ANOMALY-INDUCED INHOMOGENEOUS PHASE IN QCD-LIKE THEORIES

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LOW-ENERGY EFFECTIVE THEORY OF QCD

- Symmetry-breaking pattern:

$$\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{SU}(2)_{\mathrm{R}} \to \mathrm{SU}(2)_{\mathrm{V}}$$

- Parameterization of the coset space:

$$[\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{SU}(2)_{\mathrm{R}}]/\mathrm{SU}(2)_{\mathrm{V}} \quad \Longrightarrow \quad \Sigma \in \mathrm{SU}(2)$$

- Leading-order effective Lagrangian:

$$\mathscr{L} = \frac{f_\pi^2}{4}\left[\mathrm{tr}(D_\mu \Sigma^\dagger D^\mu \Sigma) + m_\pi^2\, \mathrm{tr}(\Sigma + \Sigma^\dagger)\right]$$

Explicit symmetry breaking

Gasser, Leutwyler, Ann. Phys. **158** (1984)

WESS–ZUMINO–WITTEN TERM

Anomalous coupling of pions has nontrivial consequences:

- Neutral pions can couple to electromagnetic fields!
- Pion fields can carry baryon number!

$$S_{\mathrm{WZW}} = -\int d^4x \left(A^{\mathrm{B}}_{\mu} - \frac{1}{2} A_{\mu} \right) j^{\mu}_{\mathrm{B}}$$

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Baryon number gauge field

Electromagnetic gauge field

Goldstone–Wilczek current

$$j^{\mu}_{\text{B}} = -\frac{1}{24\pi^2}\epsilon^{\mu\nu\alpha\beta}\,\mathrm{tr}\left[(\Sigma D_{\nu}\Sigma^{\dagger})(\Sigma D_{\alpha}\Sigma^{\dagger})(\Sigma D_{\beta}\Sigma^{\dagger}) + \frac{3i}{4}F_{\nu\alpha}\tau_3(\Sigma D_{\beta}\Sigma^{\dagger} + D_{\beta}\Sigma^{\dagger}\Sigma)\right]$$

Son, Stephanov, PRD **77** (2008)

NEUTRAL PION BACKGROUND

$$\mathscr{L} = \frac{f_\pi^2}{4}\left[\mathrm{tr}(D_\mu \Sigma^\dagger D^\mu \Sigma) + m_\pi^2\,\mathrm{tr}(\Sigma + \Sigma^\dagger)\right] + \mathscr{L}_{\mathrm{WZW}}$$

$$\Big\downarrow \quad \Sigma = e^{i\tau_3\phi}$$

$$\mathscr{H} = \frac{f_\pi^2}{2}(\boldsymbol{\nabla}\phi)^2 + m_\pi^2 f_\pi^2(1-\cos\phi) - \frac{\mu}{4\pi^2}\boldsymbol{B}\cdot\boldsymbol{\nabla}\phi$$

- Sine-Gordon Hamiltonian with a topological term!
- Favors one-dimensional modulation in direction of ***B***.
- Equation of motion: simple pendulum!

$$\partial_z^2\phi = m_\pi^2 \sin\phi$$

NEUTRAL PION BACKGROUND

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- Chiral limit (m_π=0): Hamiltonian minimized by

$$\phi(z) = \frac{\mu B z}{4\pi^2 f_\pi^2}$$

- Solution in the general case:

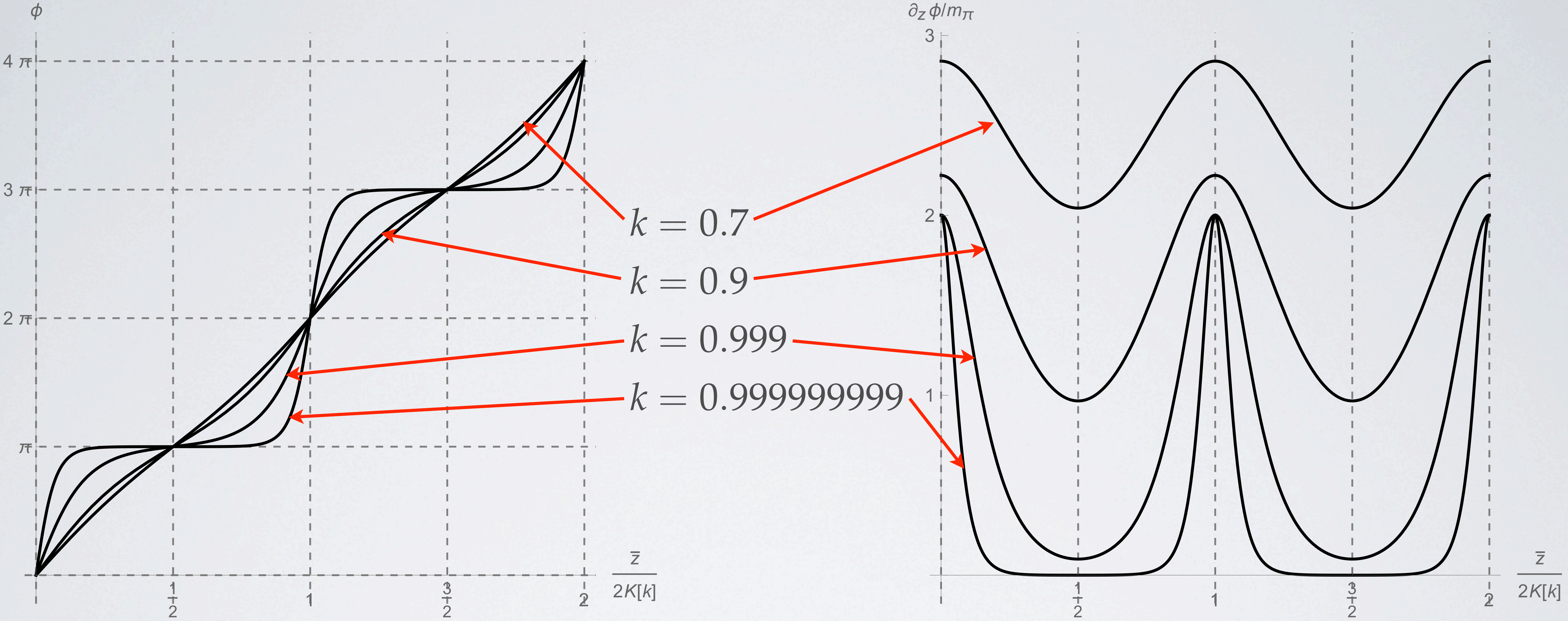
$$\cos\frac{\phi(\bar{z})}{2} = \mathrm{sn}(\bar{z}, k), \qquad \bar{z} = \frac{z m_\pi}{k}$$

Jacobi elliptic function

Elliptic modulus

CHIRAL SOLITON LATTICE

The solution is a periodic lattice of topological solitons!



$k = 0.7$

$k = 0.9$

$k = 0.999$

$k = 0.999999999$

Phase of the solution

Topological charge density

$$n_{\mathrm{B}}(z) = \frac{B}{4\pi^2}\partial_z\phi(z)$$

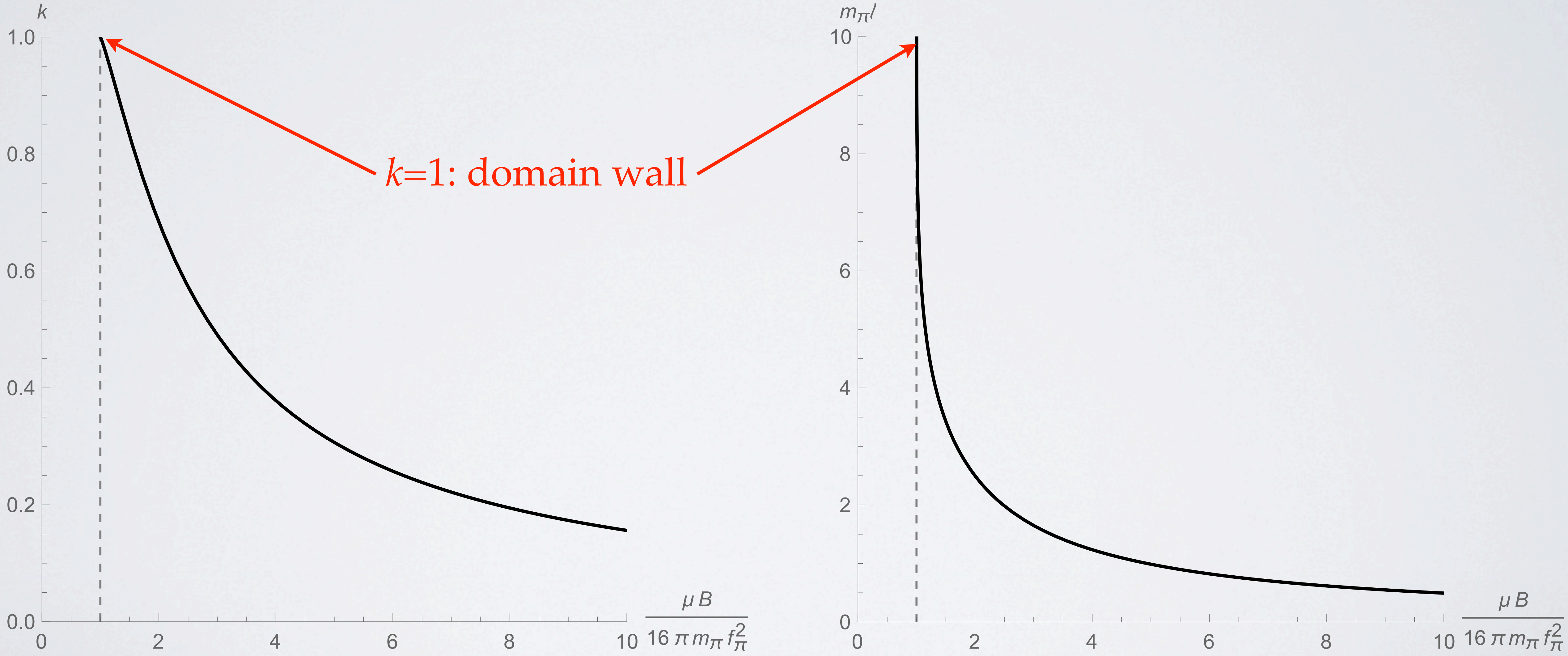
$$m(z) = \frac{\mu}{4\pi^2}\partial_z\phi(z)$$

Kishine, Ovchinnikov,
Solid State Phys. **66** (2015)

THE OPTIMUM CSL SOLUTION

The ground state is found by minimization of the Hamiltonian.

$$\frac{E(k)}{k} = \frac{\mu B}{16\pi m_\pi f_\pi^2}$$



k=1: domain wall

Optimum value of elliptic modulus

Period of the lattice

CRITICAL MAGNETIC FIELD

CSL is energetically favored
above certain critical value of magnetic field.

$$\frac{E(k)}{k} = \frac{\mu B}{16\pi m_\pi f_\pi^2} \quad \Longrightarrow \quad B_{\mathrm{CSL}} = \frac{16\pi m_\pi f_\pi^2}{\mu}$$

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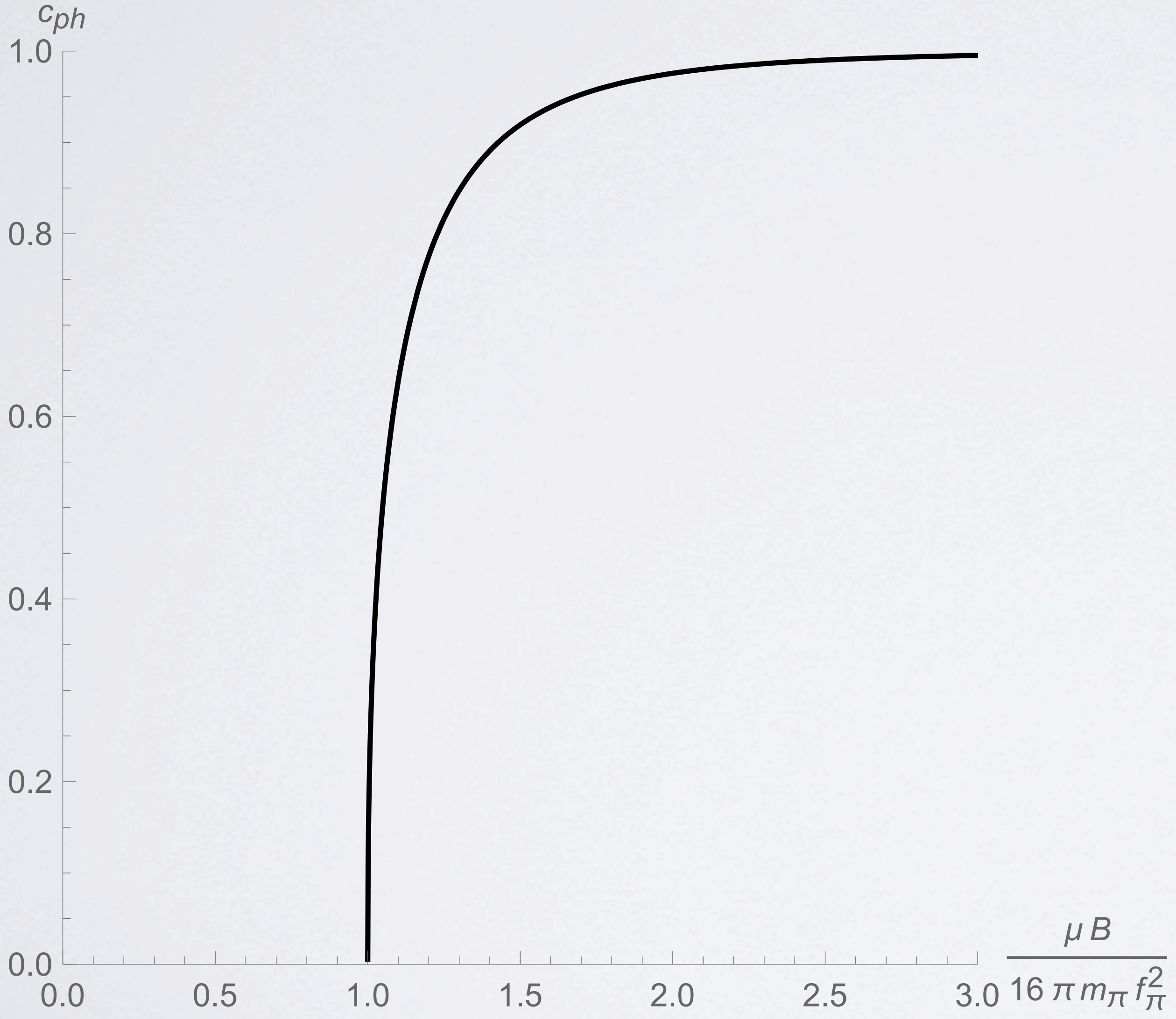
$$\left.\begin{aligned} m_\pi &\approx 140 \text{ MeV} \\ f_\pi &\approx 92 \text{ MeV} \\ \boxed{\mu = 900 \text{ MeV}} \end{aligned}\right\} \quad \Longrightarrow \quad B_{\rm CSL} \approx 0.066 \text{ GeV}^2$$

Conversion factor: $1 \text{ GeV}^2 \approx 1.7 \times 10^{20} \text{ G}$

PHONONS OF THE SOLITON LATTICE

- Linearize the sine-Gordon equation around the CSL solution.
- Results in the Lamé equation (n=1) with known spectrum.
- Two-band, gapless spectrum in accord with the Goldstone theorem.



Phonon phase velocity

$$c_{\mathrm{ph}} = \sqrt{1-k^2}\,\frac{K(k)}{E(k)}$$

CHARGED PION FLUCTUATIONS

$$\mathscr{L} = \frac{f_\pi^2}{4}\left[\mathrm{tr}(D_\mu \Sigma^\dagger D^\mu \Sigma) + m_\pi^2\,\mathrm{tr}(\Sigma + \Sigma^\dagger)\right]$$

$$\Sigma = e^{i\tau_3\phi} e^{\frac{i}{f_\pi}\vec{\tau}\cdot\vec{\pi}}$$

Expand to second order
in the field fluctuations

$$\mathscr{L}_{\text{bilin}} = \frac{1}{2}(\partial_\mu\vec{\pi})^2 + (A^\mu - \partial^\mu\phi)(\pi_1\partial_\mu\pi_2 - \pi_2\partial_\mu\pi_1) + \frac{1}{2}A^\mu A_\mu(\pi_1^2 + \pi_2^2) - \frac{1}{2}m_\pi^2\vec{\pi}^2\cos\phi$$

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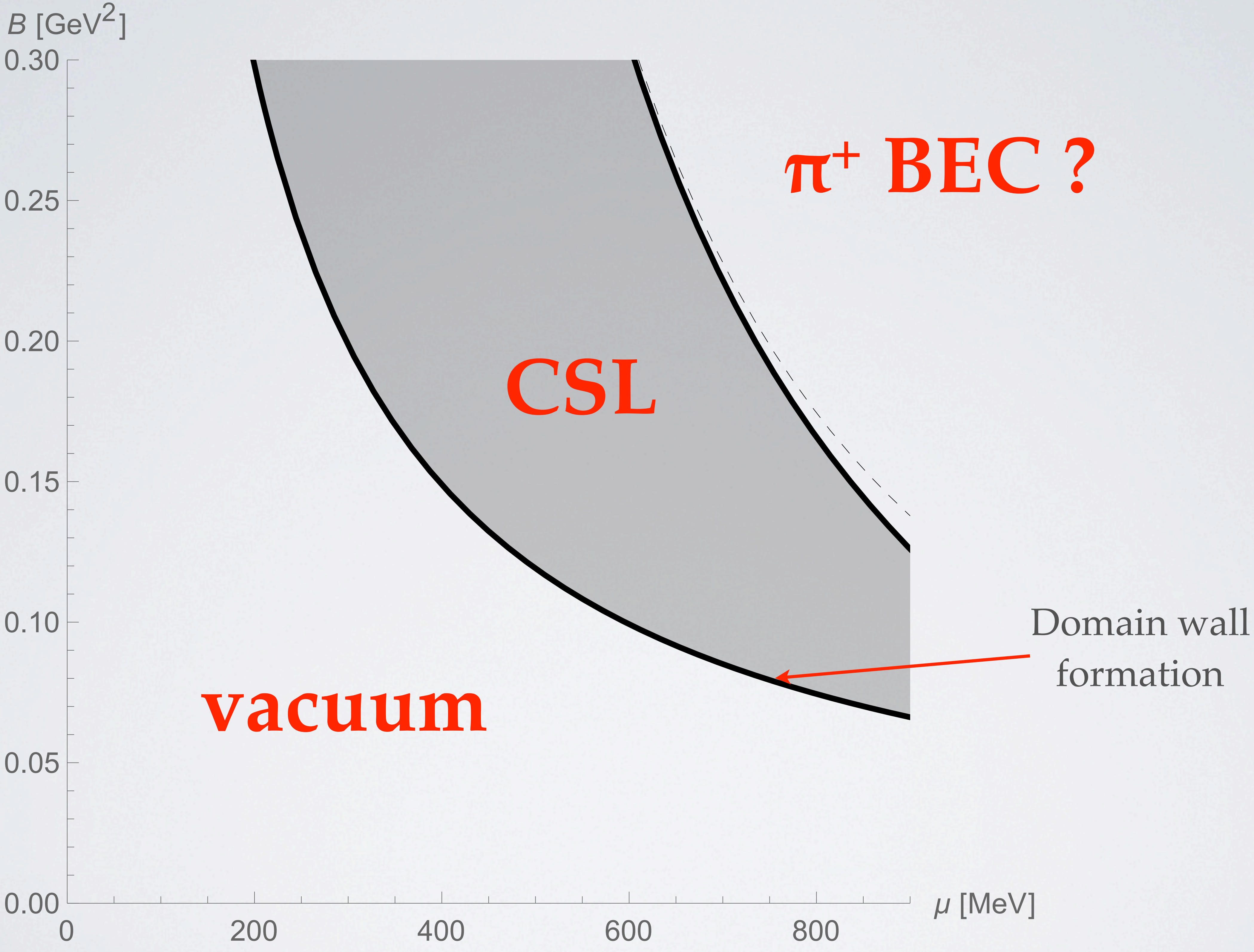
- CSL background acts as a chemical potential on charged pions!
- Chiral limit: easily solvable.
- General case: Lamé equation (*n*=2), solvable with some effort.

Li, Kusnezov, Iachello, JPA **33** (2000)
Finkel, González-López, Rodríguez, JPA **33** (2000)

PHASE DIAGRAM

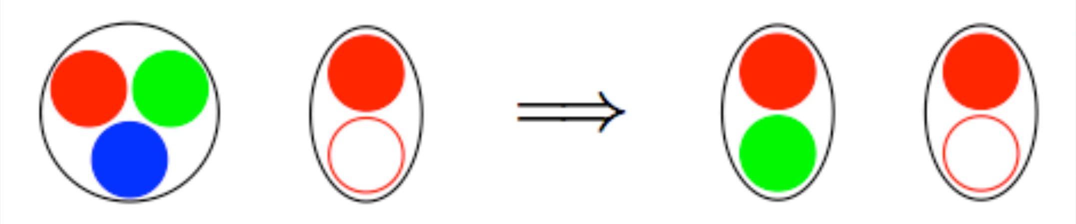


Conversion factor: $1\,\mathrm{GeV}^2 \approx 1.7 \times 10^{20}\ \mathrm{G}$

TB, Yamamoto, JHEP **04** (2017)

QCD-LIKE THEORIES AT B=0

- Quarks in a (pseudo)real representation of gauge group: theory invariant under the exchange $q_{\mathrm{L}} \Longleftrightarrow \bar{q}_{\mathrm{R}}$.
  - Real: adjoint QCD, G_2-QCD, …
  - Pseudoreal: 2cQCD, …
- Bosonic baryons in the spectrum:



- Global flavor SU(2N) symmetry for N quark flavors.
- Pseudo-NG bosons in the vacuum:
  - N^2–1 pions.
  - $N^2 \pm N$ diquarks.

ABSENCE OF SIGN PROBLEM

- For two flavors with $q_u = -q_d$, Dirac determinant is positive:

Complex conjugation

Conjugation of generators: $T_a^* = -\mathcal{P} T_a \mathcal{P}^{-1}$

$$(KC\gamma_5\mathcal{P})\mathcal{D}_u = \mathcal{D}_d(KC\gamma_5\mathcal{P})$$

Dirac operator

- Conjectured that absence of sign problem implies absence of inhomogeneous order in the phase diagram.

Splittorff, Son, Stephanov, PRD **64** (2001)

LOW-ENERGY EFT IN STRONG *B*-FIELDS

- Symmetry-breaking pattern (both real and pseudoreal theories):

$$\underbrace{\mathrm{SU}(2)\times\mathrm{SU}(2)}_{\text{Not the chiral group!}}\times\mathrm{U}(1)_Q\to\mathrm{SU}(2)_{\mathrm{diag}}\times\mathrm{U}(1)_Q$$

- Particle content: neutral pion and diquark–antidiquark pair.
- Leading-order effective Lagrangian:

$$\mathscr{L}=\frac{f_\pi^2}{4}\Big[\underbrace{(g_\parallel^{\mu\nu}+v^2g_\perp^{\mu\nu})}_{\text{Anisotropy due to }B\text{-field}}\operatorname{tr}(D_\mu\Sigma^\dagger D_\nu\Sigma)+m_\pi^2\operatorname{tr}(\Sigma+\Sigma^\dagger)\Big]+\mathscr{L}_{\mathrm{WZW}}$$

- All the couplings f_π, m_π, v depend on the *B*-field!

LOW-ENERGY EFT IN STRONG *B*-FIELDS

- Wess–Zumino–Witten term:

Baryon number
of a single quark

$$\mathscr{L}_{\mathrm{WZW}} = +\frac{ibC}{4}\epsilon^{\mu\nu\alpha\beta}F^{Q}_{\mu\nu}A^{B}_{\alpha}\,\mathrm{tr}[\tau_3(\partial_\beta\Sigma\Sigma^\dagger - \partial_\beta\Sigma^\dagger\Sigma)]$$

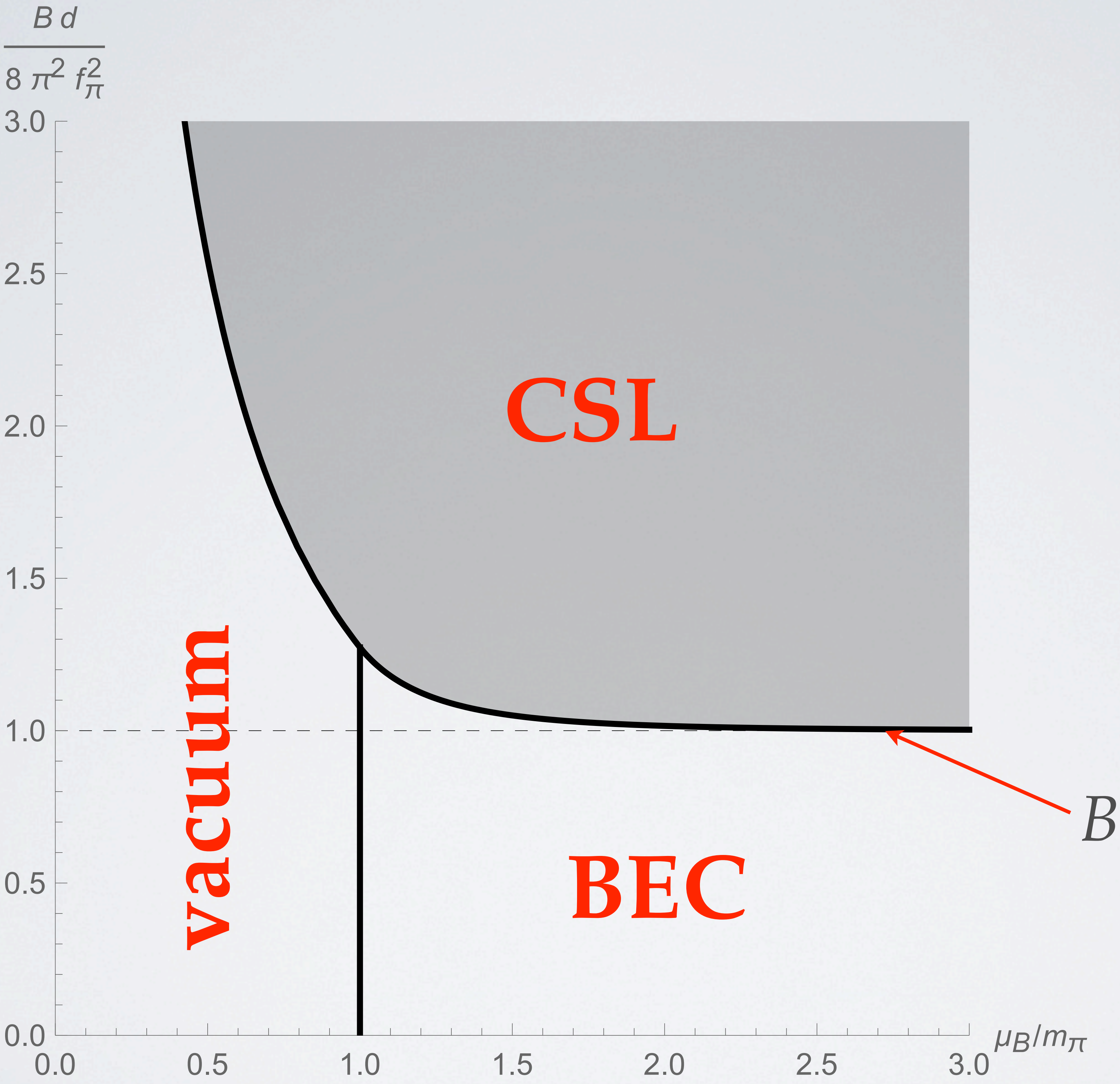
$$-\frac{C}{6}\epsilon^{\mu\nu\alpha\beta}A^{Q}_{\mu}\,\mathrm{tr}(\partial_\nu\Sigma\partial_\alpha\Sigma^\dagger\partial_\beta\Sigma\Sigma^\dagger)$$

number of quark color d.o.f.

$$C = \frac{d}{8\pi^2}(q_u - q_d)$$

TB, Filios, Kolešová, JHEP **12** (2019)

PHASE DIAGRAM



$$q_{u,d} = \pm\frac{1}{2}$$

$$b = \frac{1}{2}$$

$$B_{\rm cr} = \frac{8\pi^2 f_\pi^2}{d}$$

TB, Filios, Kolešová, PRL **123** (2019)

CAN THE CSL PHASE BE REACHED?

- Remember that f_π depends on the magnetic field!

$$B > B_{\mathrm{cr}} \qquad \Longleftrightarrow \qquad \frac{B}{[4\pi f_\pi(B)]^2} > \frac{1}{2d}$$

- One-loop result for pseudoreal QCD-like theories:

$$[4\pi f_\pi(B)]^2 = [4\pi f_\pi(0)]^2 + 2B\log 2 + \mathcal{O}(B^2)$$

Shushpanov, Smilga, PLB **402** (1997)

- In theories with large enough d, the critical field is in reach of the EFT!
- For theories such as 2cQCD, more information on $f_\pi(B)$ is needed.

CONCLUSIONS

- In sufficiently strong magnetic fields, QCD vacuum is unstable under formation of a soliton lattice of neutral pions.
- This can generate baryon densities relevant for neutron stars, although the required magnetic field seems too strong.
- The same inhomogeneous phase appears in other theories:
  - QCD with isospin chemical potential.
  - QCD-like theories with (pseudo)real quarks.
- There are many avenues for further investigation:
  - Effects of nonzero temperature?
  - Modification in finite volume?
  - Modification due to inhomogeneous magnetic fields?
  - Competition with nuclear matter?