tfrac{1}{2}(\mu_u + \mu_d - 2\mu_s) .$$

— **This talk**

- Pion condensation at $T = 0$ using two and three-flavor χpt
- Kaon condensation at $T = 0$ using three-flavor χpt



χpt at finite isospin μ_I

- **Low-energy effective theory for QCD based on symmetries and relevant degrees of freedom** [2]
- **Two-flavor QCD, pions and** $SU(2)_L \times SU(2)_R$
- **Three-flavor QCD, pions, kaons, and** η **and** $SU(3)_L \times SU(3)_R$
- **Leading-order Lagrangian and addition of quark chemical potentials**

$$\mathcal{L}_2 = \frac{f^2}{4}\mathrm{Tr}\left[\nabla_\mu \Sigma^\dagger \nabla^\mu \Sigma\right] + \frac{f^2}{4}\mathrm{Tr}\left[\chi^\dagger \Sigma + \Sigma^\dagger \chi\right] \ ,$$

$$\Sigma = e^{i\frac{\phi_a \tau_a}{f}} \ , \qquad \nabla_\mu \Sigma \equiv \partial_\mu \Sigma - i\left[v_\mu, \Sigma\right] \ ,$$

$$v_\mu = \delta_{\mu,0}\mathrm{diag}(\mu_u, \mu_d) = \delta_{\mu,0}\left(\tfrac{1}{3}\mu_B + \tfrac{1}{2}\mu_I, \tfrac{1}{3}\mu_B - \tfrac{1}{2}\mu_I\right) \ ,$$

$$\chi = 2B_0 M \qquad f_\pi \sim f \ , \qquad 2B_0 m = m_{\pi,0}^2 \sim m_\pi^2 \ .$$

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[2] Weinberg '79, Gasser and Leutwyler '84

— **Rotated ground state and pion condensation** [3]

$$\begin{aligned}\Sigma_\alpha &= e^{i\alpha\hat{\phi}_i\tau_i} = \mathbb{1}\cos\alpha + \hat{\phi}_i\tau_i\sin\alpha\ ,\\ \Sigma_\alpha^\dagger\Sigma_\alpha &= \mathbb{1}\ .\end{aligned}$$

— **Static Hamiltonian**

$$\mathcal{H}^{\text{static}} = -f^2 m_{\pi,0}^2\cos\alpha - \tfrac{1}{2}\mu_I^2 f^2\sin^2\alpha\ .$$

— **Competition between two terms, two phases**

$$\cos\alpha = 1\ ,\ \mu_I < m_{\pi,0}\ ,\quad \text{(vacuum phase - Silver Blaze property)}$$

$$\cos\alpha = \frac{m_{\pi,0}^2}{\mu_I^2}\ ,\ \mu_I > m_{\pi,0}\ ,\quad \text{(pion-condensed phase)}\ .$$

[3] Son and Stephanov Phys. Rev. Lett. 86 592 (2001)

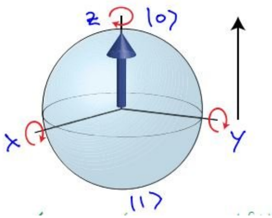
— **Naive parametrization of fluctuations**

$$
\begin{aligned}
\Sigma_\alpha &= A_\alpha \Sigma_0 A_\alpha \;, \\
A_\alpha &= e^{i\frac{\alpha}{2}\tau_1} = \cos\frac{\alpha}{2} + i\tau_1 \sin\frac{\alpha}{2} \;, \\
\Sigma &= U \Sigma_\alpha U \;, \\
U &= e^{i\frac{\phi_a \tau_a}{2f}} \;.
\end{aligned}
$$

— **Use of unrotated generators yields**

- **noncanonical kinetic term (not problematic?)**
- **divergences that cannot be cancelled by standard counterterms (disaster)**

— **Visualizing it via** $SO(3) \to SO(2)$



— **Naive parametrization**

$$\Sigma_{\text{wrong}} = U\Sigma_\alpha U = UA_\alpha\Sigma_0 A_\alpha U \ ,$$
$$U = e^{i\pi_1 J_x + i\pi_2 J_y} \ .$$

— **Rotation around the *x*-axis**

$$R_x\left[i\pi_1 J_x + i\pi_2 J_y\right] R_x^{-1} = i\pi_1 J_x + i\pi_2 J_y \cos\theta - i\pi_3 J_z \sin\theta$$
$$= i\pi_1 J_x + i\pi_2 J_y'$$

— **Fluctuations and rotated broken generators** [4]

$$\Sigma = L_\alpha \Sigma_\alpha R_\alpha^\dagger \ ,$$
$$L_\alpha = A_\alpha U A_\alpha^\dagger \ ,$$
$$R_\alpha = A_\alpha^\dagger U^\dagger A_\alpha \ ,$$
$$U = e^{i\frac{\phi_a \tau_a}{2f}} \ .$$

[4] K. Splittorff, D.T. Son, M. A. Stephanov, Phys.Rev. D64 (2001) 016003

— Leading-order Lagrangian for two-flavor χPT

$$\mathcal{L}_2^{\rm static} = f^2 m_{\pi,0}^2 \cos\alpha + \frac{1}{2} f^2 \mu_I^2 \sin^2\alpha \ ,$$

$$\mathcal{L}_2^{\rm linear} = f\left(-m_{\pi,0}^2 \sin\alpha + \mu_I^2 \cos\alpha \sin\alpha\right)\phi_1 + f\mu_I \sin\alpha\partial_0\phi_2 \ ,$$

$$\begin{aligned}
\mathcal{L}_2^{\rm quadratic} &= \tfrac{1}{2}(\partial_\mu\phi_a)(\partial^\mu\phi_a) + \mu_I \cos\alpha(\phi_1\partial_0\phi_2 - \phi_2\partial_0\phi_1) \\
&\quad -\tfrac{1}{2}\Big[(m_{\pi,0}^2\cos\alpha - \mu_I^2\cos 2\alpha)\phi_1^2 + (m_{\pi,0}^2\cos\alpha - \mu_I^2\cos^2\alpha)\phi_2^2 \\
&\quad +(m_{\pi,0}^2\cos\alpha + \mu_I^2\sin^2\alpha)\phi_3^2\Big] \ ,
\end{aligned}$$

$$\mathcal{L}_2^{\rm cubic} = \frac{(m_{\pi,0}^2 - 4\mu_I^2\cos\alpha)\sin\alpha}{6f}\phi_1(\phi_a\phi_a) - \frac{\mu_I\sin\alpha}{f}\left[\phi_1^2\partial_0\phi_2 + \phi_3^2\partial_0\phi_2\right] \ ,$$

$$\begin{aligned}
\mathcal{L}_2^{\rm quartic} &= \frac{1}{24f^2}(\phi_a\phi_a)\Big[(m_{\pi,0}^2\cos\alpha - 4\mu_I^2\cos 2\alpha)\phi_1^2 \\
&\quad +(m_{\pi,0}^2\cos\alpha - 4\mu_I^2\cos^2\alpha)\phi_2^2 + (m_{\pi,0}^2\cos\alpha + 4\mu_I^2\sin^2\alpha)\phi_3^2\Big] \\
&\quad -\frac{\mu_I\cos\alpha}{3f^2}(\phi_a\phi_a)(\phi_1\partial_0\phi_2 - \phi_2\partial_0\phi_1) \\
&\quad +\frac{1}{6f^2}\left[\phi_a\phi_b\partial^\mu\phi_a\partial^\mu\phi_b - \phi_a\phi_a\partial_\mu\phi_b\partial^\mu\phi_b\right] \ .
\end{aligned}$$

— **Next-to-leading order Lagrangian** [5]

$$\mathcal{L}_4 = \frac{1}{4}l_1\left(\mathrm{Tr}\left[\nabla\mu\Sigma^\dagger\nabla^\mu\Sigma\right]\right)^2 + \frac{1}{4}l_2\mathrm{Tr}\left[\nabla_\mu\Sigma^\dagger\nabla_\nu\Sigma\right]\mathrm{Tr}\left[\nabla^\mu\Sigma^\dagger\nabla^\nu\Sigma\right]$$
$$+\frac{1}{16}(l_3+l_4)(\mathrm{Tr}[\chi^\dagger\Sigma+\Sigma^\dagger\chi])^2 + \frac{1}{8}l_4\mathrm{Tr}\left[\nabla_\mu\Sigma^\dagger\nabla^\mu\Sigma\right]\mathrm{Tr}[\chi^\dagger\Sigma+\Sigma^\dagger\chi]$$
$$+\frac{1}{2}h_1\mathrm{Tr}[\chi^\dagger\chi]\,.$$

— **Static part**

$$\mathcal{L}_4^{\text{static}} = (l_1+l_2)\mu_I^4\sin^4\alpha + l_4 m_{\pi,0}^2\mu_I^2\cos\alpha\sin^2\alpha + (l_3+l_4)m_{\pi,0}^4\cos^2\alpha\,,$$

— **Effective potential at NLO**

$$V_1 = V_{1,\pi^+} + V_{1,\pi^-} + V_{1,\pi^0}$$
$$= \frac{1}{2}\int_p \left(E_{\pi^+} + E_{\pi^-} + E_{\pi^0}\right)$$

— **Isolate divergences by adding and subtracting divergent terms that can be calculated in dimensional regularization**

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[5] Gasser and Leutwyler '84 and '85

$$V_1^{\mathrm{div}} = -\frac{1}{2(4\pi)^2}\left[\frac{1}{\epsilon}+\frac{3}{2}+\log\left(\frac{\Lambda^2}{m_3^2}\right)\right]\left(m_{\pi,0}^2\cos\alpha+\mu_I^2\sin^2\alpha\right)^2 -\frac{1}{4(4\pi)^2}\left[\frac{1}{\epsilon}+\frac{3}{2}+\log\left(\frac{\Lambda^2}{\tilde{m}_2^2}\right)\right](m_{\pi,0}^2\cos\alpha)^2 \,.$$

— **Renormalization of parameters**

$$l_i = l_i^r(\Lambda) - \frac{\gamma_i\Lambda^{-2\epsilon}}{2(4\pi)^2}\left[\frac{1}{\epsilon}+1-\bar{l}_i\right] ,$$

— **Renormalized effective potential**

$$\begin{aligned} V_{\mathrm{eff}} &= -f^2 m_{\pi,0}^2\cos\alpha - \frac{1}{2}f^2\mu_I^2\sin^2\alpha \\ &\quad -\frac{1}{4(4\pi)^2}\left[\frac{3}{2}-\bar{l}_3+4\bar{l}_4+\log\left(\frac{m_{\pi,0}^2}{\tilde{m}_2^2}\right)+2\log\left(\frac{m_{\pi,0}^2}{m_3^2}\right)\right]m_{\pi,0}^4\cos^2\alpha \\ &\quad -\frac{1}{(4\pi)^2}\left[\frac{1}{2}+\bar{l}_4+\log\left(\frac{m_{\pi,0}^2}{m_3^2}\right)\right]m_{\pi,0}^2\mu_I^2\cos\alpha\sin^2\alpha \\ &\quad -\frac{1}{4(4\pi)^2}\left[1+\frac{2}{3}\bar{l}_1+\frac{4}{3}\bar{l}_2+2\log\left(\frac{m_{\pi,0}^2}{m_3^2}\right)\right]\mu_I^4\sin^4\alpha + V_{1,\pi^+}^{\mathrm{fin}} + V_{1,\pi^-}^{\mathrm{fin}} \,. \end{aligned}$$

Three-flavor χPT briefly

— **Adding quark chemical potential for s-quark**

$$v_\mu = \delta_{\mu,0}\mathrm{diag}(\tfrac{1}{3}\mu_B + \tfrac{1}{2}\mu_I, \tfrac{1}{3}\mu_B - \tfrac{1}{2}\mu_I, \tfrac{1}{3}\mu_B - \mu_S) \; .$$

— **Eight relevant operators at next-to-leading order in χPT**

— **Expect pion and kaon condensation in analogy with two-flavor case. Ground state rotated in a specific way**

$$\begin{aligned} \Sigma_\alpha^{\pi} &= e^{i\alpha\lambda_2} \; , \\ \Sigma_\alpha^{K^\pm} &= e^{i\alpha\lambda_5} \; , \\ \Sigma_\alpha^{K^0/\bar{K}^0} &= e^{i\alpha\lambda_7} \; . \end{aligned}$$

Pion and kaon condensation if

1. $|\mu_I| > m_\pi$,
2. $|\pm\frac{1}{2}\mu_I + \mu_S| > m_K$.

Parameter fixing in two-flavor χPT

— **Low-energy constants**

$$\bar{l}_1 = -0.4 \pm 0.6\,, \qquad \bar{l}_2 = 4.3 \pm 0.1\,,$$
$$\bar{l}_3 = 2.9 \pm 2.4\,, \qquad \bar{l}_4 = 4.4 \pm 0.2\,.$$

— **Physical masses** [6]

$$m_\pi = m_{\pi,0}\left[1 - \frac{m_{\pi,0}^2}{4(4\pi)^2 f^2}\bar{l}_3\right] = 131 \pm 3\mathrm{MeV}\,,$$
$$f_\pi^2 = f^2\left[1 + \frac{2m_{\pi,0}^2}{(4\pi)^2 f^2}\bar{l}_4\right] = \frac{128 \pm 3}{\sqrt{2}}\mathrm{MeV}\,.$$

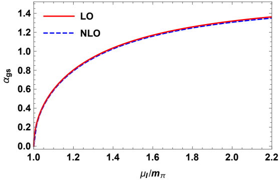
— **Parameters**

$$m_{\pi,0}^{\mathrm{cen}} = 132.4884\,\mathrm{MeV}\,, \qquad f^{\mathrm{cen}} = 84.9342\,\mathrm{MeV}\,,$$

[6] Endrodi et al '12. Talk Bastian Brandt, this session.

Results

— **Minimizing with respect to α**



— **Expansion around $\alpha = 0$**

$$
\begin{aligned}
V_{\rm eff}^{LG} &= V_{\rm eff}(\alpha = 0) + \frac{1}{2} f_\pi^2 \left[\mu_I^2 - m_\pi^2\right] \alpha^2 + a_4(\mu_I)\alpha^4 \,, \\
a_4(\mu_I^c) &> 0
\end{aligned}
$$

— **Second order transition exactly at $\mu_I = m_\pi$ since $\alpha_4 > 0$**

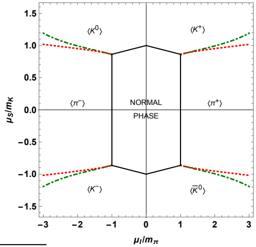
Results

— **Effective potential**

$$
\begin{aligned}
V_0 &= -f^2 B_0 (2m + m_s) \ , \\
V_0 &= -2f^2 B_0 m \cos\alpha - f^2 B_0 m_s - \frac{1}{2} f^2 \mu_I^2 \sin^2\alpha \ , \\
V_0 &= -f^2 B_0 m (1 + \cos\alpha) - f^2 B_0 m_s \cos\alpha - \frac{1}{2} f^2 \left(\tfrac{1}{2}\mu_I + \mu_S\right)^2 \sin^2\alpha \ .
\end{aligned}
$$

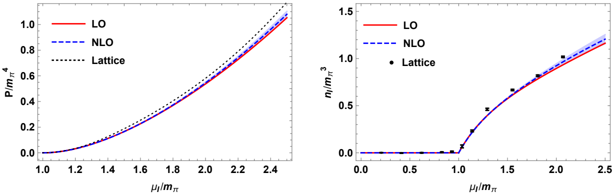
— **Silver Blaze property**

— **Transitions from the vacuum always second order. Transitions between different condensates always first order.** [7]



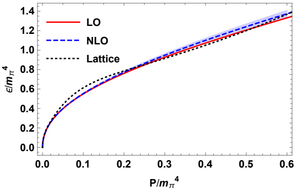
[7] Kogut and Toublain 2001.

— Pressure and isospin density



Lattice data from B. B. Brandt et al, Phys. Rev. D98 094510 (2018).

— Equation of state



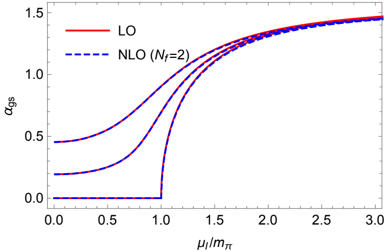
Lattice data from B. B. Brandt et al, Phys. Rev. D98 094510 (2018).

— **Quark and pion condensates**

- **Add pionic source**

$$\chi = 2B_0 m + 2i\lambda_1 j \,.$$

- **Breaks symmetry explicitly**
- **Extremizing with respect to α**



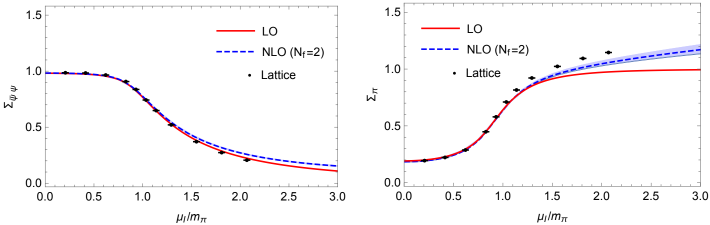
$$\langle\bar{\psi}\psi\rangle_{\mu_I} = \frac{1}{2}\frac{\partial V_{\rm eff}}{\partial m} \,, \qquad \langle\pi^+\rangle_{\mu_I} = \frac{1}{2}\frac{\partial V_{\rm eff}}{\partial j} \,.$$

— Rotation at tree level

$$\langle\bar{\psi}\psi\rangle_{\mu_I}^{\rm tree} = -f^2 B_0 \cos\alpha = \langle\bar{\psi}\psi\rangle_0^{\rm tree} \cos\alpha \,,$$
$$\langle\pi^+\rangle_{\mu_I}^{\rm tree} = -f^2 B_0 \sin\alpha = \langle\bar{\psi}\psi\rangle_0^{\rm tree} \sin\alpha \,,$$

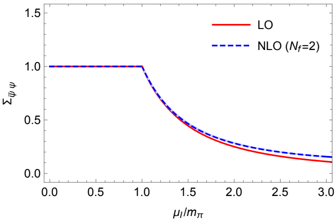
— Condensate deviations

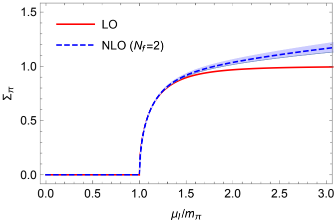
$$\Sigma_{\bar{\psi}\psi} = -\frac{2m}{m_\pi^2 f_\pi^2}\left[\langle\bar{\psi}\psi\rangle_{\mu_I} - \langle\bar{\psi}\psi\rangle_0^{j=0}\right] + 1 \,,$$
$$\Sigma_\pi = -\frac{2m}{m_\pi^2 f_\pi^2}\langle\pi^+\rangle_{\mu_I} \,,$$



Lattice data from B. B. Brandt et al, Phys. Rev. D98 094510 (2018).

— Condensate deviations





Conclusions and outlook

- First calculation of thermodynamic functions at next-to-leading order in the pion-condensed phase in two χPT
- First precision test of χpt at NLO with nonzero μ_I
- Good agreement with lattice data at $T = 0$
- Complete three-flavor results on their way
- Can map three-flavor results onto two-flavor results with renormalized parameters in the large-m_s limit
- Pions in a magnetic field [8]
- Quantum corrections to vortices

[8] Talk Prabal Adhikari, this session